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Static stability and vertical velocity: from planetary to small scale

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ABSTRACT

General relations between static stability and aspect ratio and for the "convection/advection ratio" are obtained by use of scale analysis. These relations extend known relations, and are applied to the planetary, synoptic, meso- and small scales.

1. Introduction

Vertical layering in the atmosphere is characterized by the logarithmic vertical gradient of potential temperature, viz.

$$\sigma = \frac{1}{\theta}\frac{\partial \theta}{\partial z},$$

and the name "static stability" for this parameter is apt, since a positive gradient indicates an inhibiting effect on (adiabatic) vertical motions. Intuitively, one expects this static stability to be inversely related to the aspect ratio of atmospheric motions: a statically stable atmosphere would tend to be associated with "shallow" motions, i.e., motions with height dimension h small relative to the horizontal extent L, while an unstable layering would favour "deep" convection. This is expressed by a relationship dating back to the work of Prandtl (1936), viz.

$$\frac{g\sigma}{f^2} \sim \frac{L^2}{h^2}, \tag{1}$$

which means that the static stability σ is proportional to the inverse square of the aspect ratio h/L. This was known to hold on the synoptic scale, but was shown by Burger (1958) to break down on the very large or planetary scale.

This paper contains an exploration of the validity of this relation over scales from very large to small, within the simplest physical framework of neglecting friction and turbulence, heat radiation and conduction, and atmospheric water. These do of course play important roles in various contexts and specifically in small scale motions, but the simpler physics is instructive as a broad orientation.

2. Basic order inequalities

As scale quantities, it will be convenient to use, apart from those already mentioned, the earth's radius a, the scale quantities V, W and C for horizontal and vertical wind speed and wave propagation speed respectively, and D for horizontal divergence. We also split pressure p and density ρ into a hydrostatic layering on the one hand and the part associated with the motion on the other, thus:

$$p = p_0 + p_1,$$
$$\rho = \rho_0 + \rho_1,$$

with

$$\frac{\partial p_0(z)}{\partial z} + g\rho_0(z) = 0, \tag{2}$$

and we use the same symbols to denote scale quantities. Note here incidentally that

$$\frac{\partial p}{\partial z} + \rho g = \frac{\partial p_1}{\partial z} + \rho_1 g,$$

without approximation. As final scale quantity, the "pressure height" H is defined by

$$H = -p_0 \Big/ \frac{\partial p_0}{\partial z}. \tag{3}$$

We now consider order of magnitude relationships between the scale quantities, where adjacent orders are separated by about a factor 10, and equality in order of magnitude will allow one member to be even a few times the size of the other. Thus, for the planetary scale, $L \sim a \sim 10^7$ m, while we might have $L \sim 10^6$ m for the synoptic scale, $L \sim 10^5$ m for the mesoscale (see also Emanuel (1984)) and $L \sim 10^4$ m for the small scale. Very small scales will not be considered here.

In the whole range quoted, we now have from (2) and (3) the order equality

$$\frac{p_0}{gH\rho_0} \sim 1, \tag{2a}$$

and then require the following weak inequalities:

$$\frac{h}{H}, \frac{H}{L}, \frac{L}{a}, \sigma H \lesssim 1,$$

$$\frac{C}{V}, \frac{W}{V} \lesssim 1,$$

$$\frac{DL}{V} \lesssim 1,$$

$$\frac{fV}{g} \lesssim \frac{h}{a},$$

$$\frac{\rho_1}{\rho_0} \lesssim 1,$$

and finally the strong inequalities:

$$\frac{H}{a} \ll 1,$$

$$\frac{V^2}{gh} \ll 1.$$

The additional weak inequality

$$\frac{WL}{Vh} \lesssim 1,$$

follows from writing the continuity equation in the form

$$\frac{\partial w}{\partial z} + \frac{2w}{r} + \frac{w}{\rho}\frac{\partial \rho}{\partial z} = -\nabla \cdot V - \frac{1}{\rho}\frac{\partial \rho}{\partial t} - \frac{1}{\rho} V \cdot \nabla \rho$$

and noting that the right-hand side is $\lesssim V/L$ (since ρ_1/ρ_0 and $DL/V \lesssim 1$), while the left-hand side is $\sim W/h$. (Here, as in similar situations in the sequel, we have to assume that there is no cancellation of terms on the left-hand side.)

These relationships are evidently very unrestrictive as is easily verified, and in Section 3, we shall deduce from them further relationships that will be useful when specializing to specific scales of motion.

