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A note about unsteady currents over a continental shelf

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ABSTRACT

We consider the response of a suddenly imposed wind-stress at the surface of an ocean in a shelf region where the depth vanishes at the coast. When the flow is independent of the co-ordinate along a straight coast-line, a solution is presented in a closed form which reveals a non-uniform space-time behaviour. A case study is presented and comparisons are made with the corresponding numerical solution given by Johnson and Manja.

1. Introduction

The problem of unsteady, wind-driven flow in a coastal region where the depth vanishes at the coast-line has attracted some attention in recent years. Studies have been made by Birchfield and Lunde (1978), and by Johnson and Manja (1980). In general a solution cannot be given in closed form, mainly due to the existence of propagating, damped low-frequency waves. In the present paper we draw attention to a case where a closed form of the solution can be obtained, and thus give some insight into the non-uniform space-time behaviour of the response of the ocean in the shelf region, a phenomenon which was reported by Birchfield and Lunde (1978). In order to obtain a closed form of the solution, we have to restrict the analysis to the situation of a uniform, but time-dependent wind-stress directed along a straight coast-line. We take as our starting point the governing equations derived by Johnson and Manja (1980) on a β-plane, but only the f-plane approximation will be considered. Moreover, it will be assumed that the flow is independent of the co-ordinate along the coast. It is appropriate to note that the governing equations are also given by Birchfield and Lunde (1978), the only difference being the definition of the bottom friction or bottom slip parameter. In order to solve the very general initial value problem, a Laplace transform technique was used by Birchfield and Lunde (1978), while Johnson and Manja (1980) solved the problem numerically. In order to study the various modes of oscillations (shelf waves, Rossby waves and Kelvin waves), the former method has some advantages, although numerical methods inevitably become a part of the whole problem. For this reason we decided to look for a solution in a closed and hence more tractable form.

In the case of a suddenly imposed wind-stress at the surface of an initially calm ocean, we are able to find a geostrophic solution which reveals the non-uniform behaviour of the transient shelf flow. The response at small depth is faster than the response at larger depth. The response is essentially inviscid far from the coast-line, but dominated by bottom friction close to the coast-line. Although the governing equation is a hyperbolic, second-order wave equation, we cannot explain the non-uniform behaviour as propagation along the characteristic curves; its origin is the (singular) point at which the depth vanishes in the governing equation. Finally, a case study is presented and comparisons are made with the corresponding numerical results given by Johnson and Manja (1980).

2. Formulation of the problem

Referring to Johnson and Manja (1980), the governing (non-dimensional) equations for the

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time-dependent flow in the coastal region are given by

$$fV = P_x, \tag{1a}$$

$$V_t + fU = 0. \tag{1b}$$

The Ekman-layer compatibility condition can be written

$$\alpha f U = \tau^y_x + \alpha x V_{tx} - FV_x, \tag{1c}$$

where U and V are the on-shore and long-shore components of the velocity, respectively. The bottom slope is given by α, the (constant) Coriolis parameter by f, and the long-shore wind-stress by $\tau^y = \tau^y(x,t)$. The parameter F is given here by $F = \sqrt{\tfrac{1}{2}f(1 + \alpha^2)}$. Subscripts x and t denote partial derivatives. The appropriate boundary conditions are given by

$$V(0,t) = \tau^y/F, \tag{2a}$$

$$V(x,t) \to 0 \quad \text{as} \quad x \to -\infty, \tag{2b}$$

while the initial condition is

$$V(x,0) = 0. \tag{2c}$$

By eliminating U from eqs. (1b) and (1c) we obtain

$$\alpha V_t + \alpha x V_{tx} - FV_x + \tau^y_x = 0, \tag{3a}$$

or

$$(\alpha x V_t)_x - FV_x + \tau^y_x = 0. \tag{3b}$$

This latter equation can be integrated once to yield

$$\alpha x V_t - FV = -\tau^y, \tag{3c}$$

which also satisfies the boundary condition (2a). In the case of a time-independent wind-stress (τ^y = constant), eq. (3c) has the solution

$$V(x,t) = \frac{\tau^y}{F}\left\{1 - \exp\frac{Ft}{\alpha x}\right\}, \tag{4a}$$

which shows the non-uniform behaviour in the limits $x \to 0$, $t \to 0$ and $x \to -\infty$, $t \to \infty$, respectively.

