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Resonant interactions between unstable and neutral baroclinic waves in a continuous model of the atmosphere

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ABSTRACT

Resonant interactions between a finite-amplitude marginally unstable wave and two neutral baroclinic waves are investigated in a quasi-geostrophic, continuous, infinite depth model on the beta-plane channel. Since the neutral waves are characterized by total horizontal wavenumbers less than that of the unstable wave, the kinematic resonant conditions can only be satisfied when the meridional scale of the unstable wave is sufficiently small. Employing asymptotic methods, equations are obtained governing the temporal evolution of the amplitudes and phases of the triad. The neutral waves are scaled smaller in magnitude than the unstable wave as a result of an asymptotic imbalance between the advective non-linearity and instability. Numerical solutions of the amplitude equations reveal a separation between the instability and interaction time scales. The neutral wave amplitudes vacillate on the longer interaction time scale. The unstable wave amplitude vacillates locally on the instability time scale while its envelope vacillates on the interaction time scale in direct proportion to the initial neutral wave amplitudes.

1. Introduction

During the past four decades, considerable effort has been devoted to studying the linear characteristics of a baroclinic zonal current to synoptic scale disturbances (e.g., Charney, 1947; Eady, 1949; Kuo, 1952, 1979). Although these studies have provided important results concerning growth rates, phase speeds, and vertical structures of the disturbances, they are lacking in several fundamental dynamical respects. Foremost among the deficiencies has been the linearization of the governing equations, effectively precluding the inherently non-linear phenomena of wave-mean flow interaction, and wave-wave interaction.

In an effort to understand, through analytical techniques, non-linear behavior in the atmosphere, Drazin (1970) and Pedlosky (1970, 1979b) studied the effects of weak non-linearity on the dynamics of a small, but finite amplitude baroclinic wave and associated wave-mean flow interaction. Drazin studied the problem within the context of the Eady (1949) model while Pedlosky (1970) used a model analogous to the Phillips' (1954) two-layer model. More recently, Pedlosky (1979b) performed a similar analysis within the framework of the Charney (1947) model. The three studies have shown that the weak non-linearity acts to stabilize the otherwise exponentially growing amplitude that would result from the linear theory. The equations describe a long-time vacillation between the wave amplitude and the mean flow.

Several studies have also addressed the problem of resonant wave-wave interaction among neutral waves using asymptotic methods. Longuet-Higgins and Gill (1967) studied resonant triad interactions among barotropic Rossby waves while Bretherton (1964) studied the mathematical properties of resonant interactions using an idealized model for dispersive waves. Loesch (1977) showed that in a channel configuration, resonantly interacting Rossby waves can generate zonal flows on the time scale of the interaction.

Loesch (1974a, b) combined resonant interactions with baroclinic instability by extending the Pedlosky (1970) study by allowing the initial spectrum to be made up of a triad consisting of two neutral baroclinic waves, and a wave which, according to linear theory, is marginally unstable. He found that the neutral waves extract energy indirectly from the mean flow via resonant interaction with the unstable wave. The analysis leading to this conclusion was conducted at the minimum critical shear, a point along the marginal curve where all three waves could be scaled of comparable magnitude. Loesch (1974a, b) also looked at the interaction problem away from the minimum critical shear and concluded that the neutral waves must be rescaled there to be smaller in magnitude than the unstable wave. Based on the difference in scaling, he argued that such interactions would be dynamically less important and, therefore, did not consider their dynamics in detail.

In a recent study, Mansbridge and Smith (1983) carefully examined the dynamics of the interaction considered less important by Loesch. In contrast to Loesch's assessment they concluded that, under certain initial conditions and kinematic constraints, the neutral waves do indeed have a profound effect upon the unstable wave.

Moreover, Pedlosky (1982) re-examined the weakly non-linear theory for the parameter setting anchored at the minimum critical shear required for instability. He showed that only at this point along the marginal curve does the finite amplitude analysis give rise to a forced spectrum of harmonics of the basic wave. It is unclear how these additional waves would effect the solutions given by Loesch (1974a, b).

In view of the distinct differences in behavior between the analysis at the minimum critical shear and away from the minimum, and since the two-layer model is a crude representation of the atmosphere's vertical structure, we extend Pedlosky's (1979b) analysis (for inviscid flow) by allowing the initial spectrum to consist of a triad of waves, two of which are chosen to be neutral while a third, according to linear theory, is marginally unstable. In so doing, several questions will be addressed. (i) In the continuously stratified model with β, where the marginal stability curves have no minimum, what additional constraints, if any, must be placed on the triad in order to satisfy the kinematic resonant conditions? (ii) Can the triad members be scaled of comparable magnitude as in Loesch (1974a)? If not, what are the dynamical or mathematical constraints preventing this scaling? (iii) Is the resonant triad interaction mechanism an efficient way of transfering energy from an unstable wave to neutral waves in the Charney (1947) model? (iv) Conversely, can the neutral waves significantly affect the evolution of the unstable wave?

2. The model

Consider an inviscid atmosphere characterized by a basic state constant density scale height H, and constant Brunt Väisälä frequency N^2, confined to a mid-latitude β-plane channel with rigid vertical walls a distance L apart. The basic flow is zonal and increases linearly with height from the lower rigid horizontal boundary to infinity. For quasi-geostrophic motion the governing equations, from Pedlosky (1979a), are

$$\left(\frac{\partial}{\partial t} + \lambda z \frac{\partial}{\partial x}\right) q + Q_{0y} \frac{\partial \phi}{\partial x} + \varepsilon J(\phi, q) = 0 \tag{2.1}$$

$$\frac{\partial^2 \phi}{\partial z\, \partial t} - \lambda \frac{\partial \phi}{\partial x} + \varepsilon J\left(\phi, \frac{\partial \phi}{\partial z}\right) = 0, \qquad @\, z = 0, \tag{2.2}$$

$$\frac{\partial \phi}{\partial x} = 0, \qquad @\, y = 0, 1, \tag{2.3a}$$

$$\lim_{X \to \infty} \frac{1}{2X} \int_{-X}^{X} \frac{\partial^2 \phi}{\partial y\, \partial t}\, dx' = 0, \tag{2.3b}$$

$$\int_0^{\infty} \rho_s |\phi|^2\, dz < \infty. \tag{2.3c}$$

The scales L, D, U and L/U have been used to non-dimensionalize the horizontal lengths, vertical depth, horizontal velocities and time, respectively. The non-dimensional geostrophic pressure streamfunction has been written as,

$$\Psi(x, y, z, t) = -\lambda z y + \varepsilon \phi(x, y, z, t). \tag{2.4}$$

The term $\phi(x, y, z, t)$ is the perturbation streamfunction which is superimposed upon the basic state, $-\lambda z y$. The constant ε measures the relative magnitude of ϕ to the basic flow and will be determined later. The constant λ measures the vertical shear of the basic state. The perturbation potential vorticity is given as

$$q = \nabla^2 \phi + \frac{1}{S} \frac{\partial^2 \phi}{\partial z^2} - \frac{h^{-1}}{S} \frac{\partial \phi}{\partial z}. \tag{2.5}$$

The operators ∇^2 and J are given by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

$$J(f', g') = \frac{\partial f'}{\partial x}\frac{\partial g'}{\partial y} - \frac{\partial g'}{\partial x}\frac{\partial f'}{\partial y}.$$

The non-dimensional parameters Q_{0y}, S, h and β are:

$Q_{0y} = \beta + \dfrac{\lambda}{sh}$ (meridional gradient of the basis state potential vorticity);

$S = \dfrac{N^2 D^2}{f_0^2 L^2}$ (stratification parameter);

$h = \dfrac{H}{D}$ (density scale height)

$\beta = \dfrac{\beta_* L^2}{U}$ (planetary vorticity factor);

where β_* represents the northward gradient of the Coriolis parameter f, i.e.,

$$2\Omega = f_0 + \beta_* y.$$

3. Linear theory and resonant interactions

3.1. *Linear theory*

For infinitesimal disturbances, the non-linear terms in eqs. (2.1)–(2.2) can be neglected. The problem then reduces to Charney's (1947) model of baroclinic instability and forms the basis for the weakly non-linear theory by providing the growth rates, phase speeds, and vertical structures of the disturbances. The linearization of the governing equations allows for a linear superposition of normal mode solutions, which evolve independently, and whose interaction in the finite-amplitude problem provides a mechanism for the redistribution of energy. The linear problem has already been dealt with in detail (e.g., Kuo, 1952, 1979; Miles, 1964) and, therefore, will only be summarized here to the extent necessary for the non-linear development.

