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A method for the determination of absolute geostrophic velocities in the sea

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ABSTRACT

A method utilizing hydrographic data and vertically-averaged currents for the determination of absolute geostrophic velocities in the sea is described. Relative dynamic depths are interpolated on a selected grid of points by (i) using the linear least-squares estimation for a given function with "unknown data" and (ii) solving the Poisson equation with Dirichlet boundary conditions and a finite number of linear constraints within the domain investigated. Once the fields of relative velocities are computed and the measured vertically-averaged currents are interpolated on the grid using the same procedure as for the relative dynamic depths, the absolute currents may be calculated by simply subtracting the vertically-averaged relative velocities from the vertical averages of the currents. This method is illustrated by an example from the Rijeka Bay, a small basin in the northeast Adriatic Sea. The results for the Rijeka Bay are interesting from a qualitative point of view, whereas for the exact quantitative reproduction of the Bay circulation, higher-order dynamics should be invoked. It is concluded that the method in its original form is more suitable for the open sea, and the experiment needed for its application in such an environment is briefly described.

1. Introduction

It is well-known that temperature and salinity data, taken at a number of hydrographic stations, make the determination of relative velocities in the sea possible. The procedure of computation is usually called the dynamic method, and its main assumptions are a hydrostatic equilibrium along the vertical axis and a geostrophic balance of forces in the horizontal plane. The dynamic method in its classical form has most thoroughly been described by Fomin (1964). His monograph made it clear that the main problem of the method is the transfer from relative to absolute velocities. Whereas various approaches had been proposed for solving this problem, not a single one had been accepted as satisfactory at the time Fomin's book was published.

Recently, the search for an absolute velocity field has been renewed by Stommel and Schott (1977) and by Wunsch (1977). In both papers, only hydrographic data are used for the calculation of absolute currents. Stommel and Schott supplement the geostrophic balance of forces by the condition that there is no flow across the density surfaces and that linear potential vorticity is conserved. On the other hand, Wunsch adds to the geostrophic balance the conservation of an arbitrary number of properties and the condition that the energy of the barotropic currents be minimal. The additional assumptions introduced by these authors made the calculation of absolute currents possible, leading to further investigations along the same lines.

A different approach has been suggested by Wunsch and Gaposchkin (1980). It is based on satellite measurements of the absolute topography of the sea surface. Once the elevations of the sea surface are expressed relative to the geoid, surface currents may be calculated. Combining them with the relative velocities yields the absolute current field.

Whereas methods that use hydrographic data alone supplement the geostrophic approximation with some additional assumptions, Wunsch and Gaposchkin arrive at the absolute velocities by combining geostrophic-shear data with measurements of sea-surface topography.

In this paper, another method for transforming relative into absolute velocities will be investigated. It employs directly-measured vertically-averaged currents as extra data equivalent to the satellite-altimetry data in the method by Wunsch and Gaposchkin. The method consists of 3 steps. In the first step, relative current fields are constructed by the objective analysis of hydrographic data taken in the bounded domain. In the second step, measured vertically-averaged currents are interpolated by the objective analysis on the same grid as used for relative currents. In the final step, relative velocities and vertically-averaged currents are combined to obtain the absolute velocities at every level where hydrographic data were taken.

Autocovariance functions needed for the reconstruction of both the relative and vertically-averaged currents are deduced from a unique model of flow for which horizontal and vertical eddy viscosities are taken into account. Certain parameters in the model are interpreted as random fields, admitting the construction of both autocovariance functions by standard statistical methods from the same set of hydrographic data.

Each step in our procedure for calculating absolute currents is illustrated by an example from the Rijeka Bay, a small basin in the north-east Adriatic Sea (Fig. 1). Two circumstances favor the use of this basin: (i) the Rijeka Bay is characterized by simple geometry, so that it could well be approximated by a flat-bottomed basin; (ii) we had at our disposal simultaneous hydrographic and sea-current data on a network of stations for a period of a few days when low-speed winds were registered above the Bay. However, some of the inadequacies of our data set will emerge in this paper. The fact that the experiment in the Rijeka Bay was not designed with this particular application in mind caused difficulties, and, of course, a geostrophic/hydrostatic approximation for such a small basin cannot be accepted as final. However, we believe that a simple illustrative example may facilitate the understanding of this method and the designing of adequate experiments for its application in larger sea basins.

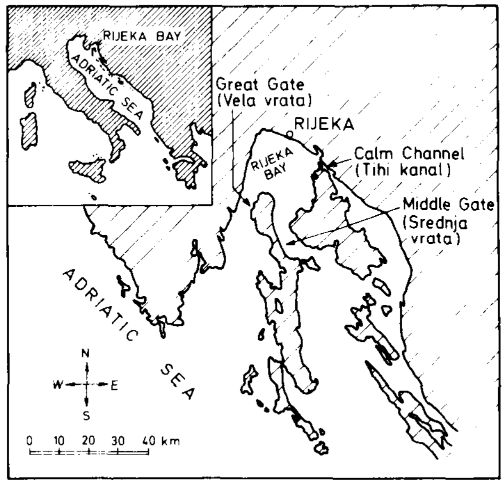

Fig. 1. The location of the Rijeka Bay.

2. Relative velocities

The basin for which currents are to be constructed is a 3-dimensional domain denoted by $\mathscr{D}$. Its boundary is denoted by $\mathscr{B}$, and we assume that it consists of vertical and horizontal segments only. Let us use the so-called p-system as the coordinate system (Kasahara (1974), p stands for pressure). The points of $\mathscr{D}$ are then denoted by $(\boldsymbol{r}, p)$, where $\boldsymbol{r} = (x, y)$, x and y being the horizontal coordinates in our selected coordinate system.

We start with the basic assumptions of the classical dynamic method, namely, the geostrophic balance of forces in the horizontal plane and the hydrostatic equilibrium along the vertical axis. The equation of horizontal motion then reads:

$$\boldsymbol{c} = -\frac{1}{f} \operatorname{grad} D \times \boldsymbol{k}, \tag{1}$$

whereas the hydrostatic equation can be expressed as:

$$\frac{\partial D}{\partial p} = \alpha, \tag{2}$$

where $\boldsymbol{c} = \boldsymbol{c}(\boldsymbol{r}, p)$ is the horizontal velocity vector, $D = D(\boldsymbol{r}, p)$ is the dynamic depth, $\alpha = \alpha(\boldsymbol{r}, p)$ is the specific volume, f is the Coriolis parameter, and $\boldsymbol{k}$ denotes the unit vector orientated vertically downward. The gradient operator, as well as the Laplace operator that appears later in the text, implies that pressure is to be held constant for partial differentiation.

