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Potential vorticity mixing by marginally unstable baroclinic disturbances

By T. WARN and P. GAUTHIER, *Department of Meteorology, McGill University, Montreal, Quebec, Canada*

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ABSTRACT

The weakly nonlinear dynamics of a baroclinic wave in a two-layer model near minimum critical shear is described in terms of a nonlinear critical layer problem which is completely integrable in the absence of dissipation. Sufficiently small amplitude supercritical inviscid disturbances are found to equilibrate to leading order, even though the absolute potential enstrophy of the lowest layer is always transient and transferred irreversibly to smaller and smaller scales. The inviscid equilibrium amplitude of the fundamental is found to be larger by a factor of $\sqrt{2}$ than the weakly dissipative value found by Pedlosky, implying that the limits $t \to \infty$ and vanishing dissipation are not interchangeable. The fundamental equilibrates when the mixing in the lowest layer results in the streamwise homogenization of the coarse-grained potential vorticity. It is also noted that coarse-grain homogenization can be established on faster time scales than the fine-grain versions discussed by Rhines and Young. With weak or vanishing supercriticality or larger initial disturbance amplitudes, the fundamental may either equilibrate or oscillate periodically, depending on the form and strength of the initial disturbance. In the oscillating case, the baroclinic vortex reversibly wraps up and un-wraps and there is no tendency to mix potential vorticity. Finally, the possibility of further instabilities similar to those discussed by Killworth and McIntyre, and Haynes for the Rossby wave critical layer problem is also noted.

1. Introduction

The present article is concerned with the evolution of marginally unstable baroclinic waves in a two-layer, quasi-geostrophic, periodic channel model at minimum critical shear. Investigations by Pedlosky (1982a, b) suggest that this configuration is of intrinsic theoretical interest as it is intermediate in complexity between traditional linear or weakly nonlinear solutions and strongly nonlinear turbulence involving many degrees of freedom. Specifically, it leads to what is probably the simplest dynamically consistent example of the wrapping up and filamentation of absolute potential vorticity contours by a developing baroclinic vortex, a process which occurs regularly in the real atmosphere and is undoubtedly an important ingredient in the process of baroclinic adjustment and nonlinear equilibration (Stone, 1978; Salmon, 1980; Vallis, 1988; Shepherd, 1988). The study of this problem would also seem to be a necessary prerequisite to the study of marginally unstable waves when the spatial periodicity constraint is removed (e.g. as in the study of the propagation of wave packets).

As observed by Pedlosky (1982a), baroclinic instability at minimum critical shear in the two-layer model is complicated by the fact that the doppler-shifted frequency of the fundamental vanishes in the lowest layer. The linear, low-level, potential vorticity advection consequently vanishes in a reference frame moving with the wave implying that the entire layer is a critical layer. As is well-known from the study of steady critical layers in shear flows (Benney and Bergeron, 1969; Davis, 1969; Maslowe, 1986), advection by the disturbance is then locally important, particularly in the presence of weak or vanishing dissipation, and the number of degrees of freedom increased dramatically. One consequence, noted by Pedlosky (1982a), is that the early single-mode theory becomes invalid at minimum critical shear, an observation which explains the discrepancies between that theory and the numerical calculations reported by Boville (1981). The latter solutions were found to depend strongly on the number of modes included in the integration.

The problem was studied further by Pedlosky (1982b), who developed the weakly nonlinear theory in the presence of a small amount of dissipation. The evolution of the marginally unstable mode was found to be described by an infinite-dimensional dynamical system involving all harmonics. Despite this complication, both numerical and analytical simplifications result since only direct interactions with the fundamental need be considered. Detailed numerical simulations of the resulting evolution equations were also reported. One important result was the observation that, when the dissipation is small, mixing by the fundamental leads, over time, to the nearly complete expulsion of the initial absolute potential vorticity gradient in the bottom layer, a result which was exploited to obtain an analytical expression for the equilibrium amplitude of the fundamental. It was found that as the dissipation tends to zero, the amplitude approaches a value which is somewhat larger than the single-mode prediction (i.e., $\pi/\sqrt{12} \approx 0.907$ versus $1/\sqrt{2} \approx 0.703$).

Pedlosky's analysis relies heavily on the presence of small but non-zero dissipation, whereas we shall be concerned primarily with the inviscid case. Uniqueness considerations then require that temporal effects be considered. In this limit the problem might be expected to be similar to the transient nonlinear critical layer problem (e.g., Warn and Warn, 1976; 1978; Stewartson 1978). One important result of the present study, motivated by these earlier works, is that Pedlosky's infinite-dimensional dynamical system can be recast into a more compact system of integro-differential equations which turn out to be exactly integrable in the absence of viscosity. Pedlosky's weakly dissipative equilibria are also easily recovered with this formulation.

When the supercriticality is large (or equivalently when the initial disturbance is small) and the flow inviscid, the fundamental is found to equilibrate even though the potential vorticity of the lowest layer remains forever transient. This is related to the fact that the fundamental sees only the coarse-grain potential vorticity which does become steady. The interactions are therefore local in wavenumber space in the sense that the small scales in the potential vorticity field exert no direct effect on the fundamental. It is also found that the inviscid equilibrium amplitude exceeds Pedlosky's weakly dissipative value by a factor of $\sqrt{2}$ (i.e., $\pi/\sqrt{6} \approx 1.283$ versus $\pi/\sqrt{12} \approx 0.907$) indicating that the order of the limits $t \to \infty$ and vanishing dissipation is important.

It is also noteworthy that while there is no fine-scale homogenization in the absence of viscosity, there is a coarse-grain equivalent. Thus if Q is the absolute potential vorticity of the lowest layer and if we define a coarse-grain average (e.g., Welander, 1955)

$$\bar{Q} = \frac{1}{\delta A} \iint_{\delta A} Q \, \mathrm{d}A,$$

where δA is an arbitrary area within the fluid then

$$\lim_{\delta A \to 0} \lim_{t \to \infty} \bar{Q} = Q_0(\psi),$$

on closed streamlines, where Q_0 is a generalized average of the initial Q along streamlines. The coarse-grain potential vorticity tends to homogenize along closed streamlines even in the absence of dissipation. It is emphasized that the order of the limit process must be respected for this result to hold. This conclusion holds for any nondiffusive passive scalar on arbitrary closed, steady, streamlines provided certain exceptional cases are excluded (see Section 6). It is the nondiffusive analog of shear-augmented, fine-grained, streamwise homogenization observed during the first stage of the relaxation of a weakly diffusive passive scalar (diffusive homogenization and the associated timescales are discussed by Rhines and Young, 1983). We note in passing that coarse-grain averages also arise naturally the prediction problem due to the existence of a limiting observational scale.

For certain initial conditions, including those which are relevant to a marginally unstable wave at large supercriticality, the generalized average has the same value on each streamline and we recover an inviscid version of the Prandtl-Batchelor theorem (Batchelor, 1956), in the sense that the coarse-grained potential vorticity gradient in the lowest layer is completely eliminated. Since coarse-grained homogenization is independent of the magnitude of the diffusion coefficient, it can be established on faster timescales than its fine-grained counterpart (provided δA is not too small).

Finally it should be mentioned that the two-layer model with uniform flow in each layer is structurally unstable in the sense that the long-time behaviour can be very sensitive to small perturbations in the physics (Pedlosky, 1981; Pedlosky and Polvani, 1987). An early indication of this was given by Holopainen (1961), who observed an O(1) shift of the neutral curve in the limit of vanishingly small Ekman dissipation. The observed sensitivity is most likely associated with the chosen background state which has been largely adopted for reasons of analytic simplicity. It is also most probably a signal of real complexity which would be observed in experiments undertaken near this state, implying that the construction of a long-time theoretical description may be difficult. Our approach is based on the observation that the difficulties are probably less acute if attention is switched from long to intermediate-time behaviour since small influences are less likely to be important. This has the advantage that the inviscid equilibration processes discussed here and in earlier works should provide a reasonably robust picture of the early stages of baroclinic instability despite the underlying structural instability of the model. A disadvantage is that there is little experimental work available on this aspect of the flow.