If normal mode solutions of the form,

$$\phi(x, y, z, t) = F(z)\, e^{ik(x-ct)} \sin n\pi y + * \tag{3.1}$$

are chosen, we obtain

$$(\lambda z - c)\left(\frac{1}{S}\frac{d^2 F}{dz^2} - \frac{h^{-1}}{S}\frac{dF}{dz} + K^2 F\right) + Q_{0y} F = 0, \tag{3.2}$$

$$c\frac{dF}{dz} + \lambda F = 0, \quad @\, z = 0, \tag{3.3}$$

where,

$$K^2 = k^2 + n^2\pi^2, \qquad n = 1, 2 \ldots,$$

and the asterisk represents the complex conjugate.

Eq. (3.2) can be transformed into the standard form of the confluent hypergeometric function (Abramowitz and Stegum, 1964) and has the solution

$$F(z) = \left(z - \frac{c}{\lambda}\right) e^{\nu z} [d_1 M(a, 2, \xi) + d_2 U(a, 2, \xi)], \tag{3.4}$$

where M and U are two linearly independent solutions, and d_1 and d_2 are arbitrary constants. The functions M and U are

$$M(a, 2, \xi) = \sum_{m=0}^{\infty} \frac{(a)_m \xi^m}{(2)_m\, m!}, \tag{3.5}$$

$$U(a, 2, \xi) = \frac{1}{\Gamma(a)}\left\{\frac{1}{\xi} + \sum_{m=0}^{\infty} \frac{\Gamma(a+m)}{\Gamma(a-1)\, m!(m+1)!} \times [\text{lin}\, \xi + \Psi(a+m) - \Psi(1+m) - \Psi(2+m)]\, \xi^m\right\}, \tag{3.6}$$

where

$$\xi = \left(z - \frac{c}{\lambda}\right)(h^{-2} + 4SK^2)^{1/2}, \tag{3.7a}$$

$$a = 1 - \frac{(S/\lambda)\, Q_{0y}}{(h^{-2} + 4SK^2)^{1/2}} = 1 - r, \tag{3.7b}$$

$$\nu = \frac{h^{-1}}{2} - \frac{(h^{-2} + 4SK^2)^{1/2}}{2}, \tag{3.7c}$$

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt; \quad \Gamma(n) = (n-1)!$$

for $n = 1, 2, \ldots$ (gamma function) (3.7d)

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x), \quad \text{(digamma function)} \quad (3.7e)$$

$$(b)_m = b(b+1)\ldots(b+m-1), \quad (b)_0 = 1$$

(Pochammers symbol). (3.7f)

The boundary condition at infinity is most easily applied by considering the asymptotic forms of M and U for large z (Abramowitz and Stegun, 1964),

$$M(a, 2, \xi) \sim \frac{\Gamma(2)\, \xi^{a-2} e^{\xi}}{\Gamma(a)}, \quad (3.8a)$$

$$U(a\, 2, \xi) \sim \xi^{-a} \frac{1}{\Gamma(a)}. \quad (3.8a)$$

The validity of each solution depends on the value of r.

For r equal to an integer, M terminates and becomes a polynomial of degree $r-1$. The solution, U, must be redefined (for instance, by the Wronskian method) and is of the form

$$U(a, 2, \xi) = \xi^{a-2} e^{\xi} + M(a, 2, \xi) \ln \xi + \ldots,$$

and therefore leads to an exponential increase for large z, requiring that $d_2 = 0$. For r not equal to an integer, the series for M does not terminate and by eq. (3.8a) leads to an exponential increase in $F(z)$ for large z requiring that $d_1 = 0$. However, U remains bounded allowing the general solution to be written as

$$F(z = \left(z - \frac{c}{\lambda}\right) e^{\nu z}$$

$$\times \begin{cases} d_1 M(a, 2, \xi) & r = 1, 2, 3 \ldots, \quad (3.9a) \\ d_2 U(a, 2, \xi) & r \neq \text{integer}. \quad (3.9b) \end{cases}$$

The phase speeds and growth rates are determined by applying (3.9a, b) to the lower boundary condition (3.3). For integer r, the boundary condition at $z = 0$ becomes

$$\xi_0^2 \left[\sum_{m=1}^{\infty} \frac{(1-r)_m \xi^{m-1}}{(2)_m (m-1)!} + \frac{\nu}{(h^{-2} + 4SK^2)^{1/2}} \times \sum_{m=0}^{\infty} \frac{(1-r)_m \xi^m}{(2)_m m!} \right] = 0, \quad @\, \xi = \xi_0, \quad (3.10)$$

where

$$\xi_0 = -\frac{c}{\lambda}(h^{-2} + 4SK^2)^{1/2}. \quad (3.11)$$

Eq. (3.10) states that $\xi_0 = 0$ ($c = 0$) is always a double root. The expression within the bracket yields polynomial solutions of degree $r-1$. For $r = 2, 3 \ldots$, the solutions are neutral waves characterized by negative phase speeds, i.e., the disturbances move westward relative to the zonal flow at the surface. Therefore, the curves $r =$ integer denote zero growth rate or marginal stability, as illustrated in Fig. 1.

For r not equal to an integer, the second solution (3.9b) is applied to the lower boundary condition. For this case, the phase speed cannot be obtained in closed form and must be determined numerically (Kuo, 1952, 1979). These studies have shown that for $r \neq$ integer, all waves are unstable when the real part of the phase speed (c_r) lies within the range of the basic flow. In addition, there are vertically trapped neutral waves propagating zonally for $r > 1$ and c_r outside of the range of the basic flow.

3.2. *Resonant interactions*

If the initial wave spectrum is chosen to be a triad, then the kinematic resonant conditions that must be satisfied for quadratic resonance to occur are,

$$k_1 \pm k_2 \pm k_3 = 0, \quad (3.12a)$$

$$n_1 \pm n_2 \pm n_3 = 0 \quad (3.12b)$$

$$k_1 c_{1r} \pm k_2 c_{2r} \pm k_3 c_{3r} = 0. \quad (3.12c)$$

In addition, we introduce a quantization constraint on the zonal wavenumbers around a latitude belt, i.e.,

$$k_j = M_j k + \delta' k, \quad (3.13a)$$

$$M_j = 1, 2, 3 \ldots, \quad (3.13b)$$

$$|\delta_j| \leq \delta', \quad (3.13c)$$

where δ' is an upper limit under which (3.12) and (3.13) can be satisfied and still allow for significant interactions (Bretherton, 1964). It should be noted that Mansbridge and Smith (1983) also impose a quantization condition on the zonal wavenumbers in the two-layer model. This is vital in their model to remove the resonance with the non-dispersive

Table 1. *The zonal and meridional wavenumbers satisfying (3.12) and (3.13) and the corresponding non-linear coupling coefficients for* $\lambda_c = 1$, $S = 1$ *and* $h = 1$

β	M_1	M_2	M_3	n_1	n_2	n_3	N_0	N_2	N_3	γ
18.30	−2	−7	9	3	−2	−1	2.65	−0.09	0.04	−11.68
18.81	−3	−5	8	3	−2	−1	5.70	−0.20	0.17	−0.84
19.50	−4	−4	8	3	−2	−1	9.56	−0.25	0.27	−1.78
20.36	−5	−3	8	3	−2	−1	13.87	−0.55	0.10	−0.89

neutral modes. In the Charney (1947) model, however, non-dispersive neutral modes occur only for integral r and will not, in general, satisfy the resonance conditions (3.12). Therefore, it should be emphasized that the sole purpose of (3.13) here is to provide a realistic constraint on the zonal wavenumbers applicable to large-scale flow in the atmosphere. Waves that satisfy (3.12) and (3.13), along with the method of solution, are given in Table 1 and Appendix A, respectively.

Since the marginal stability curves for this model have *no* minimum (see Fig. 1), and represent a discontinuous transition from regions of relatively strong instability to weak instability, with neutral waves occurring only for $r > 1$, an additional constraint will arise in finding three waves that satisfy (3.12).

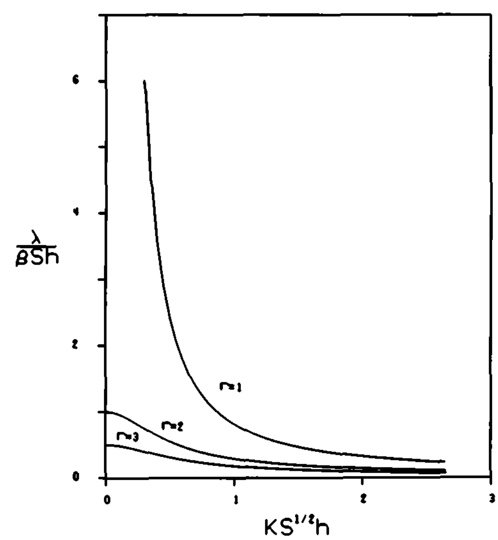

Fig. 1. The first three critical curves on which the linear growth rate vanishes in the Charney (1947) model.