By integrating (2) between the sea surface (assumed to be an isobaric surface for which pressure equals zero) and the isobaric surface p, we get:

$$D_r = \int_0^p \alpha \mathrm{d}p, \tag{3}$$

where the subscript r indicates the relative value, $D_r(\boldsymbol{r}, p) = D(\boldsymbol{r}, p) - D(\boldsymbol{r}, 0)$. The integral to the right of (3) can be determined numerically on the basis of the temperature and salinity data measured at selected hydrographic stations. Now, expressions (1) and (3) yield:

$$\boldsymbol{c}_r = -\frac{1}{f} \operatorname{grad} D_r \times \boldsymbol{k}, \tag{4}$$

from where we see that only relative currents can be calculated from the standard hydrographic data.

Here we shall define a procedure for the construction of D_r for fixed p and for a land-locked basin. Let the intersection of boundary $\mathscr{B}$ with surface $p = \text{const}$ be denoted by $\mathscr{B}(p)$, and let the relative dynamic depths have values $d_r(p)$ at $\mathscr{B}(p)$. Let $\boldsymbol{r}_k$, $1 \leqslant k \leqslant M$, be points inside $\mathscr{D}$ were the data $D_r(\boldsymbol{r}_k) = D_{r,k}$ are prescribed. Our procedure is defined by (see Appendix A for a detailed derivation):

$$\begin{aligned} &[K][\lambda(\boldsymbol{r})] = [b(\boldsymbol{r})] \\ &G_r(\boldsymbol{r}) = \sum_{k=1}^{M} \lambda_k(\boldsymbol{r})\, G_{r,k} \\ &\Delta D_r(\boldsymbol{r}) = G_r(\boldsymbol{r}) \\ &D_r|\mathscr{B}(p) = d_r(p), \qquad D_r(\boldsymbol{r}_k) = D_{r,k}, \end{aligned} \tag{5}$$

where $[K]$ is an $M \times M$ matrix, the elements of which are $K_{ik} = C(\boldsymbol{r}_i, \boldsymbol{r}_k)$; $[\lambda(\boldsymbol{r})]$ is a column of M unknown functions $\lambda_k(\boldsymbol{r})$; $[b(\boldsymbol{r})]$ is a column of M functions $b_i(\boldsymbol{r}) = C(\boldsymbol{r}, \boldsymbol{r}_i)$. The mean of the field G_r vanishes, whereas its autocovariance function is denoted by C. It may be seen that our procedure is based on (i) the linear least-squares estimate for the function G_r with "unknown data" $G_{r,k}$ and (ii) the solving of the Poisson equation with Dirichlet boundary conditions and M linear constraints inside $\mathscr{D}$. The problem (5) is a control problem with the control variables $G_{r,k}$, $1 \leqslant k \leqslant M$.

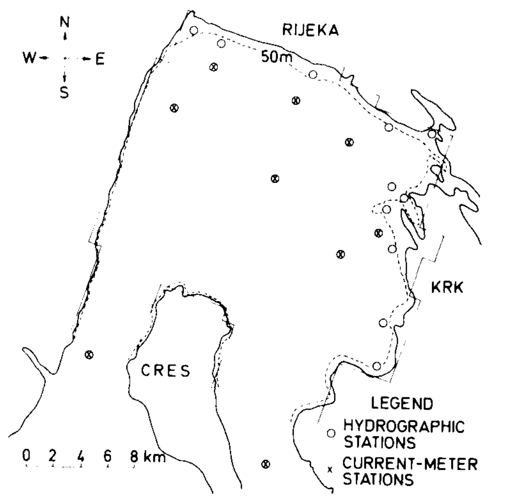

Fig. 2. The Rijeka Bay topography. The positions of the hydrographic and the current-meter stations are indicated, and the boundaries of the modelled area are shown.

The problem (5) always has a solution. The relative velocities $\boldsymbol{c}_r = -(1/f)$ grad $D_r \times \boldsymbol{k}$ are continuous and their partial derivatives of the first order are functions defined on $\mathscr{D}$.

Let us now turn our attention to the Rijeka Bay example. From Fig. 2, it appears that the Rijeka Bay can be approximated reasonably well by a flat-bottom basin. We have chosen a 60-dbar surface as the bottom of our idealized domain. Horizontally, we have schematized the boundaries of the Rijeka Bay to fit a 25 × 24 grid of points spaced 1 km apart in both x (SSW) and y (ESE) directions (Fig. 2). That has left us with two open boundaries through which the Rijeka Bay communicates with the Adriatic Sea, whereas the third opening, the Calm Channel, is supposed to be nonexistent in our idealized domain.

Relative dynamic depths D_r were computed from the hydrographic data taken almost synoptically at 19 stations in the Rijeka Bay, during 10–11 June 1976 (Jeftić, 1977). The positions of the stations, as well as the station in the Middle Gate which was not used here, are shown in Fig. 2. At most of the stations, temperature and salinity were measured at pressures of 0, 5, 10, 20, 30, 50 and 60 dbar. Where the near-bottom data were not given at exactly 50 and 60 dbar, we have interpolated sigma-t (σ_T) values onto these pressures. Relative dynamic depths were calculated from the sigma-t values alone, since such an approximation does not alter the error bounds for our measurements at low pressures.

The relative dynamic depth along the continental boundary has been obtained by the simple averaging of data from all the stations in the network. In order to determine the relative dynamic depth along the coast of the Island of Cres, we have used current-meter data from the Great Gate (Jeftić, 1977). The measured shear of currents defines the difference between the relative dynamic depths along two closed boundaries, from where the remaining boundary condition may be computed.