In Section 2 the linear theory of a marginally unstable wave near minimum critical shear is reviewed. Aspects of the intermediate-time matching conditions are also discussed. The nonlinear evolution equations are derived and discussed in Section 3 while Pedlosky's weakly dissipative equilibrium solutions are recovered in Section 4. The transient inviscid solutions are constructed and discussed in Section 5, while coarse-grain homogenization of a nondiffusive passive scalar is treated in Section 6. The results are then summarized in Section 7 where the possibility of further instabilities of the type discussed by Killworth and McIntyre (1985) and Haynes (1985) for the forced Rossby wave problem is noted.

2. Review of linear theory

We begin with a brief outline of the model and linear theory. With the exception of the discussion of the matching conditions for the nonlinear evolution stage, which does not appear to have been given previously in this context, the treatment parallels that given by Pedlosky (1970, 1979).

The model equations describe the evolution of the quasi-geostrophic motion of two layers of homogeneous fluids of differing densities. The flow is confined to a periodic channel of width L_y bounded by horizontal planes a distance D apart. In the absence of motion the interface between the fluids is assumed equidistant between the planes. The Earth's sphericity is incorporated by setting $2\Omega = f_0 + \beta' y$. If the upper and lower layers are denoted by subscripts 1 and 2 respectively, the dimensionless equations may be written

$$\frac{\partial Q_n}{\partial t} + J(\psi_n, Q_n) = 0, \tag{1}$$

where

$$Q_n = \nabla^2 \psi_n + (-1)^n F(\psi_1 - \psi_2) + \beta y,$$

J denotes the Jacobian,

$$F = f_0^2 L_y^2/(g' D/2), \quad \beta = \beta' L_y^2 / U_s,$$

g' is the reduced gravity $(= (\rho_2 - \rho_1) g/\rho_2)$ and U_s denotes a typical velocity scale. On the side walls, located at $y = 0, 1$, it is required that

$$\frac{\partial \psi_n}{\partial x} = 0, \quad \int_0^{\ell} \frac{\partial^2 \psi_n}{\partial t \partial y} \, dx = 0,$$

where $\ell = L_x / L_y$ (Phillips, 1954).

The initial state is taken to be a small departure from uniform zonal motion in each layer, so we may write

$$\psi_n = -U_n y + \mu \phi_n,$$

where $\phi_n = O(1)$ and $\mu \ll 1$, i.e., μ characterizes the amplitude of the initial disturbance. The linear disturbance equations then take the form

$$\left(\frac{\partial}{\partial t} + U_n \frac{\partial}{\partial x}\right) q_n + (\beta - (-1)^n F U_T) \frac{\partial \phi_n}{\partial x} = 0, \tag{2}$$

where $U_T = U_1 - U_2$. q_n is just Q_n with ψ_n replaced by ϕ_n without the β term. (2) has solutions of the form

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_1 \end{pmatrix} = a \begin{pmatrix} 1 \\ \gamma \end{pmatrix} W + \text{c.c.}, \tag{3}$$

where c.c. denotes the complex conjugate and

$W = e^{ik(x-ct)} \sin p\pi y$,

provided

$$c = \frac{U_1 + U_2}{2} - \frac{\beta(\alpha^2 + F)}{\alpha^2(\alpha^2 + 2F)} \pm \frac{1}{2\alpha^2(\alpha^2 + 2F)} \times \{4\beta^2 F^2 - U_T^2 \alpha^4 (4F^2 - \alpha^4)\}^{1/2}, \tag{4}$$

and

$$\gamma = \frac{\alpha^2 + F}{F} + \frac{\beta + FU_T}{F(c - U_1)}. \tag{5}$$

Here $\alpha^2 = k^2 + p^2\pi^2$ and $k = 2\pi s/\ell$, with s and p integer.

According to (4), the flow is unstable whenever U_T exceeds the critical shear U_c, where

$$U_c = \frac{2\beta F}{\alpha^2(4F^2 - \alpha^4)^{1/2}}.$$

Below we follow Pedlosky (1982a, b) and restrict our considerations to the evolution of disturbances at or near minimum critical shear*, i.e., to disturbances with $\alpha = \alpha_m$, where

$$\alpha_m^2 = \sqrt{2}\, F,$$

and U_c near U_{cm}, where

$$U_{cm} = \frac{\beta}{F}.$$

This case is of particular interest when the periodicity constraint in x is dropped and the evolution of wave packets is studied. We note that $\gamma = \gamma_m = \sqrt{2} - 1$ for this parameter setting.

We make the following observations.

(*i*) *Marginally stable flow*

For a slight departure from neutrality, $U_T = U_{cm} + \Delta$, where $|\Delta| \ll 1$, the basic state potential vorticities are

$$F(2U_{cm} + \Delta)y, \quad -F\Delta y,$$

in the upper and lower layers, respectively. The dispersion relation simplifies to

$$c^{\pm}(\alpha_m) = U_2 + \frac{(-U_{cm}\Delta)^{1/2}}{\sqrt{2} + 2} + O(\Delta),$$

at α_m, suggesting that the wave grows (propagates in a reference frame moving with speed U_2) on a timescale $t_L = |\Delta|^{-1/2}$ above (below) criticality (Pedlosky, 1970, 1979). We emphasize that this estimate holds only when the initial amplitude μ is small enough for linear theory to hold out to times $t = O(t_L)$ (see below). In the supercritical case the unstable mode will ultimately dominate in this limit, implying

$$\frac{d\phi}{dt} \sim -ikc^+\phi \quad \text{for} \quad \Delta^{1/2} t \gg 1 \tag{6}$$

Given a sufficiently small supercritical disturbance, (6) is the appropriate matching condition for the early-time nonlinear stage. It turns out, however, that there is a more general condition.

(*ii*) *Early-time behaviour*

During the early stages of the weakly nonlinear theory the disturbance evolution is determined by the equations at criticality and it is this behaviour which must be matched at longer times. Exactly at criticality the phase speeds coalesce and there is only one normal mode solution. The general solution of (2) then takes the form

$$\phi = a \begin{pmatrix} 1 \\ \gamma \end{pmatrix} W + b \begin{pmatrix} t \\ -\gamma_c/ik + \gamma t \end{pmatrix} W + \text{c.c.}, \tag{7}$$

when $U_T = U_c$. Here $\gamma_c = \partial\gamma/\partial c$. The situation is therefore similar to that encountered in the oscillator problem

$$\frac{d^2y}{dt^2} + \omega^2 y = 0,$$

when $\omega = 0$, i.e., when the two frequencies $\pm\omega$ coalesce. (7) is most easily obtained by noting that when ϕ is of the form (3) with γ given by (5), then away from the neutral curve two independent solutions are, $\phi(c^{\pm})$, where $c^{\pm}$ represent distinct roots of the dispersion relation (4). These

* Since the wavenumbers are quantized this requires that the system is "tuned" by selecting the appropriate ℓ, say.

are unsuitable on the neutral curve as they reduce to the same solution. Since $\phi(c^+)$ and

$$\frac{\phi(c^+) - \phi(c^-)}{c^+ - c^-},$$

represent independent solutions for all values of the parameters and since they reduce to $\phi(c)$ and $\partial\phi/\partial c$ on the neutral curve, (7) follows.

