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The normal modes of the spherical non-hydrostatic equations with applications to the filtering of acoustic modes

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ABSTRACT

Simple calculations suggest that hydrostatic (primitive) equation models are adequate for global numerical weather prediction or climate simulation for two-dimensional wavenumbers less than 400. This limit is approximately four times the resolution of present state of the art global models. For resolutions greater than 400, non-hydrostatic models will have to be constructed. Such models have extremely restrictive CFL limits, because of the high frequencies of the acoustic modes.

The present work suggests a simple procedure for filtering the acoustic modes from such models. The proposed method should be effective and efficient because of the very simple structures of the acoustic modes. For an isothermal basic state, the acoustic modes have universal vertical structures and the horizontal structures are essentially single spherical harmonics. A more realistic basic state introduces a minor complication, namely, that the vertical structures become horizontally scale dependent in the non-hydrostatic regime.

For a model discretization based on a spherical harmonic expansion, a simple acoustic mode filter can be constructed. This procedure simply requires the projection of the state variables onto the acoustic modes, excision of the acoustic modes and re-projection into real space. This can be done every time-step with little overhead because of the simple structures of the acoustic modes. Thus, in principle, a global non-hydrostatic model could be integrated with long time-steps and virtually no computational penalty.

1. Introduction

The dynamic component of global numerical weather prediction and climate simulation models is based on the hydrostatic or primitive equations at the present time. The hydrostatic approximation is valid when the aspect ratio (vertical scale/horizontal scale) is small. In global forecast and climate models, the aspect ratio has always been small and the hydrostatic assumption has been a negligible source of error. However, with the continuous increase in available computing power, the horizontal and vertical resolution of numerical models has been steadily improving. Ultimately, there may come a time when it will be necessary to relax the hydrostatic assumption and consider the integration of the non-hydrostatic equations.

The most serious problem with the integration of the non-hydrostatic equations is the very restrictive Courant number implied by the high frequency sound wave solutions of these equations. The anelastic approximation is frequently used in convective modelling, where the aspect ratio is O(1). Anelastic models exclude sound waves and thus can be integrated with an acceptably long timestep. Unfortunately, the anelastic approximation is not appropriate for the larger horizontal and vertical scales of a global model. There also exist convective models which permit sound waves, but circumvent the Courant number limitation by using procedures such as the split-explicit method.

The integration of the spherical non-hydrostatic equations has never been attempted. This paper examines these equations from a modelling perspective, with particular emphasis on the properties of spherical acoustic modes. One of the properties of these modes is potentially exploitable in the design of a simple, efficient acoustic mode filter. In principle, this filter can be used to exclude acoustic modes from a global model without invoking either the anelastic or hydrostatic approximations.

2. The non-hydrostatic equations

A non-hydrostatic global forecast or climate model might be based on the spherical Eulerian equations given in Pedlosky (1979, p. 316). In deriving these equations, it is assumed that the earth is spherical and the surfaces of constant gravitational potential (including centripetal effects) are spherical. The effective gravitational constant (g) is assumed constant throughout the atmosphere. The spherical coordinate system: r, distance to the earth's center; λ, longitude; ϕ, latitude are assumed.

$$\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{uw}{r}-\frac{uv}{r}\tan\phi-2\Omega v\sin\phi+2\Omega w\cos\phi+\frac{1}{\rho r\cos\phi}\frac{\partial P}{\partial\lambda}-F_u=0, \tag{2.1}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t}+\frac{wv}{r}+\frac{u^2}{r}\tan\phi+2\Omega u\sin\phi+\frac{1}{\rho r}\frac{\partial P}{\partial\phi}-F_v=0, \tag{2.2}$$

$$\frac{\mathrm{d}w}{\mathrm{d}t}-\frac{u^2+v^2}{r}-2\Omega u\cos\phi+\frac{1}{\rho}\frac{\partial P}{\partial r}+g-F_w=0, \tag{2.3}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t}-\frac{Q\theta}{C_p T}=0, \tag{2.4}$$

$$\frac{\mathrm{d}\rho}{\partial t}+\rho\left[\frac{1}{r^2}\frac{\partial r^2 w}{\partial r}+\frac{1}{r\cos\phi}\frac{\partial}{\partial\phi}v\cos\phi+\frac{1}{r\cos\phi}\frac{\partial u}{\partial\lambda}\right]=0, \tag{2.5}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t}=\frac{\partial}{\partial t}+\frac{u}{r\cos\phi}\frac{\partial}{\partial\lambda}+\frac{v}{r}\frac{\partial}{\partial\phi}+w\frac{\partial}{\partial r},$$

$$P=\rho RT,$$

$$\theta=T\left(\frac{P_0}{P}\right)^{R/C_p}.$$

Here P is the pressure, T the temperature, ρ the density and u, v and w the eastward, northward and vertical velocity components. θ is the potential temperature, R the gas constant, C_p the specific heat at constant pressure and P_0 a constant reference pressure at $z=0$. F_u, F_v, F_w are the components of the forcing and Q is the heating. Eqs. (2.1)–(2.3) are the equations of motion, (2.4) is the thermodynamic equation and (2.5) is the equation of continuity.

