Tellus (1989), *41A*, 270–274

SHORT CONTRIBUTION

Mountain-induced stagnation points in hydrostatic flow

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(Manuscript received 25 April 1988; in final form 4 August 1988)

ABSTRACT

The Bernoulli and hydrostatic relations are used to derive an exact diagnostic equation relating wind speed to the integral of vertical displacement aloft. The form of this equation does not support the "kinetic energy" concept of flow stagnation proposed by Sheppard (1956). Linear theory estimates of the displacement integral are used to predict the occurrence of stagnation points as a function of hill shape and ambient shear. For a long ridge perpendicular to a weakly sheared flow, stagnation begins aloft, thus allowing wave breaking and transition to a severe state. For a ridge aligned with the flow, waves dispersively weaken aloft and stagnation occurs first on the surface. This allows density surfaces to intersect the ground and the low-level flow to split around the hill.

1. Introduction

As one considers the airflow past higher and higher mountains, the mechanical disturbance to the ambient flow increases until the air is brought to rest at certain points in the flow field. When such a stagnation point occurs aloft, the streamlines become steeply sloping and wave breaking soon follows. When a stagnation point occurs on the solid lower boundary, an isentropic surface can intersect the boundary at a finite angle and streamline splitting can occur. Thus the prediction of qualitative flow type requires a prediction of stagnation.

The well known analysis and "kinetic energy" argument of Sheppard (1956) indicates that the lowest layer of a stably stratified fluid cannot lift over a hill without stagnating if the hill height $h > U/N$, regardless of the hill shape. Extending Sheppard's analysis for hydrostatic flow we find (1) the local height dependent term used by Sheppard cancels out, and (2) the critical hill height is quite dependent on hill shape. In a recent paper (Smith, 1988, hereafter S88) linear theory was used to estimate the onset of stagnation in unsheared flow over an axisymmetric hill. In this special case, linear theory predicts that stagnation will occur simultaneously aloft and at the lower boundary when the non-dimensional mountain height $\hat{h} = hN/U \approx 1.3$. In the present work, we extend these calculations to include other hill shapes and shearing ambient flow.

2. Determination of wind speed from the displacement field

Consider the steady flow of a stratified non-rotating Boussinesq fluid over a hill where the density far upstream is given by $\rho = \rho_0(1-(N^2/g)z)$ and the speed is $U_0(z)$. We suppose that there exists some pattern of density surface vertical displacement above the hill $\eta(x, y, z)$ which decays aloft (i.e., $\lim_{z\to\infty} \eta = 0$) and conforms to the hill (i.e., $\eta(x, y, z = h) = h(x, y)$). This latter assumption will be inconsistent if a stagnation point exists on the surface $z = h$, allowing other density surfaces to intersect it.

Following Sheppard by assuming hydrostatic balance far from the hill, Bernoulli's equation along a streamline can be written

$$u^2 = \frac{2}{\rho_0}(-p^* - \tfrac{1}{2}\rho_0 N^2 \eta^2) + U_0^2, \tag{1}$$

Where p^* is the difference between the pressure at a point (x, y, z) and the pressure at the same elevation far away (∞, ∞, z). Instead of making an assumption about p^* directly, as did Sheppard, we invoke the hydrostatic relation to determine certain properties of p^*.

$$p^*(x, y, z) = g \int_z^\infty \rho' \,\mathrm{d}z = \rho_0 N^2 \int_z^\infty \eta \,\mathrm{d}z, \tag{2}$$

where ρ' is the density anomaly. Since we wish the lower limit of the integral (2) to lie on a particular density surface it is convenient to write $\eta(x, y, z)$ in density coordinates: $\eta(x, y, z_0)$ where z_0 is the height of a particular density surface upstream, i.e., $\rho = \rho(z_0)$.