To solve (5) numerically for our working example, the autocovariance function C for the stochastic field $F_r = -(\partial\alpha/\partial p)_r$ must be known (see Appendix A). We approximate F_r by $F_{r0} = \alpha_0^2(\partial\sigma_T/\partial p)_r$, α_0 being the mean specific volume, and consider the field $E_r = (\partial\sigma_T/\partial p)_r$, where $E_r = (1/\alpha_0^2)F_{r0}$. The available data collected for the Rijeka Bay in June over various years are far from sufficient to decide the shape of the autocovariance function. However, we have at our disposal measurements taken in the Rijeka Bay at 4 transects with closely spaced stations (Jeftić, 1982) on 9 May 1981. We can group the data into pairs with a prescribed distance and compute the experimental autocovariance function. The result is plotted in Fig. 3. It suggests an oscillatory character of the autocovariance function. Therefore, we suppose that C has the form

$$\begin{aligned} C(x,0) &= 1 - l_1 x \qquad & 0 \leqslant |x| \leqslant L_1 \\ C(x,0) &= l_2 x - 1.25 \qquad & L_1 \leqslant |x| \leqslant L_2 \qquad (6) \\ C(x,0) &= 0 \qquad & L_2 \leqslant |x|, \end{aligned}$$

where the parameters are $L_1 = 6$ km, $L_2 = 10$ km, $l_1 = 0.25$ km^{-1}, and $l_2 = 0.125$ km^{-1}. The function $C(0, y)$ is defined analogously, and the autocovariance function $C(\mathbf{r})$ is simply $C(x, 0) \cdot C(0, y)$. Attention should be drawn to the fact that the experimental autocovariance function has been computed from 59 quasi-synoptic data pairs. Although this is the best data set we could muster, it certainly is far from ideal. Therefore, both the experimental autocovariance function and its approximation (6) should be treated with caution.

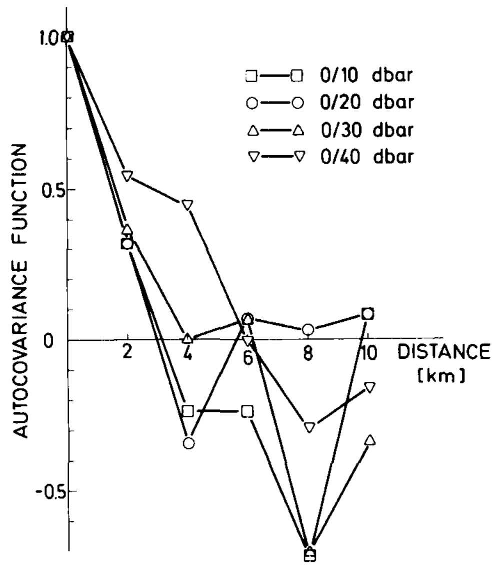

Fig. 3. The experimental autocovariance function for the relative differentials of the sigma-t value. The data were taken in the Rijeka Bay, on 9 May 1981.

Some numerical aspects of the problem (5) for our working example are presented in the Appendix B.

The results are 6 relative-current fields. Two of these, namely the 0/30 dbar and the 0/60 dbar fields, are presented in Figs. 4 and 5. It is interesting to compare the 0/30 dbar field with a subjective analysis of the respective relative dynamic depths (Orlić and Kuzmić 1980, and unpublished results). There is considerable qualitative similarity between the two: the water exchange between the Great Gate and the Middle Gate dominates the southern part of the basin, whereas separate circulation systems appear along the continental boundary of the Rijeka Bay. However, the differences between the objective and the subjective analyses are visible as well, particularly when one tries to pursue quantitative comparison. They reflect the fact that a human analyst finds it difficult to take cognizance of the boundary conditions for the whole basin, and that every analysis implies assumptions about the character or degree of the smoothness of the field being analyzed. The

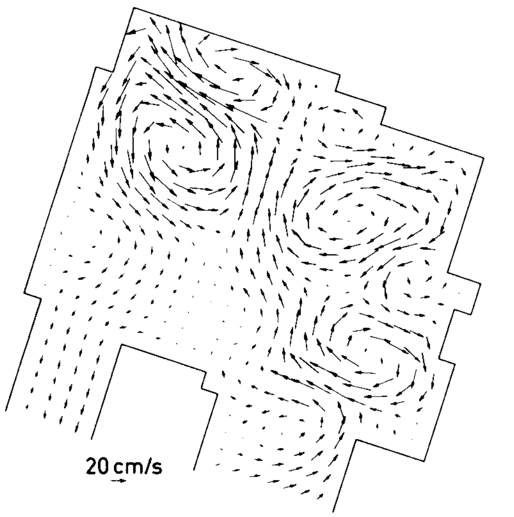

Fig. 4. The relative currents between the 0 and 30 dbar surfaces in the Rijeka Bay, 10–11 June 1976.

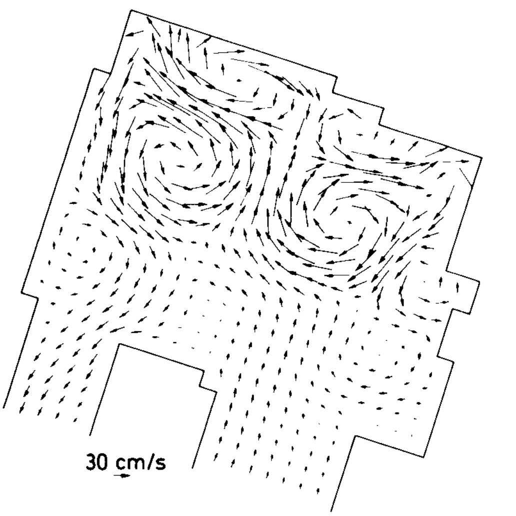

Fig. 5. As in Fig. 4 except for the 0/60 dbar pressure difference.

advantage of the objective method is its explicitness in stating the assumptions and its consistency in taking the boundary conditions into account. Let it be added that the computed relative currents (Figs. 4 and 5) are generally very high; we believe that ageostrophic and nonhydrostatic dynamics are the probable reasons, and we shall return later to this interpretation.

3. Vertically-averaged currents

Here we wish to interpolate the vertically-averaged currents that were measured in the bounded domain, the same domain that was analyzed in the previous paragraph. By integrating (1) between the sea surface ($p = 0$) and the bottom ($p = P$), and introducing the notation for a depth-mean value:

$$\langle \cdot \rangle = \frac{1}{P} \int_0^P \cdot \, \mathrm{d}p,$$

we get

$$\langle \boldsymbol{c} \rangle = -\frac{1}{f} \operatorname{grad} \langle D \rangle \times \boldsymbol{k}. \tag{7}$$

The procedure of interpolation is analogous to (5), see Appendix C:

$$[K]\,[\lambda(\boldsymbol{r})] = [b(\boldsymbol{r})],$$

$$Q(\boldsymbol{r}) = \sum_{k=1}^{2N} \lambda_k(\boldsymbol{r}) Q_k, \tag{8}$$

$$\Lambda \langle D(\boldsymbol{r}) \rangle = Q(\boldsymbol{r}),$$

$$\langle D \rangle | \mathscr{B}(0) = \langle d \rangle, \qquad \operatorname{grad} \langle D(\boldsymbol{r}_k) \rangle = -f\boldsymbol{k} \times \langle \boldsymbol{c}(\boldsymbol{r}_k) \rangle,$$

where $\langle d \rangle$ are the vertically-averaged dynamic depths along the boundary, and $\langle \boldsymbol{c}(\boldsymbol{r}_k) \rangle$ are the vertically-averaged currents prescribed within the domain at N points. The control variables are denoted by Q_k, whereas the corresponding field is denoted by Q.