A linear analysis of the equations can be performed if a few simplifying assumptions are made, following Eckart (1960). Define $z=r-a$, where a is the earth's radius. Define a motionless basic state with temperature $T_s(z)$, density $\rho_s(z)$, potential temperature $\theta_s(z)$ and pressure $P_s(z)$. Here "s" indicates basic state and T_s, ρ_s, θ_s and P_s are related by the hydrostatic relation, the definition of potential temperature and the equation of state. Define perturbation quantities about this basic state and neglect all terms which are non-linear in the perturbation quantities. In addition, neglect all terms where z appears in undifferentiated form (shallow approximation) and all the vertical Coriolis terms (traditional approximation). See Phillips (1966) for a discussion of these approximations.

The resulting linearized perturbation equations are,

$$\frac{\partial u'}{\partial t}-2\Omega v'\sin\phi+\frac{1}{a\cos\phi}\frac{\partial P'}{\partial\lambda}=0, \tag{2.6}$$

$$\frac{\partial v'}{\partial t}+2\Omega u'\sin\phi+\frac{1}{a}\frac{\partial P'}{\partial\phi}=0, \tag{2.7}$$

$$\delta_{\mathrm{H}}\frac{\partial w'}{\partial t}+\left(\frac{\partial}{\partial z}+\frac{g}{C_s^2}\right)P'-\theta'=0, \tag{2.8}$$

$$\frac{\partial\theta'}{\partial t}+N_s^2 w'=0, \tag{2.9}$$

$$\frac{\partial P'}{\partial t}+C_s^2\left[\frac{\partial w'}{\partial z}+\frac{1}{a\cos\phi}\left(\frac{\partial v'\cos\phi}{\partial\phi}+\frac{\partial u'}{\partial\lambda}\right)+\frac{w'}{g}N_s^2\right]=0, \tag{2.10}$$

where

$$N_s^2(z) = \frac{g}{\theta_s}\frac{\partial\theta_s}{\partial z}$$

is the Brunt-Väisäla frequency squared and

$$C_s^2(z) = \frac{C_p R T_s}{C_p - R}$$

is the speed of sound squared. u', v' and w' are the eastward, northward and upward velocity components and Ω is the rotation rate of the earth. δ_H is a tag which is one for the non-hydrostatic case and zero for the hydrostatic case. The perturbation variables in (2.6)–(2.10) are related to the original state variables as follows: $u' = \rho_s u$, $v' = \rho_s v$, $w' = \rho_s w$, $P' = P - P_s$ and $\theta' = (\theta - \theta_s)\rho_s g/\theta_s$.

Eqs. (2.6)–(2.10) can be separated following Eckart (1960). Assume

$$\begin{vmatrix} u' \\ v' \\ P' \\ \theta' \\ w' \end{vmatrix} = \begin{vmatrix} u(\phi)\, Z_1(z) \\ iv(\phi)\, Z_1(z) \\ 2\Omega P(\phi)\, Z_1(z) \\ 2\Omega\theta(\phi)\, Z_2(z) \\ iw(\phi)\, Z_2(z) \end{vmatrix} \exp(im\lambda - 2\Omega i\sigma_m t), \quad (2.11)$$

where m is the zonal wavenumber, $i = \sqrt{-1}$ and σ_m is a non-dimensional frequency. This gives,

$$L_1(Z_1) = (N_s^2 - 4\Omega^2\sigma_m^2\delta_H)\frac{Z_2}{N_s^2}, \quad (2.12)$$

$$\left(1 - \frac{C_s^2}{b_m}\right) Z_1 = C_s^2 L_2\left(\frac{Z_2}{N_s^2}\right), \quad (2.13)$$

$$H_m^\sigma(P) + \frac{4\Omega^2 a^2}{b_m} P = 0, \quad (2.14)$$

where

$$H_m^\sigma = \frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\left[\frac{\cos\phi}{(\sigma_m^2 - \sin^2\phi)}\frac{\partial}{\partial\phi}\right] + \frac{1}{(\sigma_m^2 - \sin^2\phi)} \times \left[\frac{m}{\sigma_m}\frac{(\sigma_m^2 + \sin^2\phi)}{(\sigma_m^2 - \sin^2\phi)} - \frac{m^2}{\cos^2\phi}\right],$$

$$L_1 = \frac{\partial}{\partial z} + \frac{g}{C_s^2}, \quad L_2 = \frac{\partial}{\partial z} + \frac{N_s^2}{g}. \quad (2.15)$$

b_m is a separation constant and H_m^σ is referred to as the horizontal structure operator.

To determine the free normal modes of (2.12)–(2.14) it is necessary to solve the three equations (together with appropriate upper and lower boundary conditions) simultaneously with the eigenvalues σ_m and b_m to be determined. Note that both eigenvalues σ_m and b_m appear in the vertical structure equations (2.12)–(2.13) and the horizontal structure equation (2.14). In the hydrostatic system ($\delta_H = 0$), the vertical structure equations do not involve the frequency. The procedure used in the hydrostatic system of solving the vertical structure equation for the separation constant and then specifying it in the horizontal structure equation to obtain the eigenfrequencies; cannot be used in (2.12)–(2.14).