Using $\mathrm{d}z = \mathrm{d}z_0 + \mathrm{d}\eta$, (2) becomes

$$p^*(x, y, z_0) = \rho_0 N^2 (I_\eta - \tfrac{1}{2}\eta^2), \tag{3}$$

where

$$I_\eta(x, y, z_0) = \int_{z_0}^\infty \eta(x, y, z_0') \,\mathrm{d}z_0'. \tag{4}$$

Combining (1) and (3)

$$u^2 = -2N^2 I_\eta + U_0^2. \tag{5}$$

The cancelling of the η^2 terms between (1) and (3) implies that speed variations predicted by (5) are associated only with non-local hydrostatic pressure variations and not with local parcel lifting. Stagnation (i.e., $u = 0$) begins when $I_\eta = U_0^2/2N^2$.

3. Linear theory prediction of stagnation

Lacking a full non-linear solution for $\eta(x, y, z_0)$ or $I_\eta(x, y, z_0)$ for use in (5) we use a linear theory in density coordinates (S88). For steady inviscid non-diffusive hydrostatic non-rotating Boussinesq flow, the equation governing the displacement field $\eta(x, y, z_0)$ in an ambient flow $U(z_0)$, $N = $ constant, is

$$U^2 \eta_{xxz_0z_0} + 2UU_{z_0}\eta_{xxz_0} + N^2(\eta_{xx} + \eta_{yy}) = 0, \tag{6}$$

where z_0 is the height of a certain density surface upstream so that $\rho = \rho(z_0)$ and upstream $U_0 = U_0(z_0)$ and $N^2 = -(g/\rho_0)(\mathrm{d}\rho/\mathrm{d}z_0)$. The lower boundary condition is

$$\eta(x, y, z_0 = z_g) = h(x, y), \tag{7}$$

and the radiation condition is used aloft. Using a double Fourier transform the solution to (6) and (7) is:

$$\eta(x, y, z_0) = \int\int_{-\infty}^{\infty} \bar{h}(k, l) f(k, l, z_0) \mathrm{e}^{i(kz + ly)} \mathrm{d}k \,\mathrm{d}l, \tag{8}$$

and the integral (4) is

$$I_\eta(x, y, z_0) = \int\int_{-\infty}^{\infty} \bar{h}(k, l) g(k, l, z_0) \mathrm{e}^{i(kx + ly)} \mathrm{d}k \,\mathrm{d}l, \tag{9}$$

where $\bar{h}(k, l)$ is the double Fourier Transform of $h(x, y)$. When

$$U_0(z_0) = U_0 = \text{constant}, \tag{10}$$

$$f(k, l, z_0) = \mathrm{e}^{imz_0},$$

where (11)

$$m(k, l) = (N/U)\left(\frac{k^2 + l^2}{k^2}\right)^{1/2} \operatorname{sgn}(k).$$

In (9)

$$g(k, l, z_0) = \frac{i}{m} f. \tag{12}$$

When the approaching flow is shearing

$$U_0(z_0) = Cz_0 + U_g, \quad C > 0, \tag{13}$$

$$f(k, l, z_0) = \left(\frac{z_e}{z_r}\right)^a \tag{14}$$

where

$$z_e = z_0 - z_r, \quad z_r = -U_g/C,$$

$$a = -\frac{1}{2} + i\sqrt{R\frac{k^2 + l^2}{k^2} - \frac{1}{4}} \operatorname{sgn}(k),$$

$$R = N^2/C^2 > \tfrac{1}{4}.$$

In (9),

$$g(k, l, z_0) = \frac{z_r}{a + 1} z_e^{a+1}. \tag{15}$$

In the limit of the Richardson Number $R \to \infty$, eqs. (13), (14), and (15) reduce to eqs. (10), (11), and (12). The double integral in (9) can be rapidly computed using a fast Fourier transform (FFT) algorithm. Small errors arise from the ill-defined exponent in (9) when $k = 0$, from the periodic nature of the FFT, and from the finite resolution of the chosen grid.

Using eqs. (5) and (13), stagnation first occurs at a particular altitude (z_0) when the non-dimensional hill height is

$$\hat{h} = \hat{h}_{\text{crit}} = \frac{h_{\text{crit}} N}{U_g} = \frac{U_g}{2QN}(1 + R^{-1/2}\hat{z}_0), \tag{16}$$

where $\hat{z} = z_0 N/U_g$ and $Q(\hat{z}_0) = \text{Max}_{z_0}(I_\eta/h_M)$, computed from (9) with (12) or (15), is independent of h_M.