The above argument suggests that the growing disturbance can be regarded as a linear combination of both the growing and decaying solutions near the marginal stability curve. The decaying mode intervenes in this limit because it is as dangerous as the unstable mode in this region of parameter space. According to (7) the linear system predicts growth even though the waves are neutral according to the (complex) phase speed at this order. This sort of algebraic amplification associated with coalescing or close frequencies has been termed "direct resonance" by Akylas and Benney (1980, 1982).

Under direct resonance the neglected terms must be reconsidered even right at neutrality if the true behaviour is to be determined. The appropriate timescale depends on the magnitude of the neglected terms, i.e., it depends on either Λ or μ. For instance, right at criticality the timescale is determined by μ which is the only small parameter remaining in the problem. Specifically, the timescale is $t_{NL} = \mu^{-1/2}$ (see Section 3). In more general circumstances the appropriate timescale is the smaller of t_L and t_{NL}.

The long time linear behaviour implied by (7) is

$$\phi \sim bt \begin{pmatrix} 1 \\ \gamma \end{pmatrix} W + \text{c.c.} + O(1) \quad \text{for} \quad t \gg 1. \tag{8}$$

which can be used to match to the early-time nonlinear stage (Akylas and Benney 1982). (8) is to be preferred over (6) since it involves no *a priori* choice concerning the dominant process. (6) is recovered in the limit $t_L \ll t_{NL}$ as the evolution is then linear on intermediate times with an $O(\Lambda)$ linear growth. The linearly unstable mode can then become dominant before the nonlinearities have a chance to intervene. Actually, as seen in the next section, expressions valid on intermediate times are not required since the nonlinear stage can be matched directly to (properly scaled) initial conditions (Akylas and Benney, 1982).

(*iii*) *Minimum critical shear*

When $U_c = U_{cm}$, (4) reduces to

$$c^{\pm}(\alpha) = U_2 + \frac{\beta(\alpha^2 - 2F^2)}{2\alpha^2 F(\alpha^2 + 2F)}(1 \pm 1),$$

i.e., $c^-(\alpha) = U_2$ for all α. The lower branch is non-dispersive at minimum critical shear (Newell, 1972), implying that all modes which interact with the fundamental, do so resonantly. While these turn out to be less dangerous than ordinary resonant triads, in the sense that their amplitudes remain smaller than the fundamental, they are nonetheless important and can influence the evolution in an important way. One consequence, noted by Pedlosky (1982a), is that the earlier weakly nonlinear single-mode theory (Pedlosky, 1970) becomes invalid at minimum critical shear due to the presence of harmonics, an observation which explains the discrepancies between the original single-mode theory and the detailed numerical simulations reported by Boville (1981).

An alternate explanation of the dynamics can also be given. Since $c(\alpha_m) = U_2$ at minimum criticality, the mean advection of the potential vorticity in the lowest layer vanishes in a reference frame moving with the wave and the entire layer is seen to be a critical layer (Pedlosky 1982a, b). Arguing by analogy with the critical layer problem in parallel shear flows, (Benney and Bergeron, 1969; Davis, 1969), the nonlinearities in the lowest layer will be enhanced, particularly in the presence of weak or vanishing dissipation, and many harmonics can be expected to be generated. This is most easily seen by noting that the long-time linear form of the streamfunction in a reference frame moving with the wave is

$$\psi_2 = -(U_2 - c)y + \mu t\gamma(W + \text{c.c.}).$$

Away from minimum criticality, the first term dominates and the streamlines in the layer remain nearly parallel. The leading order advection simply tends to translate the initial absolute potential vorticity, while higher order wave-mean flow interactions limit the growth of the disturbance. In this case the single-mode theory can be expected to hold. At minimum criticality, on the other hand, the first term vanishes and the leading order flow in the layer is given by the

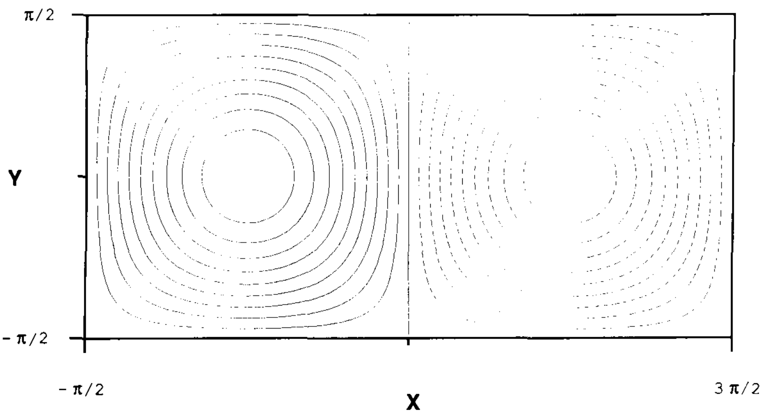

Fig. 1. Streamfunction of the fundamental. Dashed lines indicate negative values.

marginal disturbance (Fig. 1). The absolute potential vorticity of the layer will then undergo severe distortion and a wide range of harmonics will be rapidly produced.

3. Nonlinear evolution equations

The asymptotic development presented in this section is again similar in many respects to that given by Pedlosky (1982a, b). Aside from the early time matching, we depart only in the latter stages of the analysis and thereby obtain a simpler, more transparent form of his evolution equations (eg. (19) and (20) below).

As noted earlier, there are two possible timescales for the nonlinear evolution depending on the relative sizes of Δ and μ. The most general situation obtains when these are of the same order‡. To treat this case let ε, where $\varepsilon \ll 1$, be an as yet undetermined measure of the wave amplitude during the nonlinear stage (i.e., let $\psi_n = -U_n y + \varepsilon\phi_n$) and let $F\Delta = \delta\varepsilon^2$, where δ is treated as an order one quantity. The treatment is simplified if $U_2 = 0$, which involves no loss of generality. With this choice the phase speed of the marginal wave vanishes and solutions depending only on the slow time $\tau = \varepsilon t$ can be constructed. (1) then becomes

$$U_c \frac{\partial q_1}{\partial x} + 2FU_c \frac{\partial \phi_1}{\partial x} = -\varepsilon\left(\frac{\partial q_1}{\partial \tau} + J(\phi_1, q_1) + rq_1\right) - \varepsilon^2\delta\left(F^{-1}\frac{\partial q_1}{\partial x} + \frac{\partial \phi_1}{\partial x}\right), \tag{9}$$

and

$$\frac{\partial q_2}{\partial \tau} + J(\phi_2, q_2) = -rq_2 + \varepsilon\delta\frac{\partial \phi_2}{\partial x}. \tag{10}$$

While we shall be concerned primarily with the inviscid case we have, for the purposes of comparison with Pedlosky (1982b), added weak disturbance potential vorticity dissipation proportional to εr. The form of the dissipation is somewhat artificial (see Pedlosky's paper for a discussion of this point); we simply note that the formulation can result in a source of absolute potential vorticity which is absent in the inviscid system (i.e., see (14)).

An asymptotic solution to (9) and (10) may now be sought by expanding ϕ in the power series

$$\phi = \phi^{(0)} + \varepsilon\phi^{(1)} + \varepsilon^2\phi^{(2)} + \ldots,$$

where $\phi^{(0)}$ has the form of the marginal disturbance, i.e.,

$$\phi^{(0)} = A(\tau)\begin{pmatrix}1\\ \gamma_m\end{pmatrix} W_m + \text{c.c.}, \tag{11}$$

where W_m and γ_m are W and γ with $\alpha = \alpha_m$. Eq.

‡ While requiring the timescales to be of the same order might at first glance seem restrictive, it is in fact the most general choice since all neglected processes intervene at the earliest possible order. This has been called principle of maximal balance. Balances involving fewer processes then follow on considering special limits of the evolution equation.