No attempt will be made to obtain the complete eigensystem of (2.12)–(2.14). The particular properties of the eigensystem which are of interest will be examined with simpler systems.

3. Isothermal basic state

An isothermal basic state leads to a more tractable eigensystem. Assume $T_s(z) = T_0$. Then,

$$C_s^2(z) = \frac{RT_0 C_p}{C_p - R} = C_0^2, \quad (3.1)$$

$$N_s^2(z) = \frac{g^2}{C_p T_0} = N_0^2. \quad (3.2)$$

In this case, (2.12)–(2.14) can be written,

$$L_2 L_1 \begin{vmatrix} Z_1 \\ Z_2 \end{vmatrix} + \tilde{\gamma}\begin{vmatrix} Z_1 \\ Z_2 \end{vmatrix} = 0, \quad (3.3)$$

$$H_m^\sigma(P) + \frac{4\Omega^2 a^2}{C_0^2}\left[1 + \frac{C_0^2\tilde{\gamma}}{(N_0^2 - 4\Omega^2\sigma^2\delta_H)}\right] P = 0, \quad (3.4)$$

Here $\tilde{\gamma}$ is a separation constant which is related to the well-known equivalent depth $\tilde{h}$ by

$$\tilde{\gamma} = \frac{g}{C_p T_0 \tilde{h}} - \frac{N_0^2}{C_0^2}. \quad (3.5)$$

The vertical structure equations (3.3) for $Z_1(z)$ and $Z_2(z)$ are the same, but the boundary conditions are different. Eq. (3.3) can be solved as an eigenvalue problem for the vertical structure and the separation constant. The separation constant is then specified in (3.4) and the equation is then solved for the horizontal structures and eigenfrequencies.

The case of interest is when the Brunt-Väisäla frequency is much larger than the Coriolis frequency. For an isothermal atmosphere of 250°K, $N_0/2\Omega \approx 134$. The isothermal case has been discussed extensively by Eckart (1960) and relevant properties of the eigensystem are summarized below.

The vertical structure equation (3.3) is identical in the hydrostatic and non-hydrostatic cases and is indifferent to the rotation of the earth. In an unbounded isothermal atmosphere (3.3) gives rise to an external or Lamb mode, plus a continuous spectrum of internal modes. The external mode is hydrostatic and Z_2 (and hence w' and θ') are identically zero. The vertical structures are independent of the horizontal scale. In a bounded atmosphere, the external mode remains, but the internal modes are represented by a discrete spectrum which depends on the position and form of the upper boundary condition.

For a given equivalent depth $\tilde{h}$, and vertical structure, (3.4) can be solved for the frequencies and horizontal structures. There are basically three types of eigensolution to (3.4), low frequency Rossby modes, higher frequency inertia-gravity modes and very high frequency inertia-acoustic modes.

The case of no rotation is discussed thoroughly in chapter 8 of Eckart (1960). A frequency/horizontal wavenumber diagram for this case is shown in Fig. 16 (p. 108) of Eckart's book. There are external gravity modes which are really horizontally propagating non-dispersive sound waves. Their frequency increases with decreasing horizontal scale.

Internal gravity waves are lower frequency than external gravity waves and their frequencies decrease for smaller vertical scales. At small aspect ratios, internal gravity mode frequencies increase with decreasing horizontal scale; but at large aspect ratios, their frequencies are virtually independent of the horizontal scale. The primary restoring force for internal gravity modes is gravity, although they may be affected by compressibility of the medium. For small aspect ratios, they essentially propagate horizontally, but the vertical component increases as the aspect ratio increases.

Acoustic modes are also internal modes, but with a frequency larger than the external gravity mode frequency. At small aspect ratios, the acoustic mode frequencies exceed the Brunt-Väisäla frequency and are virtually independent of the horizontal scale. At large aspect ratios, the acoustic mode frequencies increase with decreasing horizontal scale. Acoustic mode frequencies tend to increase with decreasing vertical scale. The predominant restoring force for the acoustic modes is compressibility of the medium, but they are also affected by stratification. For small aspect ratios, they essentially propagate vertically and are characterized by much larger vertical than horizontal velocity components. At larger aspect ratios, the relative horizontal velocity component increases.

Examination of Fig. 16 (p. 108) of Eckart's book suggests that non-hydrostatic effects begin to be important in the isothermal case, when the external gravity mode frequency exceeds the Brunt-Väisäla frequency. This occurs when,

$$C_0 \frac{\sqrt{n(n+1)}}{a} = N_0, \tag{3.6}$$

where n is the two-dimensional or total wavenumber. For $T_0 = 250$°K, $n \approx 395$.

The effect of the earth's rotation is to modify the existing gravity and acoustic modes and also create a new set of modes, the Rossby modes. There are both internal and external Rossby modes which are characterized by decreasing frequency with decreasing vertical scale.

The principal difficulty with the numerical integration of the non-hydrostatic equations is caused by the very high frequencies of the acoustic modes. These high frequencies impose a very restrictive Courant–Friedrichs–Lewy limit on an explicit non-hydrostatic numerical integration. Because these modes propagate vertically, the limit becomes more restrictive as the vertical mesh is refined.