It could be argued that if a linear theory is used to provide $\eta(x, y, z_0)$, it would be more consistent to use the linearized Bernoulli equation instead of (1); that is

$$\rho_0 U_0 u' = -p^*, \tag{17}$$

where u' is perturbation horizontal speed. If stagnation is defined as $u' = -U_0$ and (3) is linearized then (16) is recovered but without the factor of 2 in the denominator. We choose to use the exact result (5) and (16) as it provides a common method for diagnosing linear and non-linear solutions and allows us to trace errors in u unambiguously to errors in the η field.

4. A family of ellipsoidal hills

Consider the family of hill shapes

$$h(x, y) = \frac{h_M}{(1 + (x/a_x)^2 + (y/a_y)^2)^n} \tag{18}$$

with elliptical contours. The value of Q in (16) then depends only on the aspect ratio $r = a_y/a_x$ and the exponent "n", as well as the Richardson number.

The choice (18) allows a comparison with the analytical formulae for surface pressure near elliptical hills given by Phillips (1984). According to S88, such solutions in z-coordinates with a linearized lower boundary condition can be used directly to compute I_η in density coordinates. Our FFT calculations of Q from (9) with $z_0 = 0$ agree with Phillips' values for $r = 1, \infty$ and $n = 1.0, 1.5, 2.0$, thus confirming our values of Q on the lower boundary. Phillips' method gives no estimate of Q aloft.

5. Results

For simple isolated hills, two incipient stagnation regions occur as $\hat{h} = hN/U_0$ is increased: one directly over the hill at an altitude of $\hat{z}_0 = z_0 N/U_0 \approx 4.0 \pm 0.7$ (point A) and one on the windward slope (point B). Values of $h_A = \hat{h}_{\text{crit}}$ and $h_B = \hat{h}_{\text{crit}}$ for which (16) is satisfied at points A and B are shown in Fig. 1. Stagnation on the

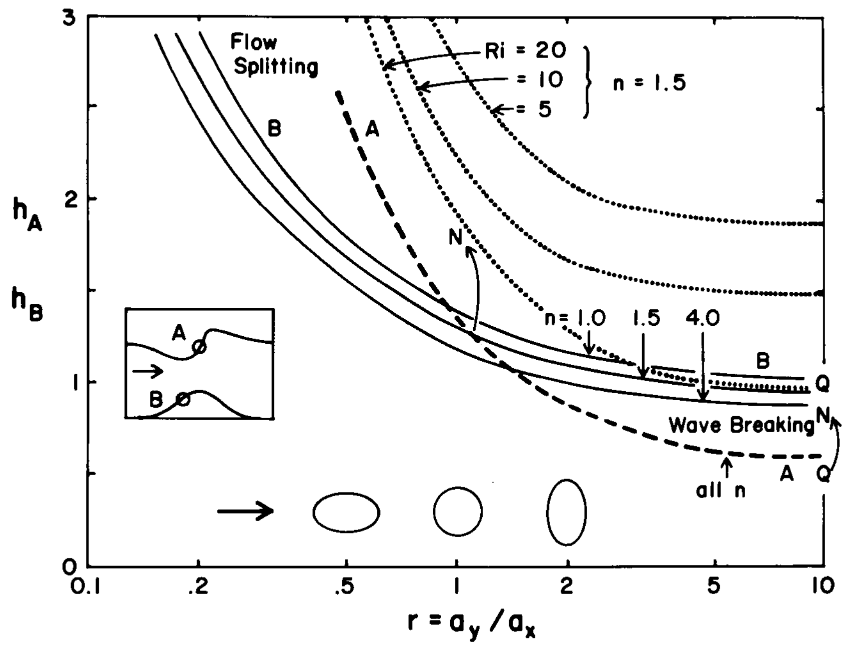

Fig. 1. The non-dimensional critical mountain height for stagnation aloft h_A and at the lower boundary h_B are shown as a function of horizontal aspect ratio ($r = a_y/a_x$). Curves for different steepness parameter (n) are shown although h_A is insensitive to its value. Two values from non-linear calculations (marked N) are shown for judging the accuracy of linear theory. The dotted curves in the upper right illustrate the effect of forward shear in delaying stagnation aloft, as a function of the Richardson number (Ri).

surface is moderately sensitive to "n" but stagnation aloft is so insensitive that the different curves overlie each other to the accuracy at which they were determined here ($\approx 5\%$). Ambient wind shear primarily influences stagnation aloft so only this variation is shown.

