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SHORT CONTRIBUTION

A note on the relation between the "traditional approximation" and the metric of the primitive equations

By ROLF MÜLLER, *Kerschensteinerstr. 10, D-6500 Mainz, FRG*

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ABSTRACT

The so-called "traditional approximation" consists in the neglect of certain terms in the equations of motion, in particular the horizontal Coriolis components. The validity of this approximation has long been discussed in the literature. In this study, it is shown that it is possible to exploit the shallowness of the atmosphere for a simplification of the spherical metric of the considered system ("Metrische Vereinfachung"). A Lagrange equation with this simplified metric may then be used to derive the equations of motion. These equations will not contain the problematic terms neglected in the "traditional approximation". Using this procedure, it is possible to include a simple friction model through a dissipation function.

1. Introduction

An important step in the derivation of the primitive equations is the so-called "traditional approximation". It is assumed that the neglect of certain terms in the equations of motion, in particular the Coriolis components proportional to the sine of colatitude (i.e., proportional to the cosine of latitude) is justified. The main consequence of this approximation is that it makes the separation of variables possible. It has been named "traditional" by Eckart (1960), who argues that it is applied throughout the literature without proper justification.

The "traditional approximation" is usually preceded by assuming that in all three components of the Navier–Stokes-equation and in the equation of continuity, the variable r (the vertical coordinate) may be replaced by the constant radius of the earth a and the derivative $\mathrm{d}/\mathrm{d}r$ through $\mathrm{d}/\mathrm{d}z$ where $r = a + z$. This assumption is justified through the fact that the radius of the earth ($a \sim 6000$ km) is large compared to the meteorologically relevant height of the atmosphere ($z \sim 30$ km), i.e., through the shallowness of the atmosphere.

Phillips (1966) has pointed out that the Navier–Stokes-equation formulated with this approximation but including the terms neglected in the "traditional approximation", violates conservation of angular momentum. He derived a set of equations which does possess an angular momentum principle by rederiving the equations of motion from a vector invariant formulation and exploiting the shallowness of the atmosphere.

In this study, a further connection will be presented between the assumption of a shallow atmosphere and the "traditional approximation". The method used here was developed by Hinkelmann in 1971 (Korb, personal communication) and named simplification of the metric ("Metrische Vereinfachung").

The procedure is as follows. The starting point is a coordinate-independent formulation of the Lagrange equation of motion. This equation will contain the metric tensor g_{ij}. Then spherical coordinates are introduced and in the metric tensor g_{ij}, the variable r will be replaced with the constant radius of the earth a. Thus the approximation is equivalent to a simplification of the metric. The equations of motion in components derived from this Lagrange equation will not contain the terms usually neglected in the "traditional approximation".

If one wishes to include friction in the model, the problem is not a pure mechanical one any more and thus not necessarily suitable for a Lagrange formalism. It is however possible, if friction is proportional to velocity (Rayleigh-friction), to include a dissipation function in the Lagrange equation (see Landau and Lifschitz, 1979) and then use the same procedure as outlined above.

2. The simplification of the metric

The Lagrange equation of motion in general coordinates q^k, $\dot{q}^k$ may be written:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial T_{\mathrm{a}}}{\partial \dot{q}^k}\right] - \frac{\partial}{\partial q^k}\bigg|_{\dot{q}^i}(T_{\mathrm{a}} - \Phi_{\mathrm{a}}) = -\frac{1}{\rho}\frac{\partial p}{\partial q^k} - \frac{\partial \mathscr{F}}{\partial \dot{q}^k} \quad (1)$$

with

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}^n \frac{\partial}{\partial q^n}.$$

Here Φ_{a} denotes the potential of attraction, and T_{a} the kinetic energy:

$$T_{\mathrm{a}} = \frac{1}{2}\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)^2 = \frac{1}{2}\boldsymbol{v}_{\mathrm{a}}^2 = \frac{g_{mn}}{2}\dot{q}_{\mathrm{A}}^n \dot{q}_{\mathrm{A}}^m, \quad (2)$$

where

$$\dot{q}_{\mathrm{A}}^i = \dot{q}^i + w_q^i. \quad (3)$$

The velocity of the coordinate system relative to the inertial frame is denoted by w_q^i in eq. (3). The Lagrange equation (1) contains a dissipation function $\mathscr{F}$ which is given by

$$\mathscr{F} := \tfrac{1}{2}\alpha_{il} g_k^l \dot{q}^i \dot{q}^k,$$

where the matrix α is defined as:

$$\alpha := \begin{pmatrix} \nu & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & \nu \end{pmatrix}.$$