There are a number of procedures, that might be considered for this problem including time-splitting and the semi-implicit algorithm. However, we have chosen to examine a procedure for completely filtering the acoustic modes from an explicitly integrated non-hydrostatic model.

The method relies on a property of the acoustic modes which will now be demonstrated. At very high frequencies, ($\sigma_m \gg 1$), the horizontal structure operator collapses to,

$$H_m^\sigma = \frac{1}{\sigma_m^2}\left[\frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\cos\phi\frac{\partial}{\partial\phi} - \frac{m^2}{\cos^2\phi}\right]. \tag{3.7}$$

The operator in (3.7) is the spherical Laplacian operator, whose eigenvectors are associated Legendre polynomials. The approximation (3.7) will be referred to as the "high frequency approximation" (see Eckart (1960), p. 271).

Approximation (3.7) implies that the eigenfunctions and eigenfrequencies of high frequency modes can be obtained from (2.6)–(2.10) simply by dropping the Coriolis terms. In other words, (2.6)–(2.7) are replaced by:

$$\frac{\partial}{\partial t}\nabla^2\chi' + \nabla^2 P' = 0, \tag{3.8}$$

where χ' is the perturbation velocity potential defined by the Helmholtz equation. The horizontal divergence term in (2.10) is replaced by $\nabla^2\chi'$, but otherwise the equations are unchanged. In the isothermal case, the vertical eigenstructures will remain unchanged, because the Coriolis terms only appear in the horizontal equation (3.4). The horizontal structures of χ', P', w' and θ' will each consist of single spherical harmonic function.

The horizontal structures and frequencies can be calculated using the high frequency approximation (3.7) and compared with those obtained from the complete horizontal structure equation (3.4). The correct horizontal structures will be compared with the approximate structures in the following way. Each individual mode (acoustic or gravity) has a structure for u, v, P, θ and w. Alternatively, the streamfunction ψ or velocity potential χ can be used instead of u and v. For a given mode with eigenfrequency σ_m, the latitudinal structure of an arbitrary state variable S can be written as a sum of associated Legendre polynomials,

$$S(\phi, m) = \sum_{n=|m|}^{\infty} S_m^n P_m^n(\sin\phi), \tag{3.9}$$

where P_m^n is the associated Legendre polynomial of the first kind of degree m and order n. The S are real expansion coefficients. The variance of S_m^n can be obtained by integrating over latitude and using the orthogonality relation for Legendre polynomials. Thus,

$$\overline{S_m^2} = \frac{1}{2}\int_{-\pi/2}^{\pi/2} S^2(\phi, m)\cos\phi\,\mathrm{d}\phi = \sum_{n=|m|}^{\infty}[S_m^n]^2. \tag{3.10}$$

Now suppose $S(m,\phi)$ is obtained from (3.4) while $\hat{S}(m,\phi)$ is obtained using the high frequency approximation (3.7). $\hat{S}(m,\phi)$ will thus consist of a single Legendre polynomial expansion coefficient, rather than a series. An error variance can be calculated by,

$$\overline{\Delta S_m^2} = \frac{1}{2}\int_{-\pi/2}^{\pi/2}[S(\phi, m) - \hat{S}(\phi, m)]_2\cos\phi\,\mathrm{d}\phi = \sum_{n=|m|}^{\infty}[S_m^n - \hat{S}_m^n]^2. \tag{3.11}$$

Because of the linearity of the problem, the eigenfunctions have arbitrary amplitudes. Consequently, for (3.11) to have any meaning, the structures $S(m,\phi)$ and $\hat{S}(m,\phi)$ must be normalized consistently.

In Fig. 1a, the correct eigenfunction calculated from (3.4) and those calculated using the high frequency approximation (3.7) are compared. The ordinate is the quantity $\sqrt{\overline{\Delta P n^2}/\overline{P m^2}}$ where P is the pressure. The abscissa is $\sqrt{n(n+1)}$, where n is the two-dimensional wavenumber of the single Legendre polynomial coefficient of $P(m,\phi)$. All calculations are for the gravest symmetric eastward propagating mode $n = |m|$. Shown are the external inertia-gravity modes (E) and two sets of internal inertia-gravity modes and internal inertia-acoustic modes. The external mode has an equivalent depth of 10 km. A_1 and G_1 are internal acoustic and gravity modes with an equivalent depth of 3 km. A_2 and G_2 are acoustic and gravity modes with an equivalent depth of 0.5 km.

The results show that the error in defining the horizontal structure of the pressure field using (3.7) is very small when the frequencies are large. This is always so for the acoustic mode and becomes increasingly true for the inertia-gravity modes as the horizontal scale decreases. Calculations were performed for other modes, and the results were much the same. The infinite sums in (3.10)–(3.11) were actually truncated at a finite value $n = |m| + N$, but there was little sensitivity to the value of N chosen.