5.1. *Unsheared flow*

At large r, stagnation first occurs aloft. This is expected, as in the hydrostatic 2-D limit (Queney, 1948) the strongest part of the wave field occurs directly above the mountain with no decay aloft. Stagnation aloft is believed to cause gravity wave breaking and transition to a severe downslope wind configuration (Clark and Peltier, 1977). Low level stagnation will also occur in the presence of wave breaking for conditions in the upper right of Fig. 1, but the linear theory prediction of h_B should be disregarded in this case as it does not take wave breaking into account. Using a non-linear hydraulic theory, Smith (1985) predicts that low level blocking will begin at $h_B = 0.985 \approx 1.0$ in the presence of wave breaking. The apparent agreement between this value of h_B and the linear theory result $h_B = 1.0$ in Fig. 1 is fortuitous.

As r decreases, both values h_A and h_B increase but h_A for stagnation aloft increases more rapidly. This is so because in 3-D flow the vertically propagating waves weaken aloft due to dispersion. For $r = 1$, this decay goes as $\eta \approx z^{-1/2}$ (S88). At a particular r (depending on n) the curves in Fig. 1 cross indicating that for smaller r, surface stagnation will occur before stagnation aloft. This allows density surfaces to intersect the ground and the low-level flow will pass around the hill. Our assumption $\eta(h) = h$ is no longer valid.

We speculate that once a surface stagnation point forms, vertical displacements become limited. An increased hill height may not generate stronger gravity waves and wave breaking may never occur aloft. The recent laboratory results of Castro (1987) support this view.

The unlimited increase of h_B for decreasing r (Fig. 1) indicates that a low-level parcel moving along the centerline of a narrow flow-aligned ridge can rise to a great height without stagnating. Lifting is not limited to a maximum value of U_0/N suggested by Sheppard (1956). This is so because the displacement field $\eta(x, y, z)$ laterally disperses so rapidly aloft that the integral of displacement above the parcel I_η remains small. According to (5), it is only I_η that matters for stagnation, not the parcel displacement η.

The availability of high speed computers will soon make it possible to solve the full non-linear field equations and establish the exact positions of the curves in Fig. 1. Two currently available values for unsheared flow are: $r = 1$, $n = \frac{3}{2}$ for which $h_B \approx 1.8$ (P. Smolarkiewicz, private communication), and $r = \infty$, $n = 1$ for which $h_A = 0.85$ (Huppert and Miles, 1969). These values indicate that our linear theory predictions of the critical $\hat{h}$ may be about 30% too low. Because we have used the exact diagnostic equation (5), we know that this error must be due to an overestimate of η and I_η by linear theory. We speculate that the crossing of the A and B curves will still occur in the exact solutions, although the value of r where crossing occurs may be modified.

5.2. *Sheared flow*

The effect of ambient forward shear is twofold. First, shear decreases the effective stability of the flow and second, the faster flow aloft requires a larger pressure rise for stagnation. The latter influence is the dominant one in Fig. 1, causing the values of h_A to rise rapidly with increasing shear (decreasing Richardson Number). In mid-latitudes, values of the Richardson Number in the troposphere typically range from 5 to 20 which, according to Fig. 1, makes wave breaking unlikely in the troposphere and severe wind events rare.

The effects of reverse shear have not been explicitly computed as the concept of a critical mountain height is no longer useful. Near the wind reversal point $z = z_{ref} > 0$, even the smallest of hills will cause local stagnation. Further information on this problem can be found in the work Booker and Bretherton (1967), Clark and Peltier (1984) and Smith (1985).

6. Acknowledgements

Useful discussions with P. Baines, J. McGregor, R. Rotunno, P. Smolarkiewicz are gratefully acknowledged. This work was supported by the National Science Foundation Division of Atmospheric Sciences Grant ATM-8605939.

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