Again, we shall illustrate the procedure by our example from the Rijeka Bay.

We had at our disposal current measurements made at 2 or 3 levels on the network of stations in the Rijeka Bay (Fig. 2), in the period 9–12 June 1976 (Jeftić, 1977). The tidal signal has been filtered out from the original time series, and resultant vectors have been averaged along the vertical axis. Of course, it would be better to base the interpolation on directly measured vertically-averaged currents (or transports). Since such measurements had not been performed in the Rijeka Bay, we had to use the approximative approach mentioned.

A fundamental difficulty should be pointed out here. As follows from (7), the measured vertically-averaged currents ought to represent the geostrophic transports only. However, one may expect these currents to be modified by frictional effects in the surface and the bottom boundary layers. Whereas such modification may not be too serious in the oceans, where the Ekman depth is only a small fraction of the total depth, it is certainly important in the shallow Rijeka Bay. We tried to partly avoid this problem by analyzing the situation when low-speed winds were registered above the Rijeka Bay. The frequency distribution for the hourly winds measured at Rijeka Airport between 9 and 12 June 1976 shows that our 4-day period was very quiet, with land and sea breezes dominating the record. About 50% of the wind speeds fall into 0 – 1 m/s class.

From current measurements in the Great Gate, we have computed the difference between the vertically-averaged dynamic depths along the 2 closed boundaries of the Rijeka Bay. One of these depths, that at the continental boundary, was arbitrarily set at zero. This caused no difficulty, since we were interested in the gradients of vertically-averaged dynamic depths, not in their absolute values.

To complete the formulation of our numerical example, we had to determine the autocovariance function C for the stochastic field $-\alpha_R = \alpha(0) - \alpha(P)$ (see Appendix C). First we approximated $-\alpha_R$ by $\alpha_{R0} = \alpha_0^2[\sigma_T(P) - \sigma_T(0)]$, which suggested the difference $[\sigma_T(P) - \sigma_T(0)]$ as a subject for further analysis. The experimental autocovariance function for this difference was computed from the same data set that had been already used for the determination of the statistics of the field $(\partial\sigma_T/\partial p)_r$. The results are shown in Fig. 6. The experimental data are well approximated by:

$$\begin{aligned} C(x,0) &= 1 - l_1 x && 0 \leqslant |x| \leqslant L_1 \\ C(x,0) &= l_2 x - \tfrac{7}{6} && L_1 \leqslant |x| \leqslant L_2 \qquad (9) \\ C(x,0) &= 0 && L_2 \leqslant |x|, \end{aligned}$$

where $L_1 = 4$ km, $L_2 = 7$ km, $l_1 = \frac{3}{8}$ km^{-1}, and $l_2 = \frac{1}{6}$ km^{-1}. Again, $C(\boldsymbol{r}) = C(x,0)\cdot C(0,y)$, where $C(0,y)$ is of the same shape as $C(x,0)$. The autocovariance function (9) is very similar to the previously-defined autocovariance function (6). Indeed, our experiments with these 2 functions showed negligible differences between the resulting fields of the vertically-averaged currents.

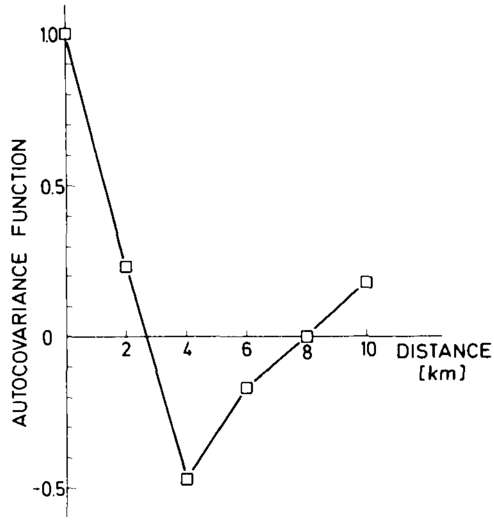

Fig. 6. The experimental autocovariance function for the 0/40 dbar difference of the sigma-t value. The data were taken in the Rijeka Bay, on 9 May 1981.

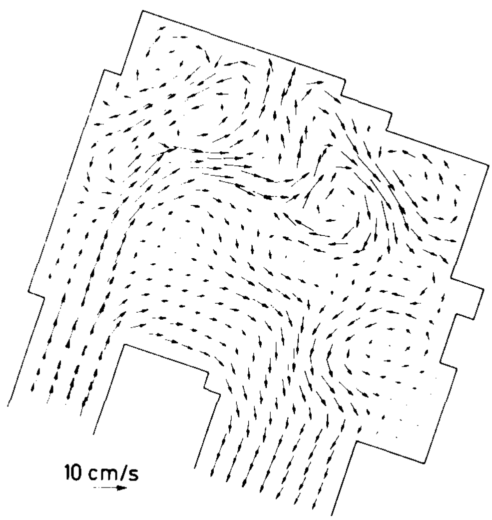

Fig. 7. The vertically-averaged currents in the Rijeka Bay, 9–12 June 1976.

The distribution of the vertically-averaged currents in the Rijeka Bay, at the beginning of June 1976, is shown in Fig. 7. We can compare this result with a somewhat different analysis of the same data set, in the northern part of the Rijeka Bay (Limić, 1984). The interpolation proposed in the cited paper satisfies the law of mass conservation and it minimizes the squared difference between the sought-after field and the field that results from the linear interpolation of measured data. Although such an approach obviously differs from the method proposed here, the correspondence between the two results is surprisingly good. Each gyre that appears in one field is also reproduced in another. Therefore, we may conclude that both methods, within the framework of their respective physics (geostrophy versus nondivergence), provide credible results.