Eigenmodes calculated from (3.8) are irrotational. To see if this is a good approximation,

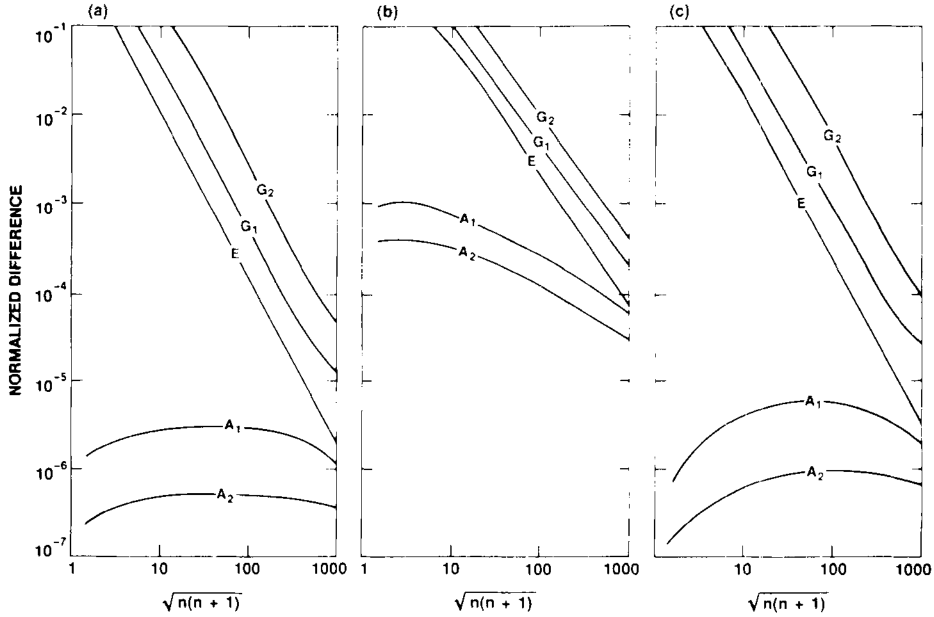

Fig. 1. Errors in the eigenstructures for the high frequency approximation as a function of horizontal wavenumber (n) for the gravest symmetric eastward propagating modes. Panel (a) Error in pressure eigenstructure for external gravity mode (E), internal gravity modes (G_1 and G_2) and acoustic modes (A_1 and A_2). Panel (b) Ratio of rotational component to divergent component in correct eigenstructure. Panel (c) Error in streamfunction eigenstructure using equations (3.14)–(3.15).

Fig. 1b plots the ratio of the rotational to the divergent wind component for the true modes obtained from (3.4). The quantity plotted is $\sqrt{\psi_m^2/\chi_m^2}$ for the same modes as in Fig. 1a. Clearly, the irrotational assumption is also a good approximation for the high frequency modes.

However, neglecting the rotational component is clearly not as good an approximation as assuming the structures of χ, P, w and θ are given by a single spherical harmonic. In deriving (3.7), $\sin^2\phi$ is neglected with respect to σ_m^2. In other words, the error in this approximation is $O(f^2/N_0^2)$ for the acoustic modes. On the other hand neglecting the Coriolis terms and hence the rotational wind component in (2.6)–(2.7) for the acoustic modes is $O(f/N_0)$ and is thus, not as good an approximation.

The rotational component of the flow for the acoustic modes can be generated from the single spherical harmonic structure of the pressure P as follows. Write eqs. (2.6)–(2.7) in terms of the streamfunction and velocity potential. Substitute in for the λ and t dependence following (2.11) and assume an infinite series in associated Legendre polynomials for the ϕ dependence. The result is,

$$a_m^n \psi_m^n + b_m^{n-1} \chi_m^{n-1} + c_n^{n+1} \chi_n^{n+1} = \sigma_m \psi_m^n, \qquad (3.12)$$

$$b_m^{n-1} \psi_m^{n-1} + c_m^{n+1} \psi_m^{n+1} + a_m^n \chi_m^n - P_m^n = \sigma_m \chi_m^n, \qquad (3.13)$$

where a_m^n, b_m^n and c_m^n are functions of m and n, which are given by Kasahara (1976). χ_m^n, ψ_m^n and P_m^n are the spherical harmonic expansion coefficients for χ, ψ and P. Now, if the structures of χ and P are given by a single spherical harmonic (high frequency approximation),

$$\psi_m^{n+1} = \frac{b_m^n \chi_m^n}{\sigma_m - a_m^{n+1}}, \quad \psi_m^{n-1} = \frac{c_m^n \chi_m^n}{\sigma_m - a_m^{n-1}}. \qquad (3.14)$$

Substitution of (3.14) into (3.13) yields a value of χ which is more accurate than that obtained from (3.8),

$$\left[a_m^n + \frac{b_m^{n-1} c_m^n}{\sigma_m - a_m^{n-1}} + \frac{c_m^{n+1} b_m^n}{\sigma_m - a_m^{n+1}} - \sigma_m\right] \chi_m^n = P_m^n. \qquad (3.15)$$

If ψ_m^{n+1}, ψ_m^{n-1} are calculated from (3.15) followed by (3.14), the rotational component of the appropriate acoustic eigenstructure consists of the two adjacent spherical harmonics (which are of opposite symmetry with respect to the equator).