(10) is seen to be satisfied to leading order on observing that

$$q_2^{(0)} = 0. \tag{12}$$

The evolution is determined at the next order, i.e., from

$$\frac{\partial q_1^{(1)}}{\partial x} + 2F\frac{\partial \phi_1^{(1)}}{\partial x} = \frac{2F}{U_c}\left(\frac{\partial \phi_1^{(0)}}{\partial \tau} + r\phi_1^{(0)}\right), \tag{13}$$

and

$$\frac{\partial P}{\partial \tau} + J(\phi_2^{(0)}, P) = -r(P + \delta y), \tag{14}$$

where P is the leading order absolute potential vorticity in the lower layer, related to $\phi^{(1)}$ by

$$\nabla^2\phi_2^{(1)} + F(\phi_1^{(1)} - \phi_2^{(1)}) = P + \delta y. \tag{15}$$

Here both (12) and

$$q_1^{(0)} = -2F\phi_1^{(0)},$$

have been used in obtaining (13) and (14). Integrating (13) with respect to x using (11) then gives

$$\nabla^2\phi_1^{(1)} + F(\phi_1^{(1)} + \phi_2^{(1)}) = \frac{2F}{ikU_c}\left(\frac{\mathrm{d}}{\mathrm{d}\tau} + r\right)AW_m + \text{c.c.} + g(y, \tau), \tag{16}$$

where $g(y, \tau)$ is arbitrary at this point. (14) and the adjoint orthogonality condition for the self-adjoint system (15) and (16) then determine the evolution of A and P. Since $(1, \gamma_m)' W_m^*$ is a solution of the homogeneous version of (15) and (16), the solvability condition may be written

$$\frac{\mathrm{d}A}{\mathrm{d}\tau} + rA = -i\sigma\langle PW_m^*\rangle, \tag{17}$$

where

$$\sigma = \frac{kU_c\gamma_m}{F},$$

and the angle brackets denote an area average. (14) and (17) fully determine the evolution once the initial conditions are specified.

If

$$\phi_1 = \mathrm{O}(\mu), \quad Q_2 = -\Delta Fy + \mu h(x, y),$$

initially, where $h = \mathrm{O}(1)$, then

$$\phi_1^{(0)} + \mathrm{O}(\varepsilon) = \mathrm{O}\left(\frac{\mu}{\varepsilon}\right),$$

$$P + \mathrm{O}(\varepsilon) = -\frac{\Delta Fy}{\varepsilon^2} + \frac{\mu}{\varepsilon^2}h(x, y) + \mathrm{O}(\varepsilon).$$

The second expression suggests that either $\varepsilon = \mu^{1/2}$ or $\varepsilon = \Delta^{1/2}$, whichever is larger. The choice $\varepsilon = \mu^{1/2}$ allows both the marginally unstable and neutral cases to be considered simultaneously. It then follows that

$$A = 0, \quad P = -\delta y + h(x, y) \quad \text{when } \tau = 0, \tag{18}$$

where $\delta = F\Delta/\mu$. δ is a measure of the relative importance of supercriticality and the initial disturbance potential vorticity of the lowest layer. Evidently the wave evolution depends only on the initial disturbance when $\delta \ll 1$. This dependence will be seen to disappear in the limit $\delta \gg 1$ (subliminal disturbances). When $\delta = \mathrm{O}(1)$ both initial conditions and supercriticality are important.

Hereafter we restrict our considerations to the case $s = p = 1$. If we introduce the new variables

$$X = kx, \quad Y = \pi(y - 1/2),$$

$$T = k\gamma_m\left(\frac{2U_c}{F}\right)^{1/2}\tau, \quad S = \pi\left(\frac{2F}{U_c}\right)^{1/2}A,$$

$$Q = \pi P + \frac{\delta\pi}{2}, \quad \nu = \frac{r}{k\gamma_m}\left(\frac{F}{2U_c}\right)^{1/2}.$$

(the primary wave scaling then agrees with Pedlosky (1982b) when $\delta = 1$), (14) and (17) are written

$$\frac{\partial Q}{\partial T} + J(\psi, Q) = -\nu(Q + \delta Y), \tag{19}$$

$$\frac{\mathrm{d}S}{\mathrm{d}T} + \nu S = -2i\langle QZ^*\rangle, \tag{20}$$

where $\psi = SZ + \text{c.c.}$, and $Z = (\mathrm{e}^{iX}\cos Y)/2$. Also

$$S = 0 \quad Q = -\delta Y + H(X, Y) \quad \text{when } T = 0, \tag{21}$$

where $H = \pi h$.

The evolution of the marginally unstable wave is determined by the infinite-dimensional nonlinear system (19) and (20) which is a compact, real-space equivalent of the infinite spectral form (2.24) of Pedlosky (1982b) when the slow spatial variations included in that study are neglected. The correspondence is most easily seen on differentiating (20) and using (19) to eliminate $\partial Q/\partial t$. (19) and (20) have a certain similarity to the equations describing the nonlinear evolution of a forced Rossby wave critical layer under special parameter settings (Stewartson, 1978; Warn and Warn, 1978). The main difference is in the form of (20), which replaces the velocity jump condition, and the form of the advecting streamfunction, ψ, which replaces the streamfunction associated with the Kelvin cat's eyes.

4. Weakly dissipative equilibria

When the flow is steady, S can be taken to be real and (19) and (20) can be written as

$$J(\psi, Q) = -\nu(Q + \delta Y), \tag{22}$$

$$\nu S^2 = \langle Q\, \psi_X \rangle, \tag{23}$$

Where $\psi = S\cos X \cos Y$. In the fundamental domain $X \in [-\pi/2, 3\pi/2]$, $Y \in [-\pi/2, \pi/2]$, the streamfunction pattern involves a pair of vortices centered at $(0,0)$ and $(\pi, 0)$ and the streamlines are simply connected and closed (Fig. 1).

The combination

$$\psi = 0, \quad Q = -\delta Y,$$

represents a solution of (22) and (23). Since it corresponds to the original unperturbed state it is unstable when $\delta > 0$ and the dissipation is weak. In this case a second solution may be found by expanding ψ and Q in power series in ν. At leading order (22) gives $Q^{(0)} = G(\psi^{(0)})$, while (23) is satisfied automatically. At the next order

$$J(\psi^{(1)}, Q^{(0)}) + J(\psi^{(0)}, Q^{(1)}) = -(Q^{(0)} + \delta Y), \tag{24}$$

$$|S^{(0)}|^2 = \langle Q^{(0)} \psi_X^{(1)} \rangle + \langle Q^{(1)} \psi_X^{(0)} \rangle. \tag{25}$$

Integrating (24) over the area enclosed by a streamline $\psi^{(0)}$ gives

$$I(\psi^{(0)}) = \iint (Q^{(0)} + \delta Y)\, \mathrm{d}X \mathrm{d}Y = 0,$$

or

$$\frac{\mathrm{d}I}{\mathrm{d}\psi^{(0)}} = \oint \frac{(Q^{(0)} + \delta Y)}{|\nabla \psi^{(0)}|}\, \mathrm{d}s = 0,$$

where the integral is taken around the bounding streamline (Rhines and Young, 1983). It then follows from the symmetry of $\psi^{(0)}$ that

$$\oint \frac{Y}{|\nabla \psi^{(0)}|}\, \mathrm{d}s = 0,$$

and so

$$G(\psi^{(0)}) \oint \frac{\mathrm{d}s}{|\nabla \psi^{(0)}|} = 0,$$

or

$$G(\psi^{(0)}) = 0.$$

The absolute potential vorticity of the lowest layer is consequently fully homogenized in agreement with the available numerical simulations (Pedlosky 1982b). We note from (22) that even when the dissipation is not weak, the generalized average of the potential vorticity on a streamline, defined as

$$\oint \frac{Q}{|\nabla \psi|}\, \mathrm{d}s,$$

vanishes.