4. Absolute currents

In Section 2, we have calculated relative velocities c_r for the domain investigated and the given pressure differences. In Section 3 the horizontal distribution of the vertically-averaged currents $\langle c \rangle$ has been determined. Recollecting now that

$$c_r(r, p) = c(r, p) - c(r, 0)$$

and integrating this equation from the sea surface to the bottom, we obtain:

$$c(r, 0) = \langle c(r) \rangle - \frac{1}{P} \int_0^P c_r(r, p) \mathrm{d}p. \tag{10}$$

Therefore, the surface currents may be calculated from (10), which makes possible the determination of the absolute current field in the sea.

For the Rijeka Bay example, we have used the simple trapezoidal rule to solve numerically the integral to the right of (10). Consequently,

$$\begin{aligned} c(r, 0) = \langle c(r) \rangle &- \frac{c_r(r, 0)}{24} - \frac{c_r(r, 5)}{12} - \frac{c_r(r, 10)}{8} \\ &- \frac{c_r(r, 20)}{6} - \frac{c_r(r, 30)}{4} \\ &- \frac{c_r(r, 50)}{4} - \frac{c_r(r, 60)}{12}. \end{aligned} \tag{11}$$

Of course, $c_r(r, 0) = 0$, whereas the other fields that are needed for the computation of surface currents are known.

The absolute currents were calculated for each level where hydrographic data had been taken; in Figs. 8 and 9 those at the sea surface and the 30 dbar isobaric surface are shown, respectively.

The main features of the surface and the mid-depth currents are the water exchange between the Great Gate and the Middle Gate in the southern part of the Bay and the numerous gyres that appear along the continental boundary. Of course, we immediately asked ourselves about the origin of such a circulation pattern. A possible explanation stems from the fact that the locations of our gyres in the Rijeka Bay coincide with the positions of submarine springs in this basin (Alfirević, 1969). As is well-known, the subma-

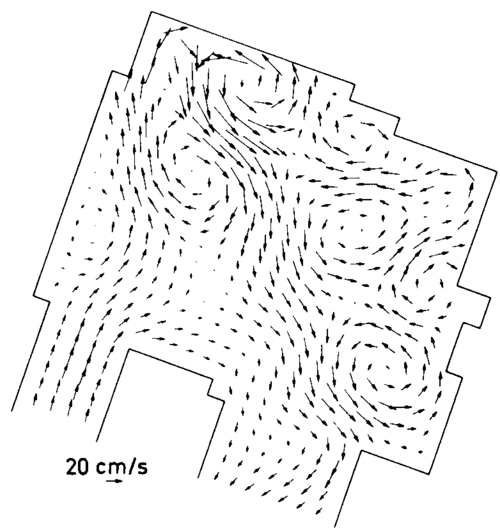

Fig. 8. The absolute velocities at the sea surface in the Rijeka Bay, 9-12 June 1976.

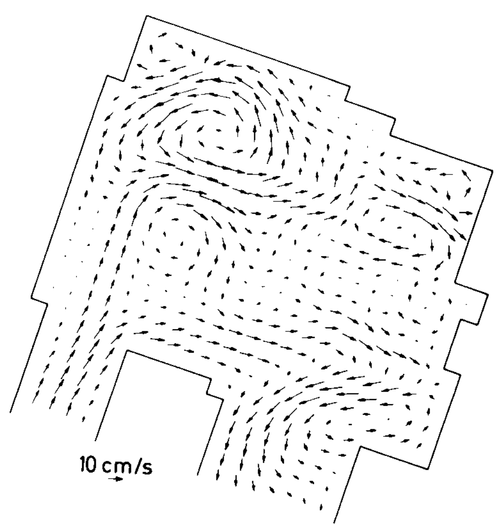

Fig. 9. As in Fig. 8 except for the 30 dbar surface.

rine springs in the Adriatic (called "vrulje") are closely connected with the circulation of the underground waters in the littoral karst area. Abundant precipitation causes water to penetrate the underground structures from which it eventually emerges into the sea. The characteristics of this water change during the seasons; the warmer part of the year in the Rijeka Bay seems to be characterized by bottom inflows of water which is denser than the ambient water. This would explain the existence of both the major circular gyres along the continental boundary of the Rijeka Bay and the semicircular flow in the southern part of the basin (Figs. 8 and 9).

Although qualitative considerations seem to support our results for the Rijeka Bay, quantitative analysis brings into focus a problem which demands adequate explanation. The calculated currents appear to be of unrealistically high magnitudes, which can most clearly be seen in the fields of the current shear (Figs 4, 5). In our attempt to understand the underlying cause, we have first taken into account the errors in the measurements of temperature and salinity. However, the minimum-energy fields, defined by a given variance of input data, did not bring the currents down to acceptable magnitudes. Secondly, we have assessed the effect of the (possibly inadequate) open-boundary conditions. Again, without resolving the point: the influence of these conditions is felt only in the southern part of the basin, whereas the northern part, where the problem is most evident, remains unaffected. Consequently, our physical assumptions remain to be reconsidered. From the time series of current measurements and the absolute currents computed here, it follows that the Strouhal number is ~ 10, whereas the Rossby number is close to one. Therefore, at least nonlinear acceleration terms should be included in the model, particularly for the northern part of the Rijeka Bay. On the other hand, taking $K_V = 10^{-3}$ m^2 s^{-1} and $K_H = 10$ m^2 s^{-1}, one easily gets 10 m as a typical depth of the bottom boundary layer, whereas the side-wall friction layers are 1 km in width. These amounts are comparable to the vertical and the horizontal dimensions of the Rijeka Bay, making frictional terms another possible cause of departures from the geostrophy. The Rijeka Bay circulation is further complicated by the influence of the submarine springs; the associated vertical accelerations may locally destroy the hydrostatic equilibrium, which certainly would affect the currents calculated on the basis of hydrostatic approximation. To summarize, we can say that the results obtained for the Rijeka Bay are interesting from the qualitative point of view, whereas higher-order dynamics should be invoked for the exact quantitative reproduction of the circulation.

5. Discussion

5.1. Difficulties encountered in the procedure of interpolation

In this paper, relative dynamic depths are interpolated via a given function from which these depths can be reconstructed. The autocovariance function is deduced from a model of flow for which horizontal and vertical eddy viscosities are taken into account. The same approach is followed for vertically-averaged currents. Consequently, the autocovariance functions for the interpolation of these currents can be determined from hydrographic data. Let us stress here some difficulties encountered in the procedure of interpolation.