The accuracy of the approximation is tested by calculating $\sqrt{\overline{\Delta\psi_m^2}/\overline{\psi_m^2}}$. This is plotted in Fig. 1c for the same modes as in Fig. 1a, b. Clearly, the rotational component of the acoustic modes, can be determined extremely accurately from (3.7), (3.15) and (3.14).

For the isothermal case, Fig. 1 shows that approximate acoustic mode structures consisting of a single spherical harmonic for P, χ, w and θ and the two adjacent spherical harmonics for ψ are an extremely accurate approximation to the correct eigenstructure. The accuracy of the approximation depends on the eigenfrequency and becomes increasingly accurate as the frequency increases. The vertical structures for the isothermal case are identical with those obtained by making the hydrostatic approximation and are independent of the horizontal scale.

4. Non-isothermal basic state

A more realistic basic state introduces some complications. The Brunt-Väisäla frequency (N_s) and speed of sound (C_s) are no longer independent of z and the simple procedure for finding the eigenfunctions in Section 3 is no longer possible.

Progress can be made, however, by making use of the high frequency approximation discussed in the previous section. The introduction of a non-isothermal basic state does not change equations (2.6)–(2.7), or the horizontal structure operator (2.15). Thus, if the frequencies are sufficiently high, approximation (3.7) should still be applicable. The procedure, then, is to make the high frequency approximation in (2.6)–(2.10), calculate the eigenfrequencies and then check *a posteriori* whether the frequencies are high enough to justify the approximation. In the cases where $\sigma_m \gg 1$, then eqs. (3.14)–(3.15) for deriving the rotational wind component will also be valid.

The system to be solved is equations (3.8) and (2.8)–(2.10) with the divergence in (2.10) replaced by $\nabla^2\chi'$. The properties of this system have been discussed by Eckart (1960). The horizontal structures are, of course, spherical harmonic functions.

There is always an external or Lamb mode as in the isothermal case. In the unbounded case there are internal gravity modes and internal acoustic modes. The spectrum of these may be continuous or discrete depending upon $\lim_{z\to\infty} T_s(z)$. In a discretized, bounded system, the spectrum is discrete, with the vertical structures and eigenvalues depending upon the top boundary condition, position of the levels and form of discretization adopted.

To demonstrate the properties of this system, eqs. (3.8) and (2.8)–(2.10) have been discretized using staggered centred differences in z with a lid condition at 40 km. The lid condition is a very simple upper boundary condition, which differs in detail from present practice in numerical modelling. The model has four levels and the basic state temperature $T_s(z)$ was obtained from the US Standard Atmosphere—1976.

Consider first the large horizontal scale limit, i.e., $n \to 0$. The vertical structures in this case are shown in Fig. 2. Panels (a) and (c) show the structures for Z_1 (corresponding to χ' and P'). Panels (b) and (d) show the structures for Z_2 (corresponding to w' and θ'/N_s^2). Panels (a) and (b) show the external mode (E) and three internal gravity modes (G_1, G_2 and G_3). Panels (c) and (d) show the three internal acoustic modes (A_1, A_2 and A_3).

The modes G_1, G_2, G_3 and E are hydrostatic in the limit $n \to 0$. However, there are important differences with the isothermal case. The external mode now has a non-zero component in Z_2 (corresponding to w' and θ'/N_s^2). Moreover, the set of internal gravity mode vertical structures are no longer identical to the set of acoustic mode structures. This occurs because the two vertical operators L_1 and L_2 no longer have constant coefficients and do not commute as they did in the isothermal case.

In the non-isothermal case, we do not have available the correct eigenstructure, with which to compare approximate solutions. Consequently, we will examine how the vertical structures change as n is increased by comparing them with the large horizontal scale eigenstructures of Fig. 2. A diagnostic, much like that used in the previous section, will be used in the comparison.

Define the vertical structure of a state variable when $n = 0$ at the levels z_k, $1 \leqslant k \leqslant K$. The