If the unstable wave is to represent a synoptic scale disturbance, we choose the curve $r = 1$ as the point of departure for the weakly non-linear problem. The short-wave side of the curve is characterized by a growth rate of $O(|\Delta|^{1/2})$ and phase speed of $O(|\Delta|)$ (Miles, 1964; Lindzen and Rosenthal, 1981), where $|\Delta| \ll 1$ is a measure of the distance from the curve $r = 1$. These synoptic scale waves are referred to as the Charney modes. In the vicinity of the long-wave side of $r = 1$, the waves have a growth rate of $|\Delta|^{3/2}$ and phase speed of $|\Delta|^{1/2}$.

If the unstable mode is chosen to be a Charney mode with a small but finite growth rate, for an *asymptotic treatment* of the interaction, the other triad members must either be neutral or have smaller growth rates. The latter case does not allow for an analytical approach, since the unstable modes on the short-wave side of $r = 1$ are *non-dispersive* to $O(|\Delta|)$ and, therefore, interact readily as an infinite set, while those on the long-wave side of $r = 1$ introduce a *second* instability time scale. For the non-dispersive unstable short waves, (3.13) does not eliminate all interactions involving them. Due to the mathematical complexity of this interaction problem, these modes are excluded from the initial spectrum. Therefore, the only unique triad amenable to weakly non-linear analysis and which provides the basis for comparison with prior studies conducted in the two-layer model (Loesch, 1974a; Mansbridge and Smith, 1983) is one consisting of a marginally unstable Charney mode and two dispersive neutral modes.

We recall, the dispersive neutral modes lie in the region $r > 1$. Therefore, in this model we have an additional constraint for the triad,

$$k_j^2 + n_j^2 \pi^2 < k_1^2 + n_1^2 \pi^2, \qquad (3.14)$$

where the subscript 1 refers to the Charney mode and $j = 2$ or 3 to the neutral modes. Due to the zonal propagation of the triad members, we expect energy exchanges within the triad to be constrained in accordance with Fjørtofts (1953) theorem. This, together with constraint (3.14), prevents the unstable wave from transferring energy to both neutral waves simultaneously. Moreover, since the wave structure is so heavily weighted by π^2, it is impossible for the unstable wave to have a meridional structure of $n_1 = 1$ and still satisfy conditions (3.12). Calculations have shown that the meridional structure of the unstable wave must be $n_1 \geqslant 3$ in order to satisfy (3.12) and (3.14).

The kinematic constraint imposed by (3.14) *severely limits* the rôle of the resonant interaction mechanism as an efficient way of transferring energy from the unstable wave to the neutral waves in this model. In view of this fact, the remainder of the paper will serve to demonstrate several of the differences and similarities in the weakly non-linear interaction dynamics between the two-layer and continuously stratified baroclinic models on the β-plane.

4. Finite amplitude problem

4.1. *Formulation*

Penetration into the unstable region is accomplished by perturbing λ, i.e.,

$$\lambda = \lambda_c(1 + \Delta), \qquad \Delta \ll 1, \tag{4.1}$$

where λ_c is the value of the shear on the marginal curve $r = 1$. Attention will be focused on the *short* wavelength side of the marginal curve allowing the differential time operator to be transformed as

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + |\Delta|^{1/2} \frac{\partial}{\partial T}, \tag{4.2}$$

where

$$T = |\Delta|^{1/2} t.$$

Furthermore, the parameter ε in eqs. (2.1) and (2.2) is chosen to be $|\Delta|^{1/2}$ and reflects the fact that the amplitude of the marginal wave is governed by mean flow variations whose changes are due to rectified fluxes of order Δ.

Incorporation of eqs. (4.1) and (4.2) into (2.1) allows it to be written as

$$\left[\delta_j \frac{\partial}{\partial t} + |\Delta|^{1/2} \frac{\partial}{\partial T} + \lambda_c(1 + \Delta)\, z \frac{\partial}{\partial x}\right] \times q + \left(Q_{0y_c} + \frac{\lambda_c}{Sh}\Delta\right)\frac{\partial \phi}{\partial x} + |\Delta|^{1/2} J(\phi, q) = 0, \tag{4.3}$$

where

$$\delta_j = \begin{cases} 0 & j = 1 \text{ (marginal wave)}, \\ 1 & j = 2, 3 \text{ (neutral waves)}, \end{cases}$$

serves as a wave "tag".

As discussed by Pedlosky (1979b), (4.3) is spacially non-uniform for the $j = 1$ mode characterized by $c_1 = 0$. For z order one, its evolutionary time scale is long compared to its advective time scale. For z small, the separation of time scales is no longer possible and the solutions cannot satisfy both boundary conditions. This singularity reflects the presence of a critical layer near the surface. To remedy this problem, and to form a uniformly valid asymptotic series, we define a new inner variable valid for small z given by $\zeta = z/|\Delta|^{1/2}$.

For the neutral members of the triad, $j = 2, 3$, the singularity does not arise since the phase speeds of these waves lie outside the range of the zonal flow. The presence of the neutral waves can be handled in two ways, each of which leads to the *same* equations governing the amplitudes of the triad. For small z, the solution for the neutral waves can be obtained by writing the solution to (4.3) in terms of $|\Delta|^{1/2}\zeta$ and expanding for small $|\Delta|$. The alternative is to introduce the inner variable ζ directly into (4.3) and solve at each order of $|\Delta|$. For consistency, and to facilitate comparison with the two-layer model, the second method is chosen.

Substituting $z = |\Delta|^{1/2}\zeta$ into (4.3) results in

$$\frac{\delta_j}{|\Delta|^{1/2}} \frac{\partial}{\partial t}\tilde{q} + \left[\frac{\partial}{\partial T} + \lambda_c(1 + \Delta)\,\zeta \frac{\partial}{\partial x}\right]\tilde{q} + \frac{\partial \tilde{\phi}}{\partial x}\left(Q_{0y_c} + \frac{\lambda_c}{Sh}\Delta\right)|\Delta|^{1/2} + J(\tilde{\phi}, \tilde{q}) = 0. \tag{4.4}$$

The appropriate boundary condition at $\zeta = 0$ is,

$$\frac{\delta_j}{|\Delta|^{1/2}} \frac{\partial^2 \tilde{\phi}}{\partial \zeta\, \partial t} + \frac{\partial^2 \tilde{\phi}}{\partial \zeta\, \partial T} - \lambda_c(1 + \Delta)\frac{\partial \tilde{\phi}}{\partial x}$$

$$+ J\left(\tilde{\phi}, \frac{\partial \tilde{\phi}}{\partial \zeta}\right) = 0, \tag{4.5}$$

where

$$\tilde{q} = |\Delta|\, \nabla^2 \tilde{\phi} + \frac{1}{S}\frac{\partial^2 \tilde{\phi}}{\partial \zeta^2} - |\Delta|^{1/2} \frac{h^{-1}}{S}\frac{\partial \tilde{\phi}}{\partial \zeta},$$

$$\tilde{\phi} = \tilde{\phi}(x, y, \zeta, t, T). \tag{4.6}$$

The weakly non-linear problem is now described by the inner region eqs. (4.4) and (4.5) and the outer region eqs. (4.3) and (2.3c) whose solutions must match at each order of $|\Delta|$. For didactic purposes, these two regions can be thought of as being analogous to the lower and upper layers in the two-layer model. The similarities and differences between the two models for the finite amplitude problem will become apparent in the non-linear development.

4.2. *Non-linear development*

The perturbation field is taken to consist of a marginally unstable wave and two neutral waves. The neutral waves are scaled to be smaller in magnitude than the unstable wave for reasons which will become apparent at $O(|\Delta|^{1/2})$.

Based on the form of (4.3), an asymptotic expansion for the outer region perturbation field is chosen to be

$$\phi(x, y, z, t, T, |\Delta|) = \phi^{(0)} + |\Delta|^{1/4} \phi^{(1)} + |\Delta|^{1/2} \phi^{(2)} + \ldots, \tag{4.7}$$

where $\phi^{(0)}$ represents the marginal wave and $\psi^{(1)}$ the neutral waves. Substituting (4.7) into (4.3) and (2.3) and collecting the O(1) and $O(|\Delta|^{1/4})$ terms reproduces the results of the linear theory, i.e.,

$$\phi^{(0)} = A_1(T)\, z\, e^{-(\beta S/2\lambda_c)z}\, e^{ik_1 x} \sin n_1 \pi y + *, \tag{4.8}$$

$$\phi^{(1)} = \sum_{j=2}^{3} A_j(T)\, F_j(z)\, e^{ik_j(x-c_j t)} \sin n_j \pi y + *, \tag{4.9}$$

where the $F_j(z)$ are given by (3.9b).