The loss of energy of the system $\mathrm{d}E/\mathrm{d}t$ is then given by:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -2\mathscr{F}.$$

The equation of continuity in general coordinates is given by:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \frac{\rho}{\sqrt{g}}\frac{\partial}{\partial q^n}(\sqrt{g}\,\dot{q}^n) = 0, \quad (4)$$

where we have assumed that the divergence of w_q^i vanishes and g denotes the determinant of the metric tensor.

The potential of attraction Φ_{a} is usually combined with the potential of the centrifugal force $(\Omega^2 l^2)/2$ (with Ω the rotational frequency of the earth and l the distance of the rotational axis) to the geopotential Φ:

$$\Phi = \Phi_{\mathrm{a}} - \tfrac{1}{2}\Omega^2 l^2.$$

A reference surface of constant Φ defines the geoid which is approximated to a high degree of accuracy by an ellipsoid of revolution with a very small eccentricity. If one were to employ a coordinate system where the surfaces of constant vertical coordinate q^3 are identical to those of constant geopotential, the horizontal derivatives of Φ would vanish.

$$\frac{\partial \Phi}{\partial q^1} = \frac{\partial \Phi}{\partial q^2} = 0, \quad \Phi = \Phi(q^3). \quad (5)$$

Hence the equations would be simplified considerably.

Thus the optimal choice for a coordinate system would be spheroidal coordinates, where the surfaces of constant vertical coordinate q^3 are ellipsoids of rotation and thus equations (5) hold. However here, as common in the meteorological literature, a less complicated, spherical coordinate system with radius r, colatitude θ and longitude ϕ is introduced. Nonetheless, it is assumed that (where g denotes the gravity acceleration of the earth):

$$\Phi(r) = g \cdot r$$

$$\frac{\partial \Phi}{\partial \phi} = \frac{\partial \Phi}{\partial \theta} = 0;$$

in other words, the ellipticity of the earth is neglected (see Phillips (1973) for a detailed discussion of this question). The geopotential Φ in spherical coordinates is given by:

$$\Phi = \Phi_{\mathrm{a}} - \tfrac{1}{2}a^2 \sin^2\theta\,\Omega^2.$$

The spherical coordinate system specifies a metric. This metric is now simplified by replacing the variable r by the constant a in the metric tensor g_{ij}. This approximation is justified through the shallowness of the atmosphere:

$$r = a + z, \quad dr = dz, \tag{6}$$

a = radius of the earth (constant)

$a \gg z$.

The metric tensor g_{ij} is then given by:

$$g_{11} = 1, \quad g_{22} = a^2, \quad g_{33} = a^2 \sin^2 \theta,$$

$$g_{ij} = 0 \quad \text{for} \quad i \neq j$$

and thus

$$\sqrt{g} = a^2 \sin \theta. \tag{7}$$

With these relations, the kinetic energy is:

$$T_a = \tfrac{1}{2}[a^2 \sin^2 \theta \dot{\phi}_A^2 + a^2 \dot{\theta}_A^2 + \dot{z}_A^2]. \tag{8}$$

If the absolute velocity is separated in the movement of the coordinate frame relative to the inertial frame and the movement of the inertial frame itself by

$$\begin{aligned} \dot{\phi}_A &= \dot{\phi} + \Omega, \\ \dot{\theta}_A &= \dot{\theta}, \\ \dot{z}_A &= \dot{z}, \end{aligned} \tag{9}$$

one obtains for the kinetic energy T_a:

$$T_a = \tfrac{1}{2}[\dot{z}^2 + a^2 \dot{\theta}^2 + a^2 \sin^2 \theta (\dot{\phi}^2 + 2\dot{\phi}\Omega)] + \tfrac{1}{2} a^2 \sin^2 \theta \Omega^2. \tag{10}$$