The amplitude of the fundamental in the weakly dissipative limit follows on noting that (24), which now takes the form,

$$J(\psi^{(0)}, Q^{(1)}) = -\delta Y,$$

can be integrated directly to obtain the zonally averaged meridional potential vorticity flux in the lowest layer, i.e.,

$$\frac{1}{2\pi} \int_0^{2\pi} \psi_X^{(0)} Q^{(1)}\, dX = -\frac{\delta Y^2}{2} + \frac{\delta \pi^2}{8},$$

A second integration gives

$$\langle \psi_X^{(0)} Q^{(1)} \rangle = \frac{\delta \pi^2}{12},$$

which with (25) implies that

$$|S^{(0)}| = \frac{\delta^{1/2} \pi}{\sqrt{12}},$$

when $\delta > 0$, in agreement with Pedlosky (1982b). Supercritical disturbances can be expected to equilibrate at this value while subcritical disturbances are expected to decay. We note that only the symmetry properties and the fact that the streamlines are simply connected, closed curves has been used. The precise form of the streamfunction is not required in the derivation, so the result may hold in a wider context.

5. The inviscid case

In the absence of dissipation, there is no loss of generality in taking S to be real (see appendix). In this case (19) and (20) may be written

$$\frac{\partial Q}{\partial T} + S\, J(\Psi, Q) = 0, \tag{26}$$

and

$$\frac{dS}{dT} = \left\langle Q \frac{\delta\Psi}{\delta X} \right\rangle, \tag{27}$$

where

$$\Psi = \cos X \cos Y. \tag{28}$$

These are to be solved subject to the initial conditions (21). Since

$$\frac{d}{dT}\langle YQ \rangle = S \left\langle Q \frac{\partial\Psi}{\partial X} \right\rangle,$$

which follows directly from (26), substitution into (27) and integrating gives

$$\frac{S^2}{2} = \langle YQ \rangle |_0^T. \tag{29}$$

As in the Rossby wave problem under special parameter settings, the present system (26) and (29) is integrable, as may be seen by introducing the new independent variable

$$\eta = \int_0^T S(T')\, dT', \tag{30}$$

and noting that (26) becomes

$$\frac{\partial Q}{\partial \eta} + J(\Psi, Q) = 0, \tag{31}$$

which is linear. In terms of the time-like variable η, the absolute potential vorticity evolves like a nondiffusive passive scalar in a steady velocity field. Given the solution of (31) we can define a "potential"

$$V(\eta) = -\langle YQ \rangle |_0^\eta, \tag{32}$$

which when used with (29) and (30) gives

$$\frac{1}{2}\left(\frac{d\eta}{dT}\right)^2 + V(\eta) = E = 0. \tag{33}$$

Given $V(\eta)$, (33) then fully determines $\partial n/\partial T$ $(= S)$: S can consequently be thought of as the velocity of a zero-energy particle in a conservative force field with a potential $V(\eta)$. Since S vanishes initially, the particle moves (and S becomes non-zero) only when there is a force at the origin, i.e., only if $V'(0)$ is non-zero. This is determined by the projection of the initial Q onto Ψ_X. Since the acceleration $(= dS/dT)$ can be taken to be positive initially, (see the appendix), we need only consider a particle confined to the half-line $\eta > 0$. Furthermore since $V(\eta) \to$ const as $\eta \to \infty$[‡] due to coarse-grain mixing, (see below and Section 6), the particle motion is rather limited and the long-time behaviour can be fully classified. Thus if $V(\eta)$ vanishes for some $\eta = \eta_m$, (which it must if $V(\infty) > 0$), and if η_m is the smallest such root then if $V'(\eta_m) > 0$ the particle is confined to a potential well. It consequently oscillates in the interval $[0, \eta_m]$ with a maximum amplitude determined by the minimum potential in the interval. The potential vorticity field and the amplitude of the fundamental are then periodic functions of time (the latter with zero mean). Large η is never reached in these cases and the potential vorticity fails to mix. Instead it reversibly wraps and unwraps. If $V'(\eta_m) = 0$, on the other hand, the particle comes to rest at η_m implying S decays to zero as $T \to \infty$. In this case, the potential vorticity partially wraps up and then becomes steady. Alternately when $V(\eta) < 0$ for all η, the particle moves continuously to the right and the long-time behaviour is one of uniform motion. The primary wave amplitude then equilibrates at a finite level while the potential vorticity, which is always unsteady, mixes in a coarse-grain sense.

To construct the potential it is necessary to first solve (31). Consider first the region $-\pi/2 < X < \pi/2$, i.e., the vortex centered at the origin and introduce what are essentially characteristic coordinates $\alpha = \alpha(X, Y)$ and $m = m(X, Y)$, where the transformation is constrained by

$$\frac{\partial X}{\partial \alpha} = -\frac{\partial \Psi}{\partial Y}, \quad \frac{\partial Y}{\partial \alpha} = -\frac{\partial \Psi}{\partial X}. \tag{34}$$

It follows immediately that $\partial\Psi/\partial\alpha = 0$, i.e., $\Psi = \Psi(m)$. α and m are therefore coordinates along and normal to the streamlines respectively. This result allows the second equation above to be written

$$\frac{\partial Y}{\partial \alpha} = -m^{1/2}(1 - m^{-1}\sin^2 Y)^{1/2},$$

on taking $m = 1 - \Psi^2$. It then follows that

$$\sin Y = -m^{1/2}\,\mathrm{sn}(\alpha|m), \tag{35}$$

$$\sin X = m^{1/2}\frac{\mathrm{cn}(\alpha|m)}{\mathrm{dn}(\alpha|m)},$$

‡ A region not necessarily reached by the particle.

where sn, cn, and dn denote Jacobi elliptic functions (Abramowitz and Stegun, 1964). (31) then reduces to

$$\frac{\partial Q}{\partial \eta} + \frac{\partial Q}{\partial \alpha} = 0, \tag{36}$$

or

$$Q = \tilde{Q}(\alpha - \eta, m), \tag{37}$$

where $\tilde{Q}$ is the initial potential vorticity (in α, m coordinates). The solution in the remainder of the domain follows on noting that (31) is invariant under the transformation $X \to \pi - X$, $Y \to Y$.

Since

$$\tilde{Q}(\alpha, m) = -\delta Y(\alpha, m) + H(X(\alpha, m), Y(\alpha, m)),$$

from (21), then if we define

$$\sin Y' = -m^{1/2}\,\mathrm{sn}(\alpha - \eta | m),$$
and
$$\sin X' = m^{1/2}\,\frac{\mathrm{cn}(\alpha - \eta | m)}{\mathrm{dn}(\alpha - \eta | m)}, \tag{38}$$

(X', Y') is the initial position of a particle currently located at (X, Y), it then follows from (37) that the solution of the initial value problem (31) can be written

$$Q(X, Y, \eta) = \begin{cases} -\delta Y' + H(X', Y') & \text{for } |X| < \dfrac{\pi}{2}, \\ -\delta Y' + H(\pi - X', Y') & \text{for } |\pi - X| < \dfrac{\pi}{2}. \end{cases} \tag{39}$$

The evaluation of (39) is facilitated by noting that addition theorems and simple identities for Jacobi elliptic functions along with (35) and (38) imply the explicit transformation

$$\sin Y' = \frac{\mathrm{cn}(\eta|m)\,\mathrm{dn}(\eta|m)\sin Y + \mathrm{sn}(\eta|m)\sin X \cos^2 Y}{1 - \mathrm{sn}^2(\eta|m)\sin^2 Y},$$
$$\sin X' = \frac{2\,\mathrm{cn}(\eta|m)\sin X\cos Y - \mathrm{sn}(\eta|m)\,\mathrm{dn}(\eta|m)\sin 2Y}{2\,\mathrm{dn}(\eta|m)\cos Y - \mathrm{sn}(\eta|m)\,\mathrm{cn}(\eta|m)\sin X \sin 2Y}, \tag{40}$$

between the primed and unprimed variables. Given $H(X, Y)$, the determination of $Q(X, Y, \eta)$ requires nothing more that the evaluation of elliptic functions and direct and inverse trigonometric functions.