The procedure generally produces a solution which is not unique. If the measuring points are spaced sufficiently apart from each other or the autocovariance function decreases with sufficient rapidity, the solution is unique. In the case of narrowly spaced measuring points and/or a slowly decreasing autocovariance function, the system (5) can be ill-conditioned, producing either no unique solution or large values of $G_{r,k}$. Then the estimated value of G_r may be unrealistic or even nonphysical due to an improper sign of physical quantities. This can be avoided by imposing additional constraints on the estimation of the field $G_r = -(g/\alpha_0)^2 (K_V/K_H)(\partial\alpha/\partial p)_r + h_r$. Additional constraints necessarily imply a reformulation of the problem (5) to an optimal control problem, as the estimation, with additional constraints, is a minimization problem (Limić and Mikelić, 1984).

The currents obtained from (5) depend on selected autocovariance functions for the field G_r. In general, the autocovariance function for G_r depends on p. To obtain an insight into the dependence of D_r on the autocovariance function, we used the following measure of error:

$$\varepsilon = \sqrt{\frac{\int_{\mathscr{S}} (c_r - s_r)^2 \, dr}{\int_{\mathscr{S}} (c_r)^2 \, dr}},$$

where c_r is obtained by (5) with the autocovariance function of (6) and s_r is obtained by (5) with another autocovariance function. By a slight change of either the shape or range of the autocovariance function, we obtain $\varepsilon = 0.5$–0.8. For the autocovariance function $C(x, y) = K(x)K(y)$, $K(x) = -\delta(x + H)/2 + \delta(x) - \delta(x - H)/2$, $H = 6$ km, we obtain $\varepsilon = 1.2$. Although the error ε is high, the patterns of c_r and s_r are the same. This feature is also pointed out by Bretherton et al. (1976). Even with functions $C(x, y) = K(x)K(y)$, K being a concave function with the range of 4–8 km (not necessarily belonging to the class of autocovariance functions), we obtain the same patterns. However, autocovariance functions with a large range or a slight decrease produce a significant distortion of the pattern of c_r.

5.2. *Design of future measurements*

The method for transforming relative into absolute velocities, advanced in this paper, has been illustrated by an example from the Rijeka Bay. However, we believe that better results with the method can be obtained in different physical environments, with adequate planning of field work. Let us describe briefly an ideal experiment that would allow the determination of absolute velocities from geostrophic-shear data and vertically-averaged currents.

The experiment should concentrate on the interior of an ocean, away from the surface, bottom and side-wall friction layers. Geostrophic and hydrostatic approximations are expected to be valid in the ocean interior. Hydrographic measurements should be closely spaced along the vertical axis, so that the accurate numerical solution of the integral given in (10) can be obtained. Vertically-averaged currents should be measured by a drop-sonde, an instrument originally developed by Richardson and Schmitz (1965). This device is dropped into the sea at a known location. It descends at a constant rate to a prescribed depth and then rises to the surface. The distance between the drop point and the point of reappearance at the surface represents the transport in a water column. Two simultaneously launched drop-sondes ought to give even better results: if one instrument samples only the surface Ekman layer and the other sinks to the top of the bottom Ekman layer, their combined information should represent the geostrophic transport in the interior of the water column. If significant tidal currents are present in the area, current-meter measurements should be used for correcting the drop-sonde data.

The fields of relative dynamic depths and vertically-averaged currents should be obtained by straightforward objective analysis, from measurements at a given number of data points, from autocovariance functions of the quantities being analyzed, and from the measurement errors. Moreover, r.m.s. errors expected in the estimates are to be calculated and illustrated in the form of error maps (Bretherton et al., 1976). It is of crucial importance to determine in advance the statistics of the fields being measured and the noise levels involved. Then, a station network which strikes an appropriate balance between accuracy and spatial coverage may be chosen, and the indirect interpolation that fixes the measured values, which we were forced to use in this paper, may be avoided. After the fields of relative velocities and vertically-averaged currents are constructed, the method described in the foregoing paragraphs enables the absolute current field (and the associated variances) to be determined.

6. Conclusions

In order to determine the absolute geostrophic velocities in the sea, one first has to interpolate measured relative dynamic depths onto a selected grid of points. In this paper, we propose a procedure by which the relative dynamic depths are obtained through solving the Poisson equation, using the linear least-squares estimation for determining the coefficients (right side) of this equation.

Next, the measured vertically-averaged currents are interpolated on the grid, using a procedure analogous to that described for relative dynamic depths. Particular attention should be paid to the source data: they ought to represent the geostrophic transports only.

Finally, the absolute currents are computed by subtracting the normalized integral of the relative velocities from the vertically-averaged currents. Sensitivity analysis of the method is required for every specific site where it is to be applied.

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8. Appendix A

Interpolation of relative dynamic depths

The pair consisting of a Poisson equation and an autocovariance function is needed for an interpolation of the relative dynamic depths. Our aim is to derive the Poisson equation for relative dynamic depths and to specify the corresponding stochastic field for which the autocovariance function has to be known.

The domain $\mathscr{D}$ representing the basin has the boundary consisting of the surface $p = 0$ (sea surface), and of the remaining part $\mathscr{B}$ (coast together with bottom). Intersections of $\mathscr{D}$ and $\mathscr{B}$ with the planes $p =$ const. are two-dimensional domains $\mathscr{D}$ (p) and their boundaries $\mathscr{B}$ (p), respectively.

(*1*) *The first model* is defined by the following suppositions.

(A) The linearized equations of stationary motion (Janowitz, 1972) may be used:

$$-f\boldsymbol{k} \times \boldsymbol{c} = \text{grad } D + K_H \Delta \boldsymbol{c} + K_V \frac{g^2}{\alpha_0^2} \frac{\partial^2 \boldsymbol{c}}{\partial p^2}, \qquad \text{(A1)}$$

$$\frac{\partial D}{\partial p} = \alpha, \qquad \text{(A2)}$$

$$\text{div } \boldsymbol{c} = 0, \qquad \text{(A3)}$$

where α_0 is the mean specific volume, g is the acceleration due to gravity, and K_H and K_V are the horizontal and vertical eddy viscosity coefficients. The boundary condition is defined by:

$$\boldsymbol{c}|\mathscr{B} = 0. \qquad \text{(A4)}$$

(B) The boundary values $d(p)$ of the dynamic depths $D(p)$ are defined at $\mathscr{B}$ (p). For a fixed p, the quantity $d(p)$ has a constant value, when restricted to any connected part of $\mathscr{B}$ (p).