As for the outer region, the expansion for the inner region, $\tilde{\phi}$, is chosen to be

$$\tilde{\phi}(x, y, \zeta, t, T, |\Delta|) = \tilde{\phi}^{(0)} + |\Delta|^{1/4} \tilde{\phi}^{(1)} + |\Delta|^{1/2} \tilde{\phi}^{(2)} + \ldots. \tag{4.10}$$

The O(1) and $O(|\Delta|^{1/4})$ problems yield, after matching with the outer region,

$$\tilde{\phi}^{(0)} = 0, \tag{4.11a}$$

$$\tilde{\phi}^{(1)} = \sum_{j=2}^{3} A_j(T)\, F_j(0)\, e^{ik_j(x-c_j t)} \sin n_j \pi y + *. \tag{4.11b}$$

Except for slight differences in notation, the inner solution at $O|\Delta|^{1/2})$ is identical to that obtained by Pedlosky (1979b) and is given as,

$$\tilde{\phi}^{(2)} = \left(\frac{1}{ik_1 \lambda_c}\frac{dA_1}{dT} + A_1 \zeta\right) e^{ik_1 x} \sin n_1 \pi y + * + \mathscr{F}_2(y, T). \tag{4.12}$$

The inner solution at $O(|\Delta|^{3/4})$ is

$$\tilde{\phi}^{(3)} = \sum_{j=2}^{3} \left[\zeta A_1 \frac{dF_j(0)}{dz} + C_j^{(3)}(T)\right] e^{ik_j(x-c_j t)} \times \sin n_j \pi y + *, \tag{4.13}$$

where, again, we have matched terms with the outer region. The $O(|\Delta|^{3/4})$ balance at the lower boundary yields,

$$-c_j \frac{dF_j}{dz} - \lambda_c F_j = 0, \quad @\, z = 0, \tag{4.14}$$

which is identical to (3.3).

Collecting terms of $O|\Delta|^{5/4})$ for the inner region yields,

$$\frac{\partial}{\partial t}\left(\frac{1}{S}\frac{\partial^2 \tilde{\phi}^{(5)}}{\partial \zeta^2}\right) = -Q_{0\zeta}\frac{\partial \tilde{\phi}^{(1)}}{\partial x}, \tag{4.15}$$

$$\frac{\partial^2 \tilde{\phi}^{(5)}}{\partial t\, \partial \zeta} = -\frac{\partial_2 \tilde{\phi}^{(3)}}{\partial T\, \partial \zeta} + \lambda_c \frac{\partial \tilde{\phi}^{(3)}}{\partial x} - J\left(\tilde{\phi}^{(1)}, \frac{\partial \tilde{\phi}^{(2)}}{\partial \zeta}\right) - J\left(\tilde{\phi}^{(2)}, \frac{\partial \tilde{\phi}^{(1)}}{\partial \zeta}\right) \quad @\, \zeta = 0. \tag{4.16}$$

The general solution is,

$$\tilde{\phi}^{(5)} = \sum_{j=2}^{3}\left[\frac{S}{c_j} Q_{0k} A_j F_j(0)\frac{\zeta^2}{2} + \zeta D_j^{(5)}(T) + C_j^{(5)}(T)\right] e^{ik_j(x-c_j t)} \sin n_j \pi y + *. \tag{4.17}$$

Application of (4.16) yields,

$$D_j^{(5)}(T)\, e^{ik_j(x-c_j t)} \sin n_j \pi y = \left[\frac{1}{ik_j c_j}\frac{dA_j}{dT}\frac{dF_j(0)}{dz} - \frac{\lambda_c}{c_j} C_j^{(3)}(T)\right] e^{ik_j(x-c_j t)} \sin n_j \pi y$$

$$+\frac{1}{ik_j c_j}\left\{\frac{i\pi}{2}F_m(0)\,[A_m A_1\,e^{i[(k_m+k_1)x-(\sigma_m+\sigma_1)t]}\times f_{m1}(y)+*+A_m A_n^* e^{i[(k_m-k_1)x-(\sigma_m-\sigma_1)t]}g_{m1}(y)+*]\right\},\quad m\neq 1,j. \tag{4.18}$$

where

$$f_{mn}(y)=(k_m n_n-k_n n_m)\sin\pi(n_m+n_n)y+(k_m n_n+k_n n_m)\sin\pi(n_m-n_n)y.$$
$$g_{mn}(y)=(k_m n_n+k_n n_m)\sin(n_n+n_m)\pi y+[k_m n_n-k_n n_m]\sin(n_m-n_n)\pi y \tag{4.19}$$

The term, $D_j^{(5)}(T)$, will turn out to be the boundary condition for $\partial\bar{\phi}^{(3)}/\partial z|_{z=0}$, required at $O(|\Delta|^{3/4})$, and which leads to the equations governing the time evolution of the neutral wave amplitudes. Both $C_j^{(5)}(T)$ and $C_j^{(3)}(T)$ are unknown and will not be required since neither term will contribute to the final amplitude equations.

Except for slight differences in notation, the order $|\Delta|^{1/2}$ outer region solution is identical to that obtained by Pedlosky (1979b) and is given by

$$\phi^{(2)}=\frac{1}{ik_1\lambda_c}\frac{dA_1}{dT}e^{-(\beta S/2\lambda_c)z}e^{ik_1x}\sin n_1\pi y+*+\Phi^{(2)}(y,T), \tag{4.20}$$

where $\Phi^{(2)}$ is the order $|\Delta|^{1/2}$ correction to the zonal flow which is determined at order $|\Delta|$.

The rationale for the choice of scaling of the neutral waves becomes apparent at this order. If the neutral waves were assumed of the same order as the marginal wave, the order $|\Delta|^{1/2}$ equations would contain additional terms due to the interaction between the neutral waves (as in Loesch (1974a) equations (4.19) and (4.21)). If the resulting equations were then multiplied by the homogeneous adjoint solution, integrated over the domain, then, in view of (3.12), we would obtain

$$\frac{dA_1}{dT}\left[Q_{0yc}\int_0^\infty\frac{e^{-zh^{-1}}F_1^2\,dz}{(\lambda_c z-c_1)^2}-\frac{\lambda_c}{S}\frac{e^{-zh^{-1}}F_1^2}{(\lambda_c z-c)^2}\right]_{z=0}-\frac{\lambda_c^2k_1\pi i}{\beta S}(k_2n_3-k_3n_2)\times\left[\frac{Q_{0yc}S}{\lambda_c}\int_0^\infty\frac{F_1F_2F_3e^{-zh^{-1}}\,dz}{(\lambda_c z-c_1)(\lambda_c z-c_2)(\lambda_c z-c_3)}-\frac{F_2F_3}{c_2c_3}\right]_{z=0}A_2^*A_3^*=0. \tag{4.21}$$

For the marginally unstable mode, the bracket multiplying dA_1/dT must vanish (Pedlosky, 1964). Therefore, the remaining term which arises from the neutral wave interactions must vanish separately. This can occur in several ways: the neutral wavevectors can be parallel in which case $(k_2n_3-k_3n_2)=0$ and interactions are precluded; the terms within the bracket may add up to zero; the amplitudes A_2 and A_3 may require rescaling so the neutral waves do not appear at this order. The aforementioned bracket when evaluated, as shown in Appendix B, reduces to

$$\frac{c_2-c_3}{\lambda_c^2}\frac{1}{(\beta/2+\lambda_c/Sh)}\frac{\beta}{2}. \tag{4.22}$$

This expression is non-zero unless $c_2=c_3$ in which case the wavevectors again become parallel. Therefore, the only condition under which (4.21) is satisfied while still allowing for interactions is to *scale* the neutral waves as in (4.7).

Note that if the triad members were chosen to consist solely of neutral waves O(1) away from the marginal curve, (4.21) would then yield 3 coupled equations describing the long-term evolution of the amplitudes of the triad appropriate for the continuously stratified model. These equations are identical in form to those obtained in prior studies of neutral resonant triads (e.g., Longuet-Higgins and Gill, 1967). If the $j=1$ mode is neutral but chosen to lie sufficiently close to the marginal curve so that the bracket multiplying dA_1/dT is $\ll 1$ and given by, say, ε_1, the equations become,

$$\varepsilon_1\frac{dA_1}{dT}=ib_1A_2^*A_3^*, \tag{4.23a}$$

$$\frac{dA_2}{dT}=ib_2A_1^*A_3^*, \tag{4.23b}$$

$$\frac{dA_3}{dT}=ib_3A_1^*A_2^*, \tag{4.23c}$$

where the constants b_i are O(1). Introducing into (4.23) the O(1) time scale τ' such that $T=\varepsilon_1\tau'$ and retaining only O(1) terms can be shown to lead to a secular growth of A_1 on the time scale τ'. A consistent balance can only be achieved again by *rescaling* A_2^* and A_3^*. When A_1 lies on the marginal curve ($\varepsilon_1=0$), a rescaling of A_2^* and A_3^* causes A_1 to act as a catalyst while A_2^* and A_3^* exchange energy between themselves.