Eq. (32) now takes the form

$$V(\eta) = -\langle Y\tilde{Q}(X', Y')\rangle + \langle Y\tilde{Q}(X, Y)\rangle, \tag{41}$$

which with (39) has been used to numerically evaluate the potentials discussed below. Actually because X' and Y' are rapidly oscillating functions of X and Y when η is large, to avoid excessive resolution requirements it was found preferable to transform to α, m coordinates and then perform the integration with respect to α first. This was easily done by noting that the Jacobian of the transformation is

$$\frac{\partial(X, Y)}{\partial(\alpha, m)} = \frac{1}{2(1 - m)^{1/2}}.$$

The asymptotic behaviour of V follows from the coarse-grain mixing arguments presented in Section 6. These suggest that

$$\langle Y\tilde{Q}(X', Y')\rangle \to \langle Y Q_0(\Psi)\rangle \quad \text{as } \eta \to \infty,$$

where $Q_0(\Psi)$ is a generalized average of the initial Q field around closed streamlines. This can be seen to vanish by symmetry. It follows that

$$V(\infty) = \langle Y\tilde{Q}(X, Y)\rangle. \tag{42}$$

The linearity of (31) allows the contributions from the supercritical shear and the initial potential vorticity disturbances to be considered separately. The behaviour when both are important is obtained by adding the resulting fields.

5.1. *Subliminal*[‡] *disturbances* ($\delta \gg 1$)

When the disturbance is small and δ is large, the contributions from the disturbance potential vorticity to the initial Q field can be ignored (H cannot be taken to be exactly zero however since the disturbance would remain identically zero for all time). In this case

$$Q = -\delta Y',$$

which has been calculated using (40). The resulting fields for $\eta = 1, 3, 5$ are shown in Fig. 2. The initially linear potential vorticity is seen to wrap up within each vortex with increasing η and the field is seen to develop increasingly small scales. The production of small scales represents a kind of laminar enstrophy cascade and is associated with the rapid variations of $\mathrm{sn}(\alpha - \eta | m)$ with m when η is large.

‡ i.e., Below the level of awareness, undetectable.

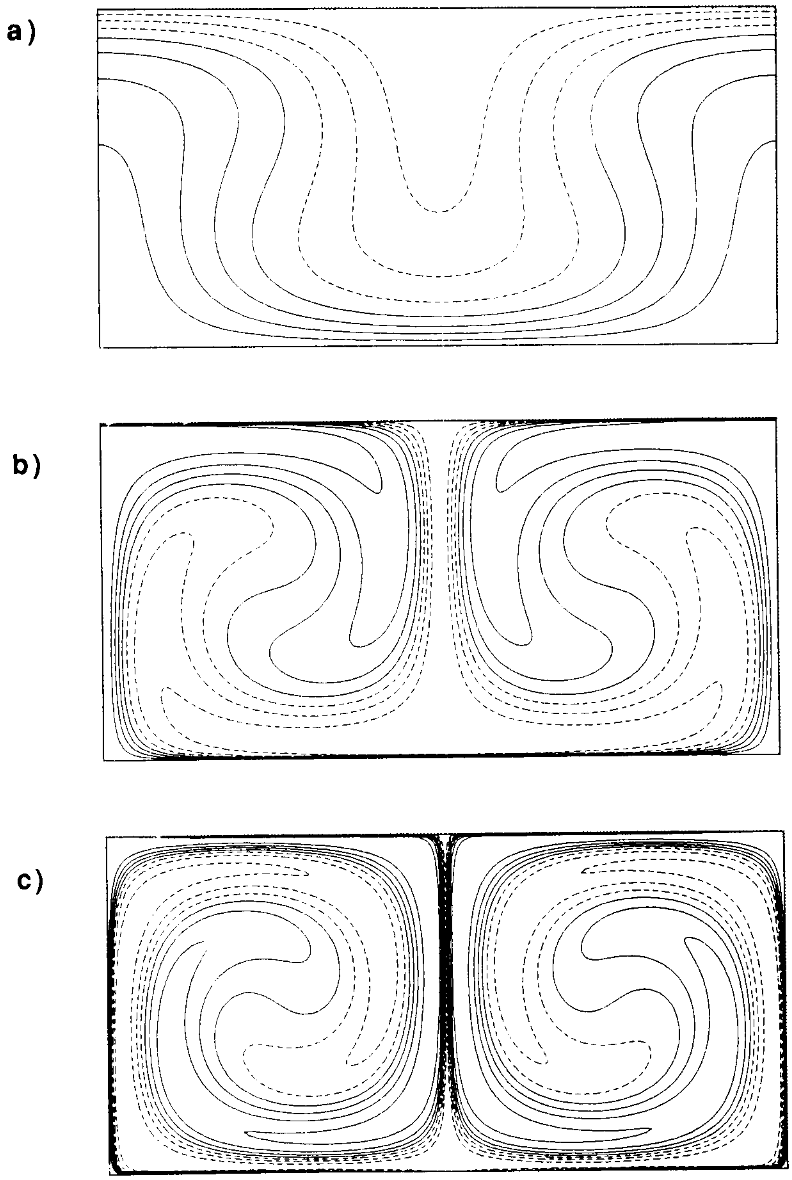

Fig. 2. Absolute potential vorticity field Q for subliminal disturbances ($\delta \gg 1$). (a) $\eta = 1$, (b) $\eta = 3$, (c) $\eta = 5$. Dashed lines indicate negative values.

The potential in this limit calculated from

$$V(\eta) = -\delta\langle Y^2 - YY'\rangle,$$
$$= -\frac{\delta\pi^2}{12} + \langle YY'\rangle, \qquad (43)$$

is shown in Fig. 3. It is seen to be everywhere negative and asymptotes to the value

$$V(\infty) = -\frac{\delta\pi^2}{12},$$

which is consistent with (42). It follows that the fundamental equilibrates at an amplitude

$$S(\infty) = \delta^{1/2}\frac{\pi}{\sqrt{6}},$$

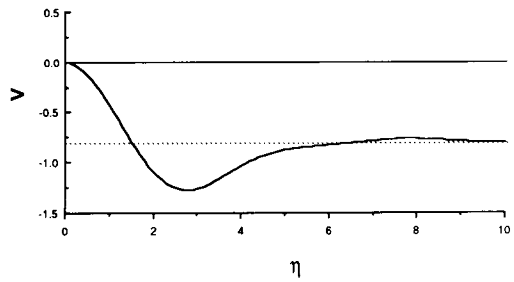

Fig. 3. Potential $V(\eta)$ for supercritical disturbances.