(C) The function $\partial D/\partial p$ is defined on the whole domain $\mathscr{D}$.

(D) The vector field $\boldsymbol{c}$ has vanishing vertical component.

(E) The quantity f is constant throughout $\mathscr{D}$.

The equation div $\boldsymbol{c} = 0$, the boundary condition (A4) and the supposition (D) imply the following representation:

$$\boldsymbol{c} = -\text{curl}(\Phi \boldsymbol{k}) \qquad \text{(A5)}$$

of the vector field $\boldsymbol{c}$. By substituting (A5) into (A1) and by applying the divergence operator to the obtained equality, we derive

$$\Delta(D - f\Phi) = 0, \qquad \text{(A6)}$$

so that $f^{-1} D$ and Φ differ by a harmonic function h_1:

$$\Phi = f^{-1} D + h_1. \qquad \text{(A7)}$$

By applying the curl operator to (A1) and by multiplying with $\boldsymbol{k}$, we obtain $\boldsymbol{k}[K_H \Delta(\text{curl } \boldsymbol{c}) + K_V (g^2/\alpha_0^2)\partial^2/\partial p^2(\text{curl } \boldsymbol{c})] = 0$. As $\boldsymbol{k}$ curl $\boldsymbol{c} = \Delta\Phi$ from (A5) and $\Delta\Phi = f^{-1}\Delta D$ from (A6), we derive an equation for D:

$$\Delta\left(K_H \Delta D + K_V \frac{g^2}{\alpha_0^2} \frac{\partial^2}{\partial p^2} D\right) = 0. \qquad \text{(A8)}$$

This equation must be supplied by boundary conditions at $\mathscr{B}$ (p). Without loss of generality, we may suppose that the function h_1 in (A7) vanishes. Then the boundary conditions corresponding to (A8) follow uniquely from the condition (A4) and the supposition (B):

$$D(p)|\mathscr{B}(p) = d(p), \qquad \frac{\partial D}{\partial n}|\mathscr{B}(p) = 0. \qquad \text{(A9)}$$

The solution $\boldsymbol{c} = -f^{-1}$ curl$(D\boldsymbol{k})$ is a unique solution satisfying (A1)–(A4).

As standard hydrographic data can be used only for a reconstruction of the relative dynamic depths, the supposition (B) has to be changed into the following one.

(B′) The boundary values $d_r(p)$ of the relative dynamic depths, $D_r(p) = D(p) - D(0)$ are defined at $\mathscr{B}(p)$.

Then the system (A8), (A9) has to be replaced by the following one:

$$\Delta^2 D_r = f_r \qquad D_r|\mathscr{B}(p) = d_r(p), \quad \frac{\partial D_r}{\partial n}\bigg|\mathscr{B}(p) = 0, \tag{A10}$$

where $f_r = -(K_V/K_H)(g^2/\alpha_0^2)\Delta[\partial^2 D/\partial p^2]_0^p$. The relative velocity field is defined by $\boldsymbol{c}_r = -f^{-1}\,\mathrm{curl}(D_r\boldsymbol{k})$.

Let us mention that system (A10) can also be used in the case of incomplete knowledge of the functions f_r and d_r, after an interpolation of these functions has been performed.

(*2*) *The second model.* Hydrographic data can easily be used for an interpolation of the functions $\partial D/\partial p$ and $\partial^2 D/\partial p^2$ by the linear least-squares estimation. However, the obtained result of interpolation should not be derived with respect to lateral variables, as such an operation produces big errors. This is the principal cause forcing us to avoid the use of the first model in the form (A10). The best we can do is to use the geostrophic approximation $\Phi = f^{-1} D$ a priori, and to insert it into the equations (A1). The resulting equation has the form $\mathrm{curl}(U\boldsymbol{k}) = 0$, where $U = K_H \Delta D + K_V(g^2/\alpha_0^2)\partial^2 D/\partial p^2$. This implies the constancy of the function U for every fixed p. Hence:

$$\Delta D = F + h_2, \qquad D|\mathscr{B}(p) = d(p), \tag{A11}$$

$$\Delta D_r = F_r + h_r, \qquad D_r|\mathscr{B}(p) = d_r(p), \tag{A12}$$

where $F = -(K_V/K_H)(g^2/\alpha_0^2)\partial^2 D/\partial p^2$, $F_r(p) = F(p)|_0^p$, h_2 is a function depending on p and $h_r(p) = h_2(p)|_0^p$. Hence the expression $\boldsymbol{c}_r = -f^{-1}\,\mathrm{curl}(D_r\boldsymbol{k})$, where D_r is defined by (A12), is the model of relative velocity field used in this work.

(*3*) *Procedure.* The right-hand side of (A12) is an unknown function to us, because the parameters K_H, K_V and h_r are unspecified. Only the function $\partial^2 D/\partial p^2$ is defined on the set $\mathscr{M}$ of M points $\boldsymbol{r}_k$, $1 \leqslant k \leqslant M$. Also the function D_r, entering the left-hand side of (A12), is defined on the same set $\mathscr{M}$. A reconstruction of the function D_r from this small set of data is based on statistical methods. We suppose:

(F) The quantities $\partial^2 D/\partial p^2$ and h_2 are a random field and a random variable, respectively, satisfying the following equation

$$\bar{F} = -\bar{h}_2,$$

where $\bar{F}$ and $\bar{h}_2$ are the corresponding statistical means.

Then the quantity D is also a random field having the mean $\bar{D}(p)$ satisfying the equation $\Delta\bar{D}(p) = 0$. This implies that $\bar{D}_r(p)$ is a harmonic function with the boundary values $\bar{D}_r(p)|\mathscr{B}(p) = d_r(p)$.