For the present case, where A_1 represents the unstable wave, the rescaling prohibits the interacting neutral waves from directly affecting the vertical phase shift of the unstable wave. *This restriction results from the inability of the weakly non-linear theory to combine, at the same order, the linear requirement for instability with the non-linear mechanism of wave interactions.*

This result differs from that obtained by Loesch (1974a) at the minimum critical shear where all three waves are scaled of comparable magnitude. A question now arises: why does the two-layer model allow for comparable scaling between the three waves only at the minimum critical shear? This question has been answered in part by Pedlosky (1982) where it was determined that the analysis in the two-layer model, at the minimum critical shear, gives rise to a forced spectrum of harmonics of the basic wave. It is only at this parameter setting where *both* the Doppler shifted frequency and basic state potential vorticity gradient vanish in the lower layer, rendering *all* forcing resonant. At all other points along the marginal curve, the singularity does not arise and the resonant terms are removed by standard techniques.

The effect that the vanishing of the linear operator has on the interaction problem can be viewed by examining the two-layer model equivalent of (4.21), i.e.,

$$\left[\frac{(U_2-c_j)P_j}{U_c+(U_2-c_j)}+\gamma_j Q_j\right]\frac{\mathrm{d}A_j}{\mathrm{d}T} - \frac{i\pi}{2}(\alpha_k m_l-\alpha_l m_k)\left[\frac{(U_2-c_j)(P_k-P_l)}{U_2+U_c-c_j} + \gamma_j(\gamma_l Q_k-\gamma_k Q_l)\right]A_k^* A_l^* = 0. \tag{4.24}$$

For details concerning the development of this equation, the reader is referred to Loesch (1974a). Briefly, the terms Q_j and P_j are the perturbation potential vorticities in the lower and upper layers, respectively, and the γ_j describe the vertical structure of the modes. The zonal speed in the lower layer is U_2. At the minimum critical shear required for instability, the marginal wave is characterized by $c_1 = U_2$ and $Q_1 = 0$. Therefore, the necessary conditions for instability are trivially satisfied since each term in (4.24) is identically zero. If the analysis is conducted *away* from the minimum, the first bracket still vanishes but now from the calcellation of the two terms thus paralleling the balance between the surface temperature gradients and basic state potential vorticity gradient that occurs in the stratified fluid.

At $O(|\Delta|^{3/4})$, the outer region governing equation is

$$\left(\frac{\partial}{\partial t}+\lambda_c z\frac{\partial}{\partial x}\right)q^{(3)} + Q_{0y_c}\frac{\partial\phi^{(3)}}{\partial x} = \frac{Q_{0y_c}}{\lambda z-c_j}F_j \times e^{ik_j(x-c_jt)}\sin n_j\pi y\frac{\mathrm{d}A_j}{\mathrm{d}T}+* + \frac{i\pi}{2}Q_{0y_c}\left\{\frac{F_mF_n(c_n-c_m)}{(\lambda_c z-c_m)(\lambda_c z-c_n)}A_mA_n \times e^{i[(k_m+k_n)x-(\sigma_m+\sigma_n)t]}f_{mn}(y)+* + \frac{F_mF_n(c_n-c_m)}{(\lambda_c z-c_m)(\lambda_c z-c_n)}A_mA_n^* \times e^{i[(k_m-k_n)x-(\sigma_m-\sigma_n)t]}g_{mn}(y)+*\right\}. \tag{4.25}$$

The indices are cycled as $j = 2$, $m = 3$, $n = 1$ and $j = 3$, $m = 1$, $n = 2$.

The equations for A_j ($j = 2, 3$) are determined by multiplying (4.25) by the homogeneous adjoint solution, denoted by $\psi_i^{(3)}$, applying (3.12), and integrating over the domain and time scale τ, viz.,

$$2Sik_j\lim_{X,J\to\infty}\frac{1}{2X\tau}\int_0^1\int_{-X}^X\int_0^\tau e^{ik_j(x-c_jt)}\sin n_j\pi y \times\frac{1}{S}\left[\frac{\mathrm{d}F_i}{\mathrm{d}z}\phi_i^{(3)}-F_i\frac{\partial\phi_i^{(3)}}{\partial z}\right]_{z=0}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau = \lim_{X,J\to\infty}\frac{1}{2X\tau}\int_0^1\int_{-X}^X\int_0^J\int_0^\tau\phi_i^{(3)} \times|\text{inhomogeneous terms}|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}\tau, \tag{4.26}$$

where

$$\phi_i^{(3)} = \frac{SF_j(z)}{(\lambda_c z-c_j)}e^{-ik_j(x-c_jt)}\sin n_j\pi y\, e^{-zh^{-1}}. \tag{4.27}$$

The terms $\phi_i^{(3)}|_{z=0}$ and $\partial\phi_i^{(3)}/\partial z|_{z=0}$ are evaluated by matching with the $O(|\Delta|^{3/4})$ and $O(|\Delta|^{5/4})$ inner region solutions. Hence,

$$\phi_i^{(3)}|_{z=0} - C_j^{(3)}(T),$$

$$\left.\frac{\partial\phi_i^{(3)}}{\partial z}\right|_{z=0} = D_i^{(5)}(T).$$

Therefore, after much algebra the result is,

$$\frac{dA_i}{dT} = iN_i A_m^* A_n^*, \tag{4.28}$$

where

$$N_i = \frac{\pi}{2}(c_m - c_n)\frac{U_m}{U_n}\bigg|_{z=0} \frac{\times\left[1 + \frac{Q_{0y_c}}{\lambda_c}\frac{1}{(\beta S/2\lambda_c + h^{-1})}\right](k_m n_n - k_n n_m).}{(hSQ_{0y_c} - \lambda_c)} \tag{4.29}$$

Having obtained the amplitude equations for the neutral waves, we focus our attention on the unstable wave.

The $O(|\Delta|)$ governing equation for the inner region is,

$$\left(\frac{\partial}{\partial T} + \lambda_c \zeta \frac{\partial}{\partial x}\right)\left[\frac{1}{S}\frac{\partial^2 \tilde{\phi}^{(4)}}{\partial \zeta^2} - \frac{h^{-1}}{S}\frac{\partial \tilde{\phi}^{(2)}}{\partial \zeta}\right] + Q_{0y_c}\frac{\partial \tilde{\phi}^{(2)}}{\partial \zeta} = 0, \tag{4.30}$$

with the boundary condition

$$\frac{\partial^2 \tilde{\phi}^{(4)}}{\partial \zeta \partial T} - \lambda_c \frac{\partial \tilde{\phi}^{(4)}}{\partial x} = -J\left(\tilde{\phi}^{(2)}, \frac{\partial \tilde{\phi}^{(2)}}{\partial \zeta}\right) - J\left(\tilde{\phi}^{(1)}, \frac{\partial \tilde{\phi}^{(3)}}{\partial \zeta}\right) \quad @\ \zeta = 0. \tag{4.31}$$

The solution to (4.30) after using (4.31) is

$$\begin{aligned}\tilde{\phi}^{(4)} = &\left[\frac{-\zeta^2}{2}\frac{\beta S}{\lambda_c}A_1 + \zeta D^{(4)}(T) + \frac{1}{ik_1\lambda_c}\frac{\partial D^{(4)}}{\partial T} - \frac{1}{\lambda_c}\frac{\partial \mathcal{F}_2}{\partial y}A_1\right] e^{ik_1 x}\sin n_1 \pi y + \frac{\pi}{2}\frac{1}{k_1\lambda_c}\\ &\times\left[F_m\frac{dF_n}{dz} - F_n\frac{dF_m}{dz}\right]_{z=0}\{A_m A_n\\ &\times e^{i[(k_m+k_n)x-(\sigma_m+\sigma_n)t]} f_{mn}(y) + * + A_m A_n^*\\ &\times e^{i[(k_m-k_n)x-(\sigma_m-\sigma_n)t]} g_{mn}(y) + *\},\end{aligned} \tag{4.32}$$

where $\sigma_j = k_j c_j$. Removal of the x-independent term in (4.31) provides the boundary condition for the correction to the zonal flow and is given by

$$\frac{\partial \chi^{(4)}}{\partial T} = \frac{-n_1\pi}{\lambda_c}\sin 2n_1\pi y\frac{d|A_1|^2}{dT}. \tag{4.33}$$