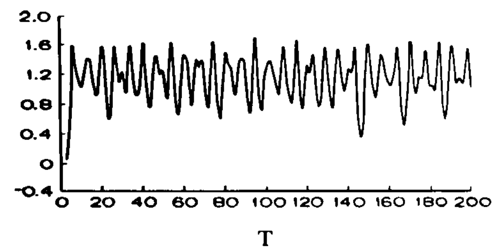

Fig. 4. Time evolution of the amplitude of the fundamental under the supercritical conditions (from Pedlosky, 1982b). $\nu = 10^{-4}$.

which is larger by a factor $\sqrt{2}$ than the weakly inviscid value‡. This behaviour may be compared with that observed in Pedlosky's (1982b) numerical simulations with small dissipation (Fig. 4). The amplitude can be seen to grow quickly during the early stages and then oscillate weakly thereafter. The failure of the system to fully relax appears to be due to insufficient resolution. We note however that the average value of the amplitude after the initial growth stage seems more consistent with the inviscid equilibrium value $\pi/\sqrt{6} \approx 1.283$, than with the weakly dissipative version, $\pi/\sqrt{12} \approx 0.907$, although this point may be debatable. The inviscid value is to be expected, even though the simulations were dissipative, because the magnitude of dissipation ($= 10^{-4}$) is much too small to have had any appreciable effect out to the times given. With adequate resolution and small enough dissipation we would speculate that the relaxation should occur in two stages; an inviscid stage associated with coarse-grain homogenization valid for times $1 \ll T \ll 0(\nu^{-1})$ associated with inviscid equilibration and a weakly dissipative stage valid for times $T \gg 0(\nu^{-1})$ when the disturbance decays to its weakly dissipative value.

‡ The results here appear to be consistent with rigorous nonlinear bounds for disturbances near criticality recently established by Shepherd (1988).

5.2. Neutral disturbances

When δ is small or vanishes the evolution is controlled by the initial disturbance potential vorticity and a range of behaviour is possible. This dependence on the details of the initial state implies that the flow is in some sense less predictable in this limit. We note that the supercritical case is in fact included in the neutral limit, since it can be recovered by taking the initial Q field to be linear in Y.

Here we choose to consider a special initial field simply to illustrate the possibility of periodic behaviour. Specifically we take

$$H(X, Y) = -2 \sin X \cos Y.$$

The resulting potential vorticity field is shown in Fig. 5 for $\eta = 0$, 1 and 2.7 while $V(\eta)$ is shown in Fig. 6. We note $V(\eta) \to 0$ as $\eta \to \infty$, which is consistent with (42). $V(\eta)$ becomes positive for $\eta \approx \eta_m$, where $\eta_m \sim 2.7$, implying that the flow is periodic. The potential vorticity wraps up until $\eta = \eta_m$, corresponding to the bottom panel of Fig. 5, whereupon it reversibly unwraps until the initial state is recovered. The circulation then changes sign and the process repeats.

6. Coarse-grain, streamwise homogenization of a passive scalar on steady, closed streamlines

As has already been noted, (31) resembles the equation for a conservative scalar transported along steady streamlines Ψ. Given weak diffusion, the equilibrium distribution is uniform within closed streamlines according to the Prandtl-Batchelor theorem (Batchelor, 1956). The transient approach to equilibrium, the so-called gradient expulsion problem, has been studied more recently by Rhines and Young (1983), who argue for a two stage process; the first involving a relatively rapid streamwise homogenization associated with shear-augmented diffusion and a second stage where cross-stream diffusion results in a uniform Prandtl-Batchelor state on the longer diffusive timescale.

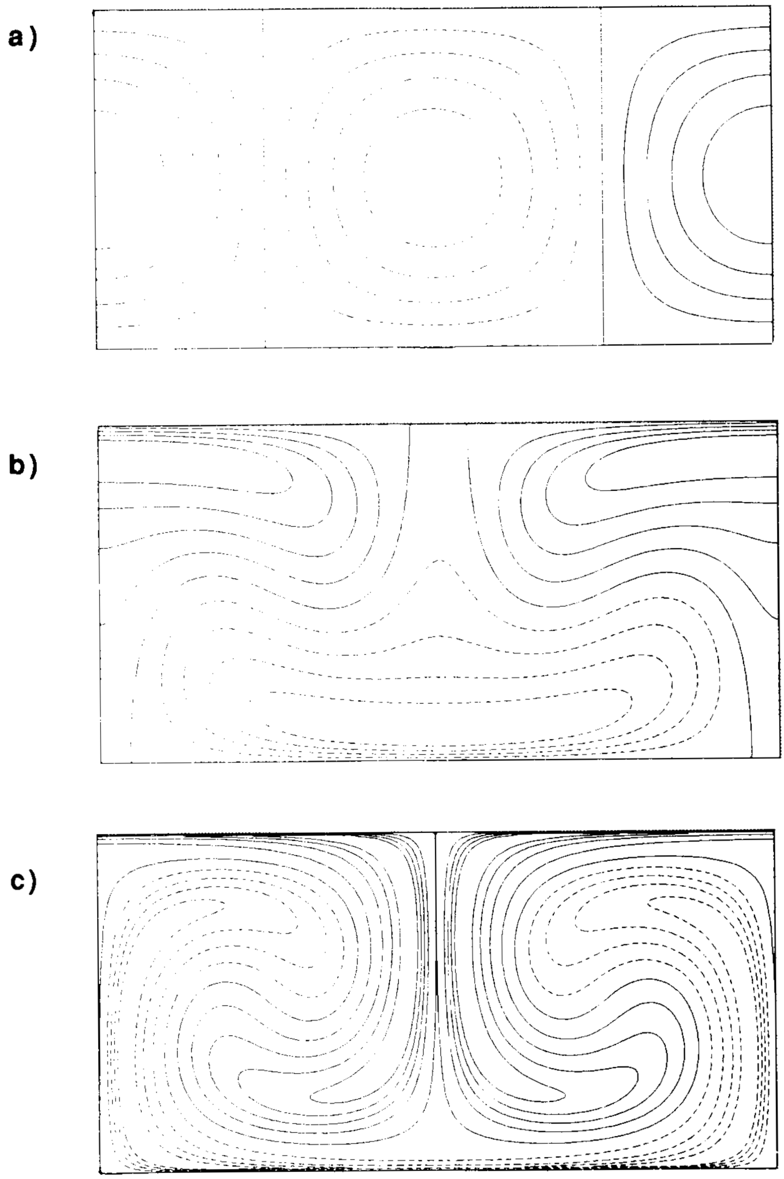

Fig. 5. As in Fig. 2 for $\delta = 0$ and $H(X, Y) = -2 \sin X \sin Y$. (a) $\eta = 0$, (b) $\eta = 1$ and (c) $\eta = 2.7$.

While there is no exact counterpart to fine-grain homogenization in the absence of diffusion, we show below that there is a nondiffusive result similar in many respects to the first relaxation stage discussed by Rhines and Young. Specifically, if $\theta = \theta(x, y, t)$ is a passive scalar transported by a steady velocity field with streamfunction ψ, then given any smooth function $f(x, y)$, then we show that, aside from certain exceptional cases to be noted below,

$$\lim_{\delta A \to 0} \lim_{t \to \infty} \overline{f(x,y)\theta} = f(x,y)\theta_0(\psi), \tag{44}$$

on closed streamlines. Here the bar denotes a coarse-grained average, (see the Introduction)

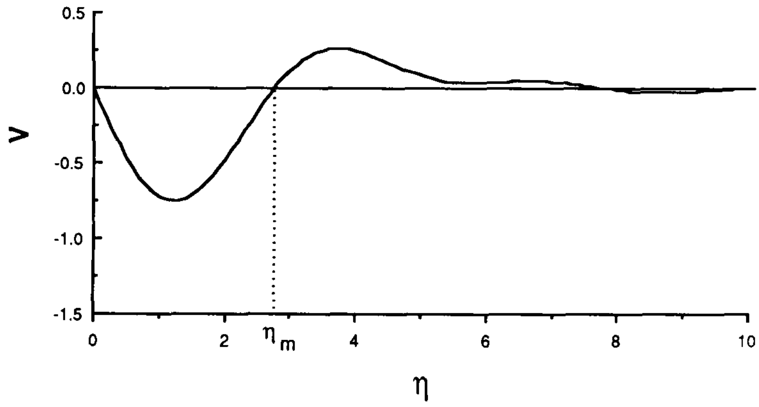

Fig. 6. Potential $V(\eta)$ when $\delta = 0$ and $H(X, Y) = -2 \sin X \sin Y$.

and θ_0 is a generalized average of the initial field around a streamline[‡]. It then follows that

$$\langle f(x, y)\, \theta(x, y, t)\rangle \to \langle f(x, y)\, \theta_0(\psi)\rangle,$$

for large t, a result that was used in Section 5. The angle brackets indicate an areal average within a region of closed streamlines.