Specifying the general coordinates as:

$$q^i = z, \theta, \phi, \tag{11}$$

$$\dot{q}^i = \dot{z}, \dot{\theta}, \dot{\phi}, \tag{12}$$

yields from equation (1) the contravariant equations of motion:

$$\begin{aligned} &\frac{d\dot{z}}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g - \nu \dot{z}, \\ &\frac{d(a^2 \dot{\theta})}{dt} - a^2 \sin \theta \cos \theta \dot{\phi}^2 - 2\Omega a^2 \sin \theta \cos \theta \dot{\phi} \\ &\quad = -\frac{1}{\rho}\frac{\partial p}{\partial \theta} - \nu \dot{\theta} a^2, \\ &\frac{d(a^2 \sin^2 \theta \dot{\phi})}{dt} + 2\Omega a^2 \sin \theta \cos \theta \dot{\theta} \\ &\quad = -\frac{1}{\rho}\frac{\partial p}{\partial \phi} - \nu \dot{\phi} a^2 \sin^2 \theta, \end{aligned} \tag{13}$$

with

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{\phi}\frac{\partial}{\partial \phi} + \dot{\theta}\frac{\partial}{\partial \theta} + \dot{z}\frac{\partial}{\partial z}.$$

Transforming to physical coordinates with (no sum convention):

$$u^k = \sqrt{g_{kk}}\, \dot{q}^k,$$

or explicitly:

$$\begin{aligned} w &= u^1 = \dot{z}, \\ v &= u^2 = a\dot{\theta}, \\ u &= u^3 = a \sin \theta \dot{\phi}, \end{aligned}$$

one obtains the final form of the equations of motion:

$$\begin{aligned} &\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g - \nu w, \\ &\frac{dv}{dt} - \frac{u^2}{a}\cot \theta - 2\Omega \cos \theta u \\ &\quad = -\frac{1}{a\rho}\frac{\partial p}{\partial \theta} - \nu v, \\ &\frac{du}{dt} + \frac{uv}{a}\cot \theta + 2\Omega \cos \theta v \\ &\quad = -\frac{1}{a \sin \theta}\frac{1}{\rho}\frac{\partial p}{\partial \phi} - \nu u, \end{aligned} \tag{14}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{u}{a \sin \theta}\frac{\partial}{\partial \phi} + \frac{v}{a}\frac{\partial}{\partial \theta} + \frac{\partial}{\partial z}.$$

The transformation of the equation of continuity is quite simple. With eqs. (7), (11) and (12) one obtains for eq. (4):

$$a \sin \theta \frac{d\rho}{dt} + \rho \frac{\partial}{\partial z}(a \sin \theta \dot{z}) + \rho \frac{\partial}{\partial \theta}(a \sin \theta \dot{\theta}) + \rho \frac{\partial}{\partial \phi}(a \sin \theta \dot{\phi}), \tag{15}$$

or in physical coordinates:

$$\frac{d\rho}{dt} = \rho \left[\frac{\partial w}{\partial z} + \frac{1}{a \sin \theta}\left(\frac{\partial}{\partial \theta}(\sin \theta v) + \frac{\partial u}{\partial \phi}\right)\right]. \tag{16}$$

3. Conclusions

It has been demonstrated that it is possible to obtain the result of the "traditional approximation" through the assumption of a shallow atmosphere applied to the metric of the system. The problematic terms neglected in the "traditional approximation" do not appear at all in the set of equations (14) derived here. It should be pointed out that a set of equations in components containing these terms has no properly defined metric. It should also be noted that the metric defined above through the "Metrische Vereinfachung" is not a Euclidian one any more.

The Lagrange formalism employed here does not generally allow the treatment of dissipation. In spite of this fact, it was shown that the inclusion of a simple friction model (Rayleigh-friction) is possible.

It must be emphasized, however, that the arguments presented above do not provide a physical justification for the neglect of the terms in question. The aim of this note is to clarify *what* is actually done in the "traditional approximation", not *why* it may be done. It is always necessary to consider the actual physical situation to judge the applicability of this approximation. Through the study by Queney (1950) and in the discussion between Veronis (1968) and Phillips (1968), it has been clarified that the terms in question may be neglected if the buoyancy frequency N (Brunt–Väisälä-frequency) is large compared to the rotational frequency Ω of the earth:

$N^2 \gg \Omega^2$.

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Note added in proof:

The author has learned recently that Hinkelmann has published the technique of simplifying the metric not only in his lecture notes but also in the following reference: Hinkelmann, K. 1969. *Lectures on numerical short-range weather prediction.* Hydrometeoizdat, Leningrad, 306–375.

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