The remaining unknown functions are determined by matching and are

$$J_2(y, T) = \Phi^{(2)}(y, 0, T) \tag{4.34a}$$

$$\chi^{(4)}(y, T) = \frac{\partial \Phi^{(2)}}{\partial z}(y, 0, T), \tag{4.34b}$$

$$D^{(4)}(T) = \frac{-\beta S}{2\lambda_c^2}\frac{1}{ik_1}\frac{dA_1}{dT}. \tag{4.34c}$$

The boundary condition for $\phi^{(4)}$ is

$$\begin{aligned}\phi^{(4)}(x, y, 0, t, T) = &\left[\frac{\beta S}{2\lambda_c^3}\frac{1}{k_1^2}\frac{d^2A_1}{dT^2} - \frac{1}{\lambda_c}\frac{\partial \Phi^{(2)}}{\partial y}(y, 0, T)A_1\right] e^{ik_1 x}\sin n_1\pi y + *\\ &+ \frac{\pi}{2}\frac{1}{k_1\lambda_c}\left[F_m\frac{dF_n}{dz} - F_n\frac{dF_m}{dz}\right]_{z=0}\{A_m A_n\\ &\times e^{i[(k_m+k_n)x-(\sigma_m+\sigma_n)t]} f_{mn}(y) + * + A_m A_n^*\\ &\times e^{i[(k_m-k_n)x-(\sigma_m-\sigma_n)t]} g_{mn}(y) + *\}.\end{aligned} \tag{4.35}$$

The $O(|\Delta|)$ outer region equation is

$$\begin{aligned}\lambda_c z\frac{\partial}{\partial x}q^{(4)} + \frac{\partial \phi^{(4)}}{\partial x}Q_{0y_c} = &\frac{\Delta}{|\Delta|}\beta i k_1 z\, e^{-(\beta S/2\lambda_c)z}A_1\\ &\times e^{ik_1 x}\sin n_1\pi y + * - ik_1 z A_1 e^{-(\beta S/2\lambda_c)z} e^{ik_1 x}\\ &\times \sin n_1\pi y\frac{\partial M_2}{\partial y} + * - A_1\frac{Q_{0y_c}}{\lambda_c}ik_1\frac{\partial \Phi^{(2)}}{\partial y}\\ &\times e^{-(\beta S/2\lambda_c)z}e^{ik_1 x}\sin n_1\pi y + *\\ &+ \frac{i}{2}\frac{Q_{0y_c}F_m F_n(c_n - c_m)}{(\lambda_c z - c_n)(\lambda_c z - c_m)}\{A_m A_n\\ &\times e^{i[(k_m+k_n)x-(\sigma_m+\sigma_n)t]} f_{mn}(y) + * + A_m A_n^*\\ &\times e^{i[(k_m-k_n)x-(\sigma_m-\sigma_n)t]} g_{mn}(y) + *\} - \frac{\partial M_2}{\partial T} + \frac{Q_{0y_c}}{\lambda_c^2}\\ &\times n_1\pi\sin 2n_1\pi y\frac{d|A_1|^2}{dT}e^{-(\beta S/\lambda_c)z},\end{aligned} \tag{4.36}$$

where

$$M_2(y, z, T) = \frac{\partial^2 \Phi^{(2)}}{\partial y^2} + \frac{1}{S}\frac{\partial^2 \Phi^{(2)}}{\partial z^2} - \frac{h^{-1}}{S}\frac{\partial \Phi^{(2)}}{\partial z}. \tag{4.37}$$

The last two terms in (4.36) must be removed in order to prevent $\psi^{(4)}$ from growing linearly in x. This requirement together with (4.33) results in the equations governing the correction to the zonal flow. These are,

$$-\frac{\partial}{\partial T} M_2 = -Q_{0y_c}\frac{n_1\pi}{\lambda_c^2}\sin 2n_1\pi y\,\frac{\mathrm{d}|A_1|^2}{\mathrm{d}T}\,\mathrm{e}^{-(\beta S/\lambda_c)z}, \tag{4.38a}$$

$$\frac{1}{\lambda_c}\frac{\partial}{\partial T}\frac{\partial\Phi^{(2)}}{\partial z} = -\frac{n_1\pi}{\lambda_c^2}\sin 2n_1\pi y \times \frac{\mathrm{d}|A_1|^2}{\mathrm{d}T} \qquad @\, z=0, \tag{4.38b}$$

$$\frac{\partial^2\Phi^{(2)}}{\partial y\,\partial T} = 0 \qquad @\, y=0,1. \tag{4.38c}$$

According to (4.38a), the zonal flow changes are governed solely by the rectified portion of the self-interaction of the unstable wave. If the $O(|\Delta|^{1/2})$ correction to the zonal current is represented as $\mu_1 = -\partial\Phi^{(2)}/\partial y$, then (4.38) yields

$$\frac{\partial}{\partial T}\int_0^\infty \mathrm{e}^{-zh^{-1}}\mu_1\,\mathrm{d}z = 0, \tag{4.39}$$

indicating that each latitude the total zonal momentum is conserved. The neutral waves can only effect the zonal flow indirectly via interaction with the unstable wave. This is in marked contrast to Loesch (1974b) in that the equations governing the zonal flow correction contain additional terms arising from the rectified portion of the neutral wave interactions which, through rectified Reynold's stresses, are responsible for the latitudinal redistribution of zonal momentum.

The first three terms on the right-hand side of (4.36) are identical to those obtained in the single wave theory and have been discussed by Pedlosky (1979b). The fourth term is a wave Reynold's stress produced by the interaction between the neutral waves.

Removal of terms in (4.36) having a projection on the linear operator results in the equation governing the complex amplitude of the unstable wave, viz.,

$$\frac{\mathrm{d}^2A_1}{\mathrm{d}T^2} - \frac{\Delta}{|\Delta|}\sigma^2 A_1 + A_1|N_0(|A_1|^2 - |A_1(0)|^2)| - \gamma A_2^* A_3^* = 0, \tag{4.40}$$

where

$$\sigma = \frac{\sqrt{2}\,\lambda_c^2 k_1}{SQ_{0y_c}}, \tag{4.41a}$$

$$N_0 = \frac{2k_1^2\lambda_c^2}{\beta S}\left\{\frac{-Q_{0y_c}n_1^2\pi^2}{S(\beta+Q_{0y_c})^2} + \frac{64\pi^4 n_1^4 Q_{0y_c}}{\lambda_c^2} \times \sum_j{}' \frac{\dfrac{\beta}{\beta+Q_{0y_c}} - \dfrac{j^2\lambda_c}{Q_{0y_c}|(S/\lambda_c)Q_{0y_c} - \alpha_{2j}|}}{(j^2-4n_1^2\pi^2)\left(\dfrac{\beta Q_{0y_c}S}{\lambda_c^2} - j^2\right)}\right\} \tag{4.41b}$$

$$\alpha_{2j} = \frac{1}{2h} - \left(\frac{h^{-2}}{4} + j^2 S\right)^{1/2}, \tag{4.41c}$$

$$\gamma = \frac{\pi\lambda_c k_1}{S}\,\frac{(k_2n_3 - n_2k_3)(c_3-c_2)}{\beta/2 + \lambda_c/Sh}\,|U_2U_3|_{z=0} \tag{4.41d}$$

Eq. (4.40), together with (4.28), governs the evolution of the complex amplitudes of the triad.

5. Results

Writing the complex wave amplitudes, A_j, in terms of their moduli and phases, i.e..

$$A_j = R_j\mathrm{e}^{i\theta_j}$$

we have

$$\frac{\mathrm{d}^2R_1}{\mathrm{d}T^2} - R_1\left(\frac{\mathrm{d}\theta_1}{\mathrm{d}T}\right)^2 - \frac{\Delta}{|\Delta|}\sigma^2R_1 + N_0R_1[R_1^2 - R_1^2(0)] - \gamma R_2R_3\cos(\theta_1+\theta_2+\theta_3) = 0, \tag{5.1a}$$

$$R_1\frac{\mathrm{d}^2\theta_1}{\mathrm{d}T^2} + 2\frac{\mathrm{d}\theta_1}{\mathrm{d}T}\frac{\mathrm{d}R_1}{\mathrm{d}T} + \gamma R_2R_3\sin(\theta_1+\theta_2+\theta_3) = 0, \tag{5.1b}$$

$$\frac{\mathrm{d}R_2}{\mathrm{d}T} = N_2R_3R_1\sin(\theta_1+\theta_2+\theta_3), \tag{5.1c}$$

$$\frac{\mathrm{d}\theta_2}{\mathrm{d}T} = N_2\frac{R_3R_1}{R_2}\cos(\theta_1+\theta_2+\theta_3), \tag{5.1d}$$

$$\frac{\mathrm{d}R_3}{\mathrm{d}T} = N_3R_1R_2\sin(\theta_1+\theta_2+\theta_3), \tag{5.1e}$$

$$\frac{\mathrm{d}\theta_3}{\mathrm{d}T} = N_3\frac{R_1R_2}{R_3}\cos(\theta_1+\theta_2+\theta_3). \tag{5.1f}$$