To prove (44) we note that any conservative scalar θ transported by a velocity field with a steady but otherwise arbitrary closed streamline will evolve according to

$$\theta = \tilde{\theta}(\alpha - t, m),$$

provided the transformed variables α and m are defined according to (34) with Ψ replaced by ψ. Since θ is periodic in α within closed streamlines, it has the representation

$$\theta = \sum_n \theta_n(m)\, e^{in\omega(m)(\alpha - t)}, \tag{45}$$

where $\omega(m) = 2\pi/T(m)$ and $T(m)$ is the recirculation time on a streamline, i.e.,

$$T(m) = \oint \frac{ds}{|\nabla\psi|}.$$

The integral of $f(x, y)\, \theta(x, y, t)$ over a small area element δA may be written

$$\iint f(x, y)\, \theta(x, y, t)\, dx dy = \iint f(\alpha, m)\, \tilde{\theta}(\alpha - t, m)\, D\, d\alpha dm$$

where D is the Jacobian of the transformation. (45) and an integration by parts on the m variable implies, with reasonable smoothness conditions on f, D and θ, that

$$\int_{m - \delta m/2}^{m + \delta m/2} f\theta D\, dm = f(\alpha, m)\, \theta_0(\psi)\, D(\alpha, m)\, \delta m + O(\delta m^2) + O\left(\frac{1}{t\omega'(m)}\right),$$

for large t, small δm, provided $\omega'(m) \neq 0$. δm denotes the range of m values within δA. The main contribution to the areal average comes from the term $n = 0$ in (45) which involves

$$\theta_0(\psi) = \frac{\oint \theta(\alpha, m)\, d\alpha}{\oint d\alpha}, = \frac{1}{T(m)} \oint \frac{\theta}{|\nabla\psi|}\, ds. \tag{46}$$

since $ds = |\nabla\psi| d\alpha$. θ_0 is an invariant and can consequently be calculated from the initial data. (44) then follows provided the order of the limits is respected.

The condition $\omega'(m) \neq 0$ will usually be satisfied on interior streamlines although when it vanishes stationary phase arguments can be applied provided some higher derivative of ω is nonzero. (44) then continues to hold although with a modified error estimate. The rate of relaxation will then generally be slower. The argument fails if ω is constant over a finite region as does the fine-grained version based on shear-

[‡] We revert here to x, y, t variables.

augmented diffusion. The flow is then essentially equivalent to solid body rotation and the advection is ineffective (Rhines and Young, 1983). (44) also fails at the boundary between open and closed streamlines where ω vanishes. The coarse-grain average then also depends on the distribution of θ on exterior streamlines.

The above estimate suggests that when $\omega' \neq 0$, $\bar{\theta}$ will be homogenized when $t|\omega'|\delta m \gg 1$ and $\delta m \ll 1$. Since $|\omega'|/\omega = |T'|/T$ and $\Delta T = O(T' \delta m)$ homogenization can be expected when

$$\frac{t}{T} \gg \frac{T}{|\Delta T|} \quad \frac{\Delta T}{T} \ll 1.$$

ΔT denotes the range of recirculation times associated with the streamlines crossing the area being averaged. The timescale for coarse-grain homogenization therefore depends on δA. For a small but finite area and sufficiently weak diffusion, coarse-grain relaxation can be expected to occur on a faster timescale than the fine-grain processes discussed by Rhines and Young.

As noted by a reviewer the result (44) is perhaps less surprising when one realizes that molecular diffusion is itself really just a form of coarse-grain averaging, an analogy which is strengthened by the observation that the coarse-grained potential enstrophy is not conservative (Welander, 1955). Of course it should be kept in mind that only streamwise coarse-grain homogenization is possible since the restriction to steady flow completely eliminates communication between streamlines. Molecular processes, on the other hand, lead to full homogenization.

7. Conclusions

The present study suggests that the evolution and equilibration of marginally unstable waves in a two-layer model at minimum critical shear can profitably be studied along the lines of conventional nonlinear critical layer problems. In addition to providing a compact description, the resulting integro-differential equations are easier to manipulate and more amenable to standard mathematical techniques than are their spectral counterparts. The inviscid evolution equations which prove to be exactly integrable provide an illustration of this. It is also expected that the resulting simplifications will carry over to other more complex problems which would otherwise be difficult to treat (e.g., the study of wave packets or the interaction of the unstable wave with forcing).

It should be noted, however, that there are potential pitfalls with our approach. While the perturbation methods employed here (and in earlier works) have been successfully applied to a wide variety of problems, they are nonrigorous and can lead to misleading or even erroneous results. The scaling assumptions place strong constraints on the nature of the solution, constraints which may only be met for limited times. One possible alternate scenario, discussed by Killworth and McIntyre (1985) and Haynes (1985) for the forced Rossby wave critical layer problem, is that the flow may develop further instabilities during the nonlinear development stage. These are associated with the strong potential vorticity gradients that appear as the vortex wraps up. The situation is similar for the flows discussed here, particularly in the inviscid supercritical case where there is no limit to the increasingly fine scales that are developed in the potential vorticity field. A nonrigorous application of the necessary condition for the instability of a parallel flow in the two-layer model, which states (under the appropriate boundary conditions) that the absolute potential vorticity gradient of the background flow must change sign somewhere in the flow domain (Pedlosky, 1979), suggests that the flow may be unstable to further disturbances. Since the flow is locally parallel in each vortex near the walls away from $X = \pm \pi/2$ when η is large (see Fig. 2c for $\eta = 5$), and since a relatively large potential vorticity gradient which oscillates rapidly with y is ultimately produced, then if the effects of the boundaries are deemed unimportant, the necessary condition would seem to be satisfied implying the flow may be locally unstable. The long-time validity of our solutions is then doubtful. One possibility is that the subsequent flow becomes turbulent with an enstrophy cascade leading to a kind of partial statistical homogenization of the coarse-grained potential vorticity. The final equilibration, if one exists, might be a state in which the coarse-grained enstrophy is minimized in some sense. As in pipe flow, complete homogenization seems unlikely due to the boundary constraints.

When the supercriticality is not large and the flow time-periodic, the potential vorticity wraps up and then unwraps and the true nature of the evolution probably depends on the degree of the wrap-up and the strength of the resulting potential vorticity gradients. When the wrap-up is slight, the gradients will be limited and it seems reasonable to suppose that the flow will remain stable. In this case the solutions presented here should remain representative. When the wrap-up is strong, instabilities and perhaps turbulence might again be anticipated and the long-time validity of the solutions presented here is again in doubt. It may be that the flow evolves in a manner similar to the supercritical case. Although we have made no attempt to investigate these questions, they would seem to be of considerable importance.

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9. Appendix

When $\nu = 0$ (19) and (20) imply that

$$\begin{aligned}\frac{d^2S}{dT^2} &= 2i\langle J(SZ + \text{c.c.}, Q)\, Z^*\rangle, \\ &= 2iS\langle J(Z, Q)\, Z^*\rangle, \\ &= -2S\left\langle |Z|^2 \frac{\delta Q}{\delta Y}\right\rangle. \end{aligned} \qquad \text{(A-1)}$$

Since dS/dT can be taken to be real and positive initially (by translating the X-coordinate), and since S vanishes initially, (A-1) implies that S remains real for all T.

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