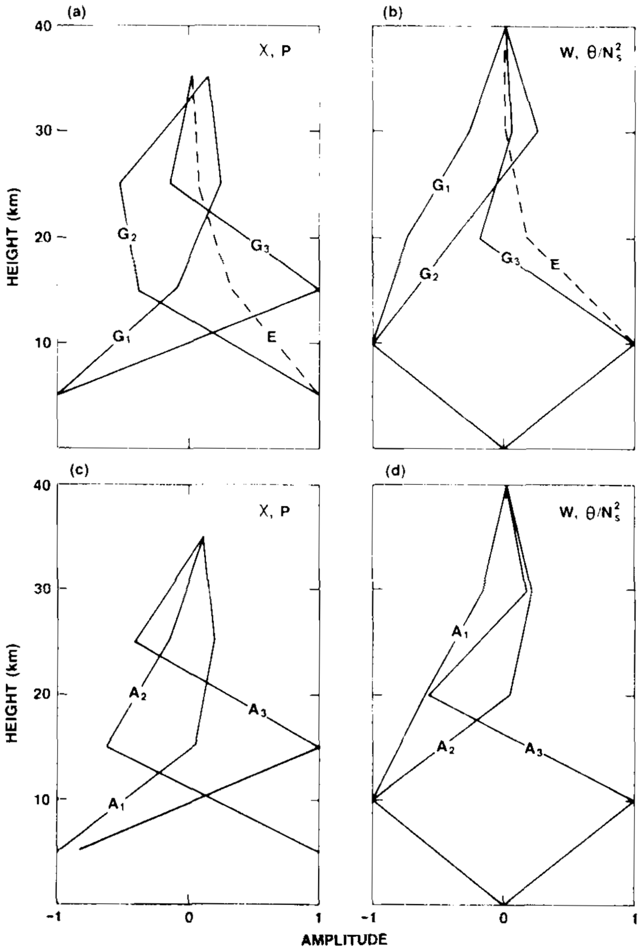

Fig. 2. Vertical structures for 4 level modes of Section 4 as $n \to 0$. Panels (a) and (c) show vertical structures of χ and P components and panels (b) and (d) show w and θ/N_s components. External mode (E), internal gravity modes (G_1, G_2, G_3) and acoustic modes (A_1, A_2, A_3) are plotted.

variance over the vertical levels is defined to be:

$$\overline{S_0^2} = \sum_{k=1}^{K} S_0^2(k),$$

where S is some state variable. If S_n is the structure of the same variable, when the horizontal wavenumber $= n$, then the difference can be defined (provided a consistent normalization is used) by,

$$\overline{\Delta S_n^2} = \sum_{k=1}^{K} [S_n(k) - S_0(k)]^2.$$

In Fig. 3a is plotted $\sqrt{\Delta P_n^2/P_0^2}$ where P is the pressure, as a function of $\sqrt{n(n+1)}$ for the modes G_2, G_1, A_1, A_2 and E. In Fig. 3b are shown the non-dimensional frequencies of the same five modes as a function of $\sqrt{n(n+1)}$. The dashed lines in panel (b) show the frequencies in the hydrostatic case ($\delta_H = 0$).

Examining first panel (b), it is evident that the frequencies of the modes A_1 and A_2 are large, which justifies *a posteriori* the use of the high frequency approximation. The high frequency approximation is also justified for the modes E,

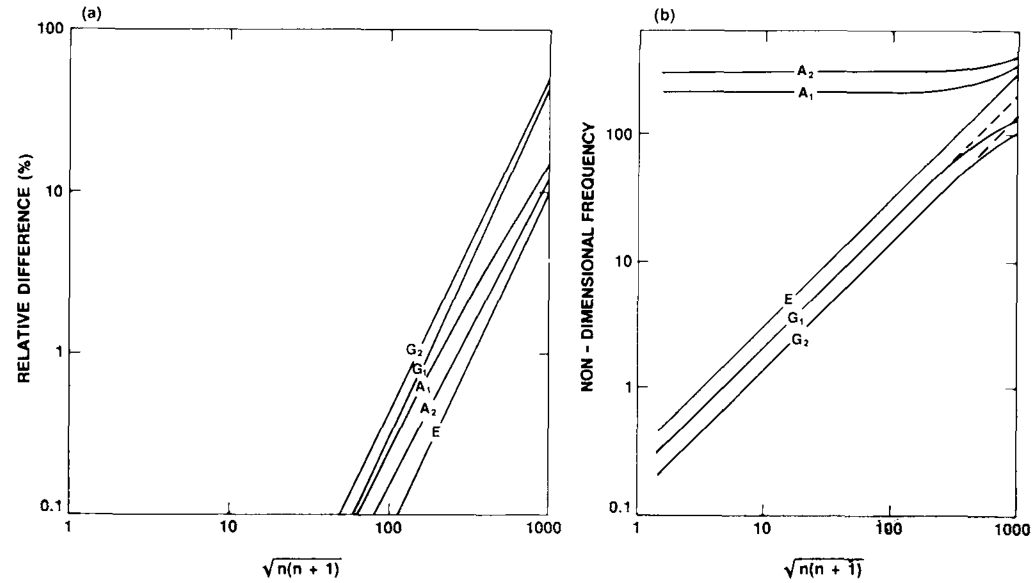

Fig. 3. Panel (a) Difference between vertical structure calculated at horizontal wavenumber (n) and the vertical structure of Fig. 2. Differences are shown for E, G_1, G_2, A_1 and A_2. Panel (b) Frequencies of the same modes as a function of (n). The dashed lines show the hydrostatic approximation.

G_1 and G_2 for large values of n. From panel (b), it is also evident that non-hydrostatic effects become important after $n \approx 400$, consistent with (3.6).

In panel (a), it can be seen that the vertical structures differ significantly from those in Fig. 2, for large values of n. Thus, it is not possible to use universal (independent of horizontal scale) vertical structures as can be done in both the hydrostatic non-isothermal case and the non-hydrostatic isothermal case. At small horizontal scales, the change in vertical structure must be taken into account in this case.

5. An acoustic mode filter

A global model based on the non-hydrostatic equations (2.1)–(2.5) has never been constructed. At the present resolutions of global models ($n < 200$), it is clearly not a priority. However, with the current rate of increase in computer power, global models with $n > 400$ will be feasible in a few years and the relaxation of the hydrostatic constraint will become an important consideration. A simple procedure for integrating such models efficiently could be based on the analysis of Sections 3 and 4.