These equations are identical in form to those obtained by Mansbridge and Smith (1983); the differences arise in the coupling coefficients. As shown in their study, (5.1) have first integrals that yield,

$$\frac{-2}{\gamma}\left[R_1^2\frac{d\theta_1}{dT}-\left(R_1^2\frac{d\theta_1}{dT}\right)_{T=0}\right]=\frac{1}{N_2}|R_2^2-R_2^2(0)|$$
$$=\frac{1}{N_3}|R_3^2-R_3^2(0)| \quad (5.2a)$$

$$|R_2^2-R_2^2(0)|=|R_3^2-R_3^2(0)|\frac{N_2}{N_3}. \quad (5.2b)$$

For the triads considered, $N_2/N_3 < 0$, which follows from constraint (3.13). This, in conjunction with (5.2b) requires that the amplitude of one neutral wave increase (decrease) as the other one decreases (increases), thereby severely limiting the amount of energy that the neutral waves can extract from the unstable wave. Therefore, the discussion of the results that follow can only be compared with the "no energy cascade" case of Mansbridge and Smith (1983).

The evolution of the real amplitudes, R_j, and phases, θ_j, of the triad are described by (5.1) and have been solved numerically using the Runge Kutta-Verner 5th and 6th order method (Hull et al., 1976). For all of the integrations performed, we have imposed two requirements: (i) the initial vertical tilt of the unstable wave corresponds to that obtained from linear theory, i.e. $(1/R_1)\,dR_1/dT|_{T=0} = \sigma_1$; (ii) the initial energy contained in the perturbation field be less than 25% of that in the basic state.

In the absence of neutral waves, the unstable wave is exemplified by an evolution given in Fig. 2. The growth of R_1 beyond its initial value is severely limited by the meridional requirement $n_1 \geqslant 3$ since the smaller exponential growth rates associated with such meridional structures are more quickly overcome by the non-linear stabilization. For the triads listed in Table 1, the non-linear coupling coefficient, N_0, is approximately two orders of magnitude larger than the linear growth rate σ.

A typical evolution of the amplitudes of the triad, illustrating the effect of the neutral modes on the unstable Charney mode is given in Fig. 3. In this case, the initial conditions on the Charney mode are as in Fig. 2 and $R_2(0) = R_3(0) = R_1(0)$. Extrema in R_2 and R_3 occur when $\theta_1 + \theta_2 + \theta_3 = \pm N\pi$ (eqs. (5.1c) and (5.1e)) and correspond to extrema in the envelope of R_1. In contrast to Fig. 2, the vacillation of the Charney mode is now characterized by two time scales: a shorter local time scale associated with the instability (as in Fig. 2) and a longer time scale associated with the interaction. The neutral waves, like the envelope of R_1, vacillate *only* on the longer interaction time scale. If $R_2(0) > R_1$ and/or $R_3(0) > R_1(0)$ the

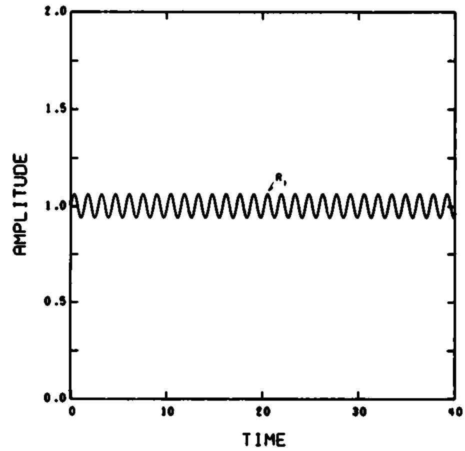

Fig. 2. The time evolution of R_1, in the absence of interactions, for the initial conditions: $R_1(0) = 1$, $dR_1/dT|_{T=0} = 0.276$, $\theta_1(0) = 0$, $d\theta_1/dT|_{T=0} = 0$, and parameter values: $\beta = 19.5$, $M_1 = -4$, $M_2 = -4$, $M_3 = 8$, $n_1 = 3$, $n_2 = -2$ and $n_3 = -1$.

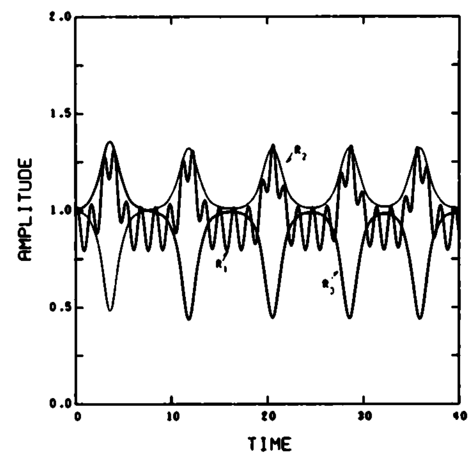

Fig. 3. The time evolution of the amplitudes, R_j. The model parameters and initial conditions for R_1 and θ_1 are the same as in Fig. 2 with $R_2(0) = 1$, $R_3(0) = 1$, $\theta_2(0) = 0.023$ and $\theta_3(0) = 0.047$.

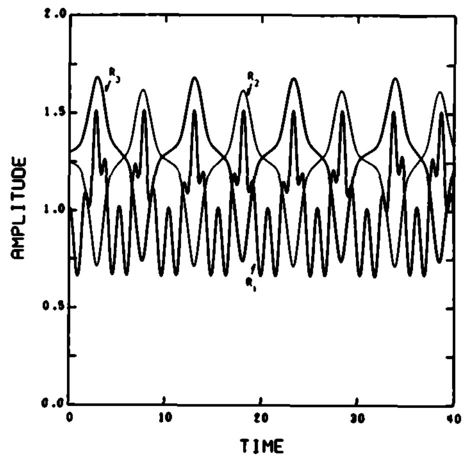

Fig. 4. As in Fig. 3 except that $R_2(0) = 1.25$ and $R_3(0) = 1.3$.

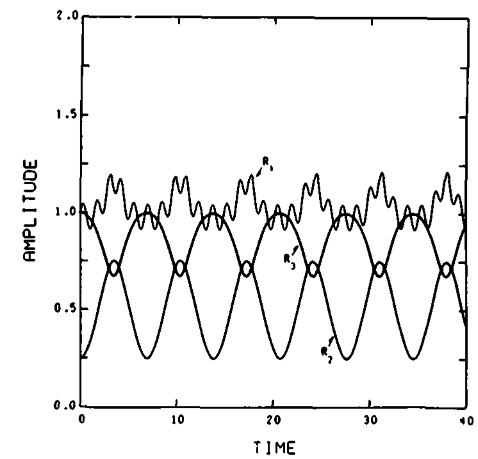

Fig. 6. As in Fig. 3 except that $R_2(0) = 0.25$ and $R_3(0) = 1$.

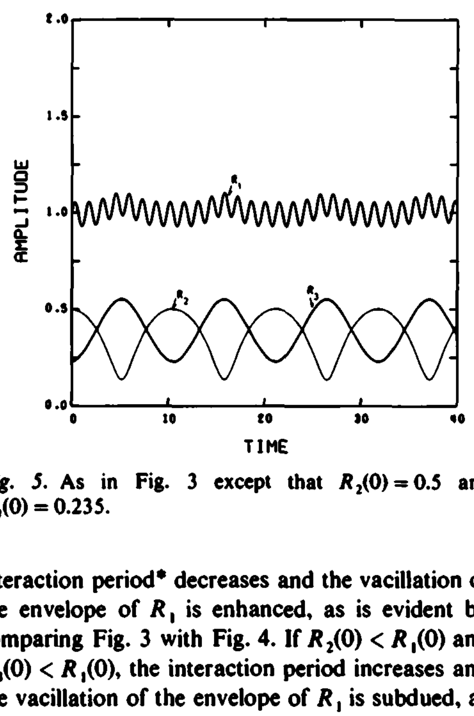

Fig. 5. As in Fig. 3 except that $R_2(0) = 0.5$ and $R_3(0) = 0.235$.

interaction period* decreases and the vacillation of the envelope of R_1 is enhanced, as is evident by comparing Fig. 3 with Fig. 4. If $R_2(0) < R_1(0)$ and $R_3(0) < R_1(0)$, the interaction period increases and the vacillation of the envelope of R_1 is subdued, as is evident by comparing Fig. 3 with Fig. 5. The evolution of R_1 resembles, for a longer part of the interaction period, that obtained in the non-interaction case. However, as shown in Fig. 6, if $R_2(0) < R_1(0)$ and $R_3(0) = R_1(0)$ or $R_2(0) = R_1(0)$ and $R_3(0) < R_1(0)$, the vacillation of the envelope of R_1 is reduced (as in Fig. 5) but the interaction period decreases (as in Fig. 4).