For that, we first note that, as a result of the above, certain order relationships hold between terms in the basic equations. We look for order relations in which the two sides will balance over the whole range of scales concerned, which means that there is always at least one term of dominant magnitude on each side. Within this limitation, we aim at retaining the minimum number of terms that would still cover all scales considered. For instance, it follows from

$$\frac{C}{V}, \frac{WL}{Vh} \lesssim 1$$

that in the Eulerian individual derivative of velocity, the horizontal advection dominates in order of magnitude, so that the partial time derivative and the vertical advection or convection need not feature in the order statement.

3. Consequential order equalities

Application of the above considerations to the horizontal equation of motion yields

$$-\frac{1}{\rho}\nabla p \sim (V \cdot \nabla V)_n + f\mathbf{k} \times V$$

(standard notation and geometric or z-system coordinates are used throughout). In terms of the scale quantities this implies

$$\frac{p_1}{L\rho_0} \sim \frac{V^2}{L} + fV,$$

and thus

$$\frac{p_1}{fVL\rho_0} \sim 1 + \frac{V}{fL}.$$

This allows us to write, from the vertical equation of motion,

$$-\frac{1}{\rho}\frac{\partial p_1}{\partial z} \sim \frac{\rho_1}{\rho} g,$$

i.e.,

$$\frac{p_1}{h\rho_0} \sim \frac{\rho_1 g}{\rho_0},$$

and thus

$$\frac{\rho_1}{\rho_0} \sim \frac{p_1}{gh\rho_0} \sim \frac{fVL}{gh}\left(1+\frac{V}{fL}\right).$$

From these and eq. (2a), we obtain a further inequality

$$\frac{p_1}{p_0} \sim \frac{p_1}{gH\rho_0} \sim \frac{h}{H}\frac{\rho_1}{\rho_0} \leqslant \frac{\rho_1}{\rho_0}.$$

Therefore, we get from the thermodynamic equation,

$$w\sigma \sim -\frac{1}{\rho} \boldsymbol{V}\cdot\nabla\rho$$

i.e.,

$$W\sigma \sim \frac{V\rho_1}{L\rho_0},$$

and thus

$$\frac{WL}{Vh} \sim \frac{1}{\sigma h}\frac{\rho_1}{\rho_0} \sim \frac{fVL}{g\sigma h^2}\left(1+\frac{V}{fL}\right). \tag{4}$$

In the continuity equation,

$$\nabla\cdot\boldsymbol{V} \sim -\frac{1}{\rho}\frac{\partial \rho w}{\partial z} - \frac{1}{\rho}\boldsymbol{V}\cdot\nabla\rho,$$

i.e.,

$$D \sim \frac{W}{h} + \frac{V}{L}\frac{\rho_1}{\rho_0},$$

and thus

$$D \sim \frac{W}{h} + W\sigma \sim \frac{W}{h}.$$

The vorticity equation yields

$$(\zeta+f)\nabla\cdot\boldsymbol{V} \sim -\boldsymbol{V}\cdot\nabla(\zeta+f) - \boldsymbol{k}\cdot\nabla\frac{1}{\rho}\times\nabla p,$$

i.e.,

$$fD + \frac{V}{L}\frac{W}{h} \sim \frac{V^2}{L^2} + \frac{fV}{a} + \frac{1}{L^2}\frac{\rho_1}{\rho_0}\frac{p_1}{\rho_0},$$

and thus

$$\begin{aligned}\frac{WL}{Vh}\left(1+\frac{V}{fL}\right) &\sim \frac{V}{fL} + \frac{L}{a} + \frac{1}{fVL}\frac{\rho_1}{\rho_0}\frac{p_1}{\rho_0}\\ &\sim \frac{V}{fL} + \frac{L}{a} + \frac{fVL}{gh}\left(1+\frac{V}{fL}\right)^2\\ &\sim \frac{V}{fL} + \frac{L}{a} + \frac{fVL}{gh}\left(1+\frac{V^2}{f^2L^2}\right)\\ &\sim \frac{V}{fL}\left(1+\frac{V^2}{gh}\right) + \frac{L}{a}\left(1+\frac{fVa}{gh}\right)\\ &\sim \frac{V}{fL} + \frac{L}{a}.\end{aligned} \tag{5}$$