Let C be the autocovariance function of F_r. Then the linear least squares estimation of $F_r + h_r$, from data $G_{r,k} = F_r(\boldsymbol{r}_k) + h_r$, has the form:

$$F_r(\boldsymbol{r}) + h_r = \sum_{k=1}^{M} G_{r,k}\lambda_k(\boldsymbol{r}), \tag{A13}$$

where the functions λ_k are solutions of the matrix system:

$$\sum_{k=1}^{M} K_{ik}\lambda_k(\boldsymbol{r}) = b_i(\boldsymbol{r}), \qquad 1 \leqslant i \leqslant M, \tag{A14}$$

and $K_{ik} = C(\boldsymbol{r}_i, \boldsymbol{r}_k)$, $b_i(\boldsymbol{r}) = C(\boldsymbol{r}, \boldsymbol{r}_i)$. The expression (A13) actually does not determine an interpolation of $F_r + h_r$, as $G_{r,k}$ are unspecified. Nevertheless, this expression, with M unspecified parameters $G_{r,k}$, $1 \leqslant k \leqslant M$, is inserted into (A12), yielding a solution D_r of (A12), being a linear function of M parameters $G_{r,k}$:

$$D_r(p) = D_0(p) + \sum_{k=1}^{M} G_{r,k} D_k(p). \tag{A15}$$

Here, the function D_0 and D_k are solutions of the following systems:

$$\Delta D_0 = 0, \qquad \Delta D_k = \lambda_k, \qquad D_0|\mathscr{B}(p) = d_r(p), \qquad D_k|\mathscr{B}(p) = 0. \tag{A16}$$

The unspecified parameters $G_{r,k}$ have to be determined from the M data $D_r(\boldsymbol{r}_k, p)$, $1 \leqslant k \leqslant M$. The solution obtained (A15) is the interpolation of the relative dynamic depths to be used in this work.

Let us point out that a relative autocovariance function C of the field F_r is actually used in our procedure. In fact, the first factor K_V/K_H in the expression for the field F_r is unknown, and the autocovariance function is constructed from the field $(g^2/\alpha_0^2)\partial^2 D/\partial p^2$. This causes no difficulties in our procedure, as the parameters $G_{r,k}$ are determined from data after the expression (A15) is obtained. Of course, for an analysis of errors, the absolute autocovariance function must be used.

9. Appendix B

Open boundaries and numerical procedure

A numerical realization of the procedure of Appendix A, when applied to the Rijeka Bay, is presented here. For the sake of this, the domain $\mathcal{D}$, representing the Rijeka Bay, is modelled by the disk for which the polyhedrons Ω form the basis; the vertical boundary is the surface Γ. The boundary Γ is divided into 3 parts: the coastal parts Γ_1, Γ_2 and the open part Γ_3 through which the Bay communicates with the rest of the sea.

The boundary values of the relative dynamic depths are not known in the present case, in contrast to the supposition (B) of Appendix A. Therefore we have first to determine these boundary values. Let the functions d_r, when restricted to Γ_i, be denoted by d_i, $1 \leqslant i \leqslant 3$. The difference $\delta(p) = d_1(p) - d_2(p)$ causes a current through the Gates having the differential flux, between the depths p and $p + \mathrm{d}p$, equal to $s(p)\mathrm{d}p$. From the data on the velocity field in the Gates, the values of $s(p)$ are estimated and δ is interpolated by a linear function. Then only one of the functions between d_1 and d_2 has to be determined from data. Let $m(p)$ be the arithmetic mean of the data $D_r(\boldsymbol{r}_k, p)$ at all the measuring points $\boldsymbol{r}_k$, $1 \leqslant k \leqslant M$, in the basin. The estimates of d_r are then taken to be

$$d_1(p) = m(p) + 0.5\,\delta(p),$$

$$d_2(p) = m(p) - 0.5\,\delta(p).$$

The essential difference between the models of Appendix A and the problem considered here is the existence of the open boundary Γ_3, at which the density of flux $s(p)$ is defined. This difference can be easily overcome by replacing D_0 of (A16) with a solution of

$$\Lambda D_0 = 0, \qquad D_0|\Gamma_i = d_i, \quad 1 \leqslant i \leqslant 2, \quad \boldsymbol{k}\ \mathrm{curl}(D_0 \boldsymbol{k})|\Gamma_3 = s. \tag{B1}$$

It is supposed here that the third term of the boundary conditions in (B1) admits a unique solution of the system (B1). Having a solution D_0 of (B1), the procedure of Appendix A can be used without any other adaptation.

It remains now to describe our numerical procedure for the solving of the Poisson equations. The polyhedron Ω is divided into squares having hedges equal to 1 km. Let $\mathcal{S}$ be the mesh consisting of square vertices. The mesh $\mathcal{S}$ contains 467 knots. The Poisson equations are solved by using the finite element method with bilinear elements. A solution D_r obtained by this method is continuous, but its partial derivatives have jumps at square hedges, so that the velocity field calculated from D_r is not continuous. Therefore, the velocities at the interior knots of the mesh $\mathcal{S}$ are obtained by averaging over 4 neighbouring squares.

10. Appendix C

Interpolation of vertically-averaged velocities

A procedure for the determination of the vertically-averaged velocity field is presented in this appendix. The procedure is based on the suppositions (A)–(F), and consequently on the second model of Appendix A.

Instead of a set $\mathcal{M}$ of M points for which the relative dynamic depths are given, we have now a set $\mathcal{N}$ of N points where the vertically-averaged velocities are measured.

To avoid superfluous considerations with respect to the first model of Appendix A, we start here from the beginning, with the second model as defined by (A11). After applying the averaging operation to (A11), we have

$$\Lambda\langle D\rangle = R + \langle h_2\rangle,$$

$$\langle D\rangle|\mathscr{B}(0) = \langle d\rangle,$$

where $R = -(K_V/K_H)(g^2/\alpha_0^2)[\partial D/\partial p]_0^p$. The supposition (F) implies $\bar{R} = -\langle \bar{h}_2\rangle$, so that a procedure analogous to the one presented in Appendix A can be used here. Let us emphasize only that a relative autocovariance function corresponding to the field $\partial D/\partial p|_0^p$ is used now. The final expressions have the form:

$$\langle D\rangle = V_0 + \sum_{k=1}^{2N} Q_k V_k, \tag{C1}$$

$$\langle c \rangle = -f^{-1}\,\mathrm{curl}(V_0 \boldsymbol{k}) - f^{-1} \sum_{k=1}^{2N} Q_k\,\mathrm{curl}(V_k \boldsymbol{k}), \tag{C2}$$

where the functions V_0 and V_k are the solutions of

$$\begin{aligned} \Delta V_0 &= 0, & \Delta V_k &= \lambda_k, \\ V_0|\mathscr{B}(0) &= \langle d \rangle, & V_k|\mathscr{B}(0) &= 0; \end{aligned} \tag{C3}$$

the functions λ_k are solutions of the problems (A14), and parameters Q_k, $1 \leqslant k \leqslant 2N$, have to be determined from N data on $\langle \boldsymbol{c}(\boldsymbol{r}_k) \rangle$, $1 \leqslant k \leqslant N$.

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