Of the three general aspects of the interaction behaviour found by Mansbridge and Smith (1983) in the two-layer model, only *one* emerges in this study. They find that the period of vacillation of a neutral wave is greatest (least) when the mean value of the unstable wave amplitude, R_1, over that period is least (greatest). This is evident here although in many cases is not nearly as pronounced. They also find that the difference between the values of successive local maxima and minima of the neutral waves is greatest (least) when the rate of change of R_1 is least (greatest). Our integrations do not produce any such systematic relationship. Finally, they show that the ratio of the number of local maxima of the neutral waves to the unstable wave is, for a given triad, independent of the initial conditions. Our solutions are initial condition dependent and our ratios are essentially reversed. For example, in Figs. 3 and 4, the ratios are approximately 1:6 and 1:7, respectively. In Mansbridge and Smith (1983), in their "no energy cascade" case, Figs. 3 and 5 show ratios of approximately 4:1 and 3:1, respectively. The differences can only be accounted for by the difference in triads (especially due to the meridional constraint $n_1 > 3$), parameters, and vertical structure between the two models.

* A careful inspection of Figs. 3–6 reveals that the vacillations, on both time scales, are quasi-periodic. The relative deviations from periodic behavior are generally small but increase with $R_2(0)$ and $R_3(0)$.

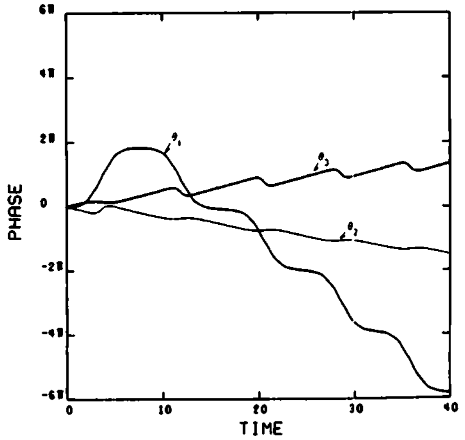

Fig. 7. The time evolution of the phases, θ_j, for the same initial conditions and model parameters as in Fig. 3.

The Charney mode has a minor effect on the evolution of the neutral modes. As can be seen from the figures, the instability period of the Charney mode goes "unnoticed" by the neutral modes. Since this more local vacillation is due to the wave-mean flow interaction, it is further evidence that the mean flow is unable to transfer significant energy to the neutral modes via the Charney mode. However, the presence of the unstable wave is vital in determining the periods of vacillation of the neutral modes. The Charney mode *acts primarily as a catalyst* for the energy exchange between the neutral modes.

In Fig. 7, the evolution of the phases, θ_j, for the same initial conditions as in Fig. 3, is depicted. Since $\theta_1(0)$ and $d\theta_1/dT|_{T=0}$ are zero, θ_1 will remain zero in the absence of interactions. However, as Fig. 7 illustrates, the evolution of θ_1 is drastically changed by the resonant interaction with the neutral waves. This is valid for all the triads in Table 1 under a wide range of initial conditions. As found by Mansbridge and Smith (1983), a generalization of the behavior of the θ_j, similar to that given for the amplitudes R_j, is also not possible in the context of the Charney (1947) model.

6. Summary and discussion

The preceding analysis, in the context of the Charney (1947) model, has dealt with the resonant interactions between a marginally unstable mode and two dispersive neutral modes. The equations governing the amplitudes and phases of the triad members are identical in structure to those obtained by Mansbridge and Smith (1983) whose analysis in the two-layer model was conducted away from the minimum critical shear. The structure of the equations, however, differs markedly from that obtained by Loesch (1974a) in the two-layer model. Loesch pivoted the analysis about the minimum critical shear, a parameter setting which allows for strong wave-wave and triad-mean flow interactions due to the comparable scaling between all three members of the triad. It has been shown that this scaling is prohibited in the Charney model, requiring the neutral modes to be scaled so as to affect the Charney mode at higher order. This scaling parallels that required in the two-layer model at points along the marginal curve away from the minimum. Therefore, the neutral waves are scaled as in Mansbridge and Smith. This scaling requirement arises from the inability of the weakly non-linear theory to combine, at the same order, the linear requirement for instability with the non-linear term arising from the interaction between the neutral modes. By delegating the neutral modes to a higher order than the Charney mode, the evolution of the mean field correction is identical in form to that obtained in the single wave theory and, therefore, precludes any latitudinal redistribution of zonal momentum. The neutral modes are indirectly linked to the zonal flow correction through an additional term that arises in the equation for the Charney mode.

A major difference between the Charney (1947) model and two-layer model analysis is in the wavenumbers that can satisfy the kinematic resonant conditions. It has been shown that in the Charney model, where the neutral modes have phase speeds c_{2_r} and c_{3_r} less than zero and horizontal wavenumbers which must satisfy $K_j^2 < K_1^2$, the meridional structure of the interacting Charney mode must be $n_1 > 3$. These constraints *prevent* the simultaneous growth of both neutral modes, thereby severely limiting the amount of energy transferred from the Charney mode to the neutral modes. The limited energy transfers which occur within a triad correspond, in the two-layer model, to the "no energy cascade" case of Mansbridge and Smith (1983). Despite limited triad energetics, the unstable mode is essential in determining the vacillation period of the neutral modes, while the neutral modes, even though small in amplitude, act to enhance the vacillation of the unstable mode. In general, the time scale associated with the instability is shorter than that associated with the interaction. The neutral mode amplitudes vacillate only with the longer interaction period. The unstable mode amplitude vacillates locally with the shorter instability period while its envelope, vacillates with the longer interaction period. The strength of the envelope vacillation increases in proportion to the initial amplitudes of the neutral modes. In contrast to Mansbridge and Smith (1983), the ratio of the number of local maxima of the unstable amplitudes to that of a neutral amplitude is, for a given triad, initial condition dependent and, essentially, reversed.

Application of the foregoing results to the atmosphere is not readily apparent since the scaling requirement for the neutral modes is an artifact of the weakly non-linear theory. Furthermore, for the triads considered, the kinematic resonance conditions impose meridional constraints which restrict the meridional scale of the unstable wave and yield zonal scales of neutral waves comparable with other unstable waves. Although triads involving more than one unstable wave have been excluded from the present study, their inclusion could easily overwhelm and/or alter the dynamics of the type of triad considered. This problem poses serious difficulties and is left for future consideration.

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8. Appendix A

The solution of the kinematic resonant conditions (3.12), subject to the constraint (3.13), proceeds in two steps. First, the phase speeds are determined numerically using Muller's method (Arfken, 1978). The dispersion curve plots are then overlayed such that the origin of one plot is symmetrically displaced so as to coincide with a point on the other dispersion curve. The intersection of the two curves will be the point where conditions (3.12a, b) are met provided (3.12c) is satisfied. The parameter δ' in (3.13) is chosen to be 0.25. Clearly, the number of triads satisfying (3.12) and (3.13) decrease as δ' decreases. Although the choice of δ' is arbitrary, the interaction dynamics for a given triad is qualitatively similar to that occurring for other triads, obviating the need to consider a wide range of model parameters in calculating wavenumbers that satisfy (3.12) and (3.13). Wavenumbers representative of those satisfying (3.12) and (3.13) are given in Table 1.

9. Appendix B

Using (3.3) and (3.9a) allows the bracket multiplying $A_2^* A_3^*$ in (4.21) to be written as,

$$\frac{c_3 - c_2}{\lambda_c^2}\left[\frac{Q_{0y_c} S}{\lambda_c}\int_0^\infty f_{23}(z)\, e^{-Ez}\, dz - f_{23}(0)\right], \quad \text{(B.1)}$$

where

$$f_{23}(z) = U_2 U_3, \quad \text{(B.2)}$$

$$E = \frac{\beta S}{2\lambda_c} + h^{-1} + \nu_2 + \nu_3. \quad \text{(B.3)}$$

Substituting (3.9a) into (3.3) yields the relation

$$\frac{df_{23}}{dz} = -(\nu_2 + \nu_3)\, U_2 U_3 \qquad @\, z = 0. \quad \text{(B.4)}$$

The integral in (B.1) can now be evaluated by repeated integration by parts and using (B.4) to form a geometric series with the sum

$$\frac{1}{1 + (\nu_2 + \nu_3)/E} \quad \text{(B.5)}$$

After some simplification eq. (B.1) becomes

$$\frac{c_3 - c_2}{\lambda_c^2}\frac{1}{(\beta/2 + \lambda_c/Sh)}\frac{\beta}{2}$$

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