If we finally eliminate W from (4) and (5), the result is

$$\frac{V}{fL} + \frac{L}{a} \sim \frac{fVL}{g\sigma h^2}\left(1+\frac{V}{fL}\right)^2 \sim \frac{fVL}{g\sigma h^2}\left(1+\frac{V^2}{f^2L^2}\right),$$

i.e.,

$$\frac{g\sigma h^2}{f^2L^2}\left(\frac{V}{fL}+\frac{L}{a}\right) \sim \frac{V}{fL}\left(1+\frac{V^2}{f^2L^2}\right),$$

which, with B (see Phillips (1963), Pedlosky (1979)), and Ro defined as

$$B = \frac{g\sigma h^2}{f^2L^2},$$

$$\mathrm{Ro} = \frac{V}{fL},$$

can be written as

$$B\left(1+\frac{L}{a}\frac{1}{\mathrm{Ro}}\right) \sim 1+\mathrm{Ro}^2. \tag{6}$$

This relation generalizes the Prandtl relation (1), which can be written as $B = 1$, to both the planetary and the small scales.

Note further that, with the "convection/advection ratio" A defined as

$$A = \frac{WL}{Vh},$$

(4) can be written as

$$A \sim \frac{\mathrm{Ro}}{B}(1+\mathrm{Ro}). \tag{7}$$

4. Application to different scales

The two key relations (6) and (7) can be specialized to various scales. For the planetary scale $L/a \sim 1$ and $\mathrm{Ro} \ll 1$, and thus

$B \sim \mathrm{Ro} \ll 1,$

$A \sim 1.$

For the synoptic scale $L/a \sim \mathrm{Ro} \ll 1$, and thus

$B \sim 1,$

$A \sim \mathrm{Ro} \ll 1.$

For the mesoscale $L/a \ll 1$ and $\mathrm{Ro} \sim 1$, and thus

$B \sim 1,$

$A \sim 1.$

Finally, for the small scale $L/a \ll 1$ and $\mathrm{Ro} \gg 1$, and therefore

$B \sim \mathrm{Ro}^2 \gg 1,$

$A \sim 1.$

The generalized relation (6) thus accommodates values of the static stability which need not become excessively large for small aspect ratios or excessively small for large aspect ratios. The relation (7) shows that vertical advection is relatively small on the synoptic scale only.

Application of the above relations to the basic equations on the various scales from planetary to small, is now a straightforward exercise. This will be sketched here, with the approach that only dominant terms are retained.

Because fVL/gh and $V^2/gh \ll 1$ on all scales considered, the anelastic form of the continuity equation

$$\nabla \cdot \boldsymbol{V} + \frac{1}{\rho}\frac{\partial \rho w}{\partial z} = 0$$

holds if $h/H \sim 1$, i.e. for "deep" systems, and the "shallow" or incompressible form

$$\nabla \cdot \boldsymbol{V} + \frac{\partial w}{\partial z} = 0$$

for $h/H \ll 1$, which can occur on the mesoscale and small scale.

In accordance with our simple physics, the adiabatic relation

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0$$

holds for all scales considered, but the horizontal individual derivative of p can be omitted for "shallow" systems.

The hydrostatic relation applies down to the mesoscale, but on the small scale for "deep" systems where $h/L \sim 1$, the term $\mathrm{d}w/\mathrm{d}t$ is retained in the vertical equation of motion.

The Coriolis term $fk \times V$ features in the horizontal equation of motion on all scales except the small scale. On the planetary and synoptic scales, this equation is stationary with the geostrophic terms dominating. This stationary character applies to the vorticity equation too on the planetary scale, but not on the synoptic scale, where it has a particularly convenient form due to the relative smallness of vertical advection ($A \ll 1$). The "solenoidal" term is omitted on all scales, again since fVL/gh and $V^2/gh \ll 1$. The form of the divergence equation parallels that of the vorticity equation with the pressure term added, except on the synoptic scale where it is simply $-f\zeta + (1/\rho)\nabla^2 p = 0$.

Interestingly, for the mesoscale, the set of equations thus obtained constitutes the "primitive" equations, and in the vorticity equation, no beta term occurs for this scale.

5. Conclusion

In summary, we have seen, within the limitations of the simple physical framework adopted, that:

the key relations (6) and (7) are crucial in determining the ratios of terms in the basic meteorological equations over a wide range of scales;

relations (6) and (7) generalize known relations of Prandtl (1936) and Burger (1958), and avoid excessively large or small values of static stability with varying aspect ratio;

the "convection/advection ratio" A is small on the synoptic scale only;

the primitive equations apply primarily on the mesoscale;

the vorticity equation has a particularly simple form only on the synoptic scale;

the continuity equation is "anelastic" on all scales considered;

on small scales, the dynamics of "deep" and "shallow" systems differ, as reflected in the continuity and thermodynamic equations and the vertical equation of motion.

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