Daley (1980) described several procedures for integrating the spherical hydrostatic (primitive) equations efficiently. These procedures were all based on dividing the inertia-gravity modes up into a fast set and a slow set. For a given timestep, the fast modes were those which would have a Courant number greater than one, if integrated explicitly. The slow modes were the remaining modes. All procedures discussed in Daley (1980) projected the model state variables onto the fast modes to obtain the fast mode amplitudes. The various methods differed only in how they treated the fast modes.

The procedure of interest here is method A, in which the fast gravity modes were filtered or excised completely from the model. This method was numerically stable with a long timestep. As applied in Daley (1980), method A took into account the full horizontal and vertical structures of the fast gravity modes. These structures were different for each mode and changed with the vertical structure. Despite this complexity of the projection procedure, method A was highly efficient, comparing favourably with competing semi-implicit procedures.

The acoustic mode filter is essentially the application of method A to the acoustic modes. It will be assumed, in the lack of any evidence to the contrary, that acoustic modes do not play any significant role in the atmospheric dynamics of interest. Thus, they can be excised completely from the model. However, it will be necessary to excise all the acoustic modes, not just a subset. Although there are more modes to treat in the non-hydrostatic problem, the very simplicity of the acoustic mode structures is an enormous asset. In the isothermal case, the modes have universal vertical structures and all acoustic modes have single spherical harmonic horizontal structures for χ, P, w, and θ, together with the two adjacent spherical harmonics for ψ. Thus, each acoustic mode is represented by 6 degrees of freedom per vertical level (4, if the irrotational approximation can be made). The use of a realistic basic state temperature does not change the horizontal acoustic mode structures, but only permits universal vertical structures for $n \lesssim 400$. When non-hydrostatic effects become important, different vertical structures must be used for each horizontal wavenumber. This last condition does not make the method any more costly, but may require more storage.

In a model based on a spherical harmonic expansion; projection onto the acoustic mode, excision, and re-projection into real space, would require virtually no overhead. Thus, it should be possible to take long explicit timesteps by excising the acoustic modes every timestep. As a by-product, the technique could also be used as an initialization procedure for eliminating the acoustic modes at initial time.

It cannot be guaranteed *a priori* that an acoustic mode filter will work in practice. Time integration methods that circumvent a strict CFL limit, such as the semi-implicit or split-explicit methods, generally divide the terms of the governing equations into two groups. The largest group of terms (mostly non-linear and forcing) is treated explicitly, but a small group of terms thought to involve high frequencies, are specially treated. In the case of the semi-implicit method this second group is treated implicitly, while in the split-explicit method it is integrated with a short timestep. In most cases, the separation of the high frequency terms is not perfect; there remains high frequency terms (usually non-linear) which are treated with the explicit group. Models can be stably integrated with these techniques as long as the major part of the high frequency terms are especially treated. Similar arguments apply in the case of the gravity mode filter (Method A) discussed by Daley (1980). The division into fast and slow modes was defined with respect to a linearization about a simple basic state at rest. The fast modes that were excised were only approximations to modes defined with respect to more complex basic states. However, most of the high frequency behaviour of the model projected onto these approximate fast modes and hence the method was stable.

The acoustic normal modes derived in Section 3 are also approximate. After eq. (2.5) the "shallow" approximation is made, the vertical Coriolis terms are dropped and the equations are linearized about a simple state of rest. In Section 3, the "high frequency" approximation is made. In principle, at least, it should be possible to derive normal modes of eqs. (2.1)–(2.5) under much less restrictive assumptions. However, it seems likely that the major part of the high frequency acoustic wave activity in global non-hydrostatic model would be projected on the approximate acoustic modes of Section 3. The circumstantial arguments of the previous paragraph suggest that in this case, the acoustic mode filtering technique should be stable when applied to the integration of (2.1)–(2.5).

The present analysis has been very preliminary. Many aspects remain to be examined. Orographic effects obviously cause complications in the non-hydrostatic equations. Treatment of this problem by transformation to terrain following coordinates has been discussed by Miller and White (1984). Other potential problems are the parameterization of physical processes and the treatment of small-scale explicitly resolved gravity waves which play a role in convection. No model based on the spherical non-hydrostatic equations, exists at this time, so the ideas presented here cannot yet be tested. However, the time when such a model will be required might be only a few years off, so it is not too early to start thinking about these equations. The next step is to build a global model based on eqs. (2.1)–(2.5) and test the acoustic mode filter and other possible integration schemes.

A final comment can be made concerning hydrostatic (primitive) equation models. In the application of normal mode initialization to these models, considerable storage space is required for storing the horizontal structures of the high frequency inertia-gravity modes. The problem gets worse as the horizontal resolution of the model increases. The present analysis suggests that the horizontal structures of the highest frequency modes can be approximated quite accurately by single spherical harmonics (two harmonics in the case of ψ). For a model based on a spherical harmonic expansion (or even one that is not), a simplified horizontal structure implies enormous potential savings in storage space. It is also conceivable that WKB methods could be used to generate simplified horizontal structures at somewhat lower frequencies.

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