In the general case with a time-dependent wind-stress, the solution can be deduced by means of the Laplace transform technique (or the Lagrange method of variation of parameters)

$$V(x,t) = -\int_0^t \frac{\tau^y(x,t')}{\alpha x} \exp\left\{\frac{F}{\alpha x}(t - t')\ \mathrm{d}t'\right\}, \tag{4b}$$

which can be solved by simple quadrature when τ^y is given. The time-dependence of $V(x,t)$ will not be that of the imposed wind-stress. The response of the ocean to a suddenly imposed wind-stress is thus related to the well-known phenomenon of "spin-up" of a homogeneous, rotating fluid, but the response is not uniform in space. It is also important to note that we are able to find a geostrophic solution which vanishes at time $t = 0$, but this may not be the case in general (cf. Birchfield and Lunde, 1978).

The governing equation (3a) is a second-order hyperbolic (wave) equation with characteristic curves given by the straight lines x = constant and t = constant, respectively. The initial values of this equation are given by known values on two characteristics, and the equation is singular at $x = 0$. In the case of a constant wind-stress, the long-shore velocity is constant along the straight lines t/x = constant. Thus, all values of V between its initial and final value seem to "start" at the singular point $x = t = 0$. We emphasize that the characteristic curves are not defined at $x = 0$. We can give an explanation of the non-uniform transient motion in the case of a suddenly imposed wind-stress at time $t = 0$ by noting the effect of the bottom stress represented by the term multiplied by F in eq. (3a). If this term is set equal to zero, the resulting equation represents an essentially inviscid approximation to the problem. The solution is in fact given by the first term in the Taylor expansion of eq. (4a) for small values of t/x, i.e. $V(x,t) = -(t\tau^y/\alpha x)$. For larger values of t/x, however, the bottom stress must be included, and $V(x,t)$ is given by (4a). Thus, it seems that there is an "outer" region which is essentially inviscid ($t/x \ll 1$), and an "inner" region where bottom stress has obtained a non-zero value ($t/x \gg 1$), (see Fig. 1). It is not an easy task to explain this phenomenon directly in terms of propagation along the characteristic curves of the governing hyperbolic equation. The phenomenon was also reported by Birchfield and Lunde (1978).

3. A case study

In this section we shall give one example for the case of a time-dependent wind-stress referred to as

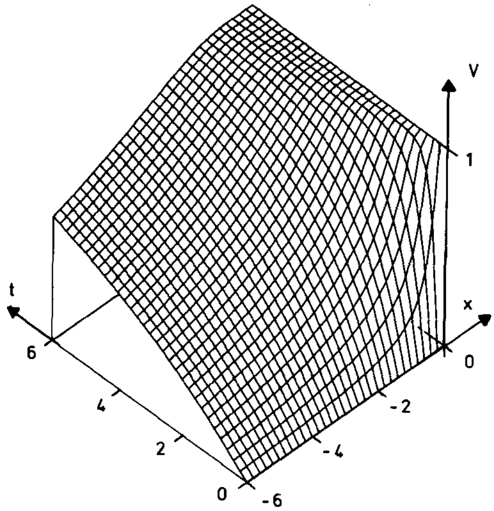

Fig. 1. A three-dimensional plot of the (dimensionless) long-shore current V in the case of a suddenly imposed (steady) wind-stress at the surface of a calm ocean. The parameter values are: $\alpha = 1, f = 1$ which gives $F = 1$. The (dimensionless) wind-stress is set equal to $\tau^y = 1$. The inviscid and viscid parts of the solution can readily be identified.

case (ii) by Johnson and Manja (1980) (cf. their eq. (16)):

$$\tau^y(t) = -\left\{\frac{1}{2} + \sin \pi\left(t - \frac{1}{2}\right) + \frac{1}{2}\sin 3\pi\left(t - \frac{1}{2}\right)\right\}, \tag{5a}$$

which is depicted in Fig. 4 of their paper. With the parameter values $\alpha = 1, f = 1$ so that $F = 1$, we have after some simple algebra the solution for the long-shore velocity

$$\begin{aligned} V(x,t) = &-\frac{1}{2}\left\{1 - \exp\frac{t}{x}\right\} \\ &+ \left\{\cos \pi t - \pi x \sin \pi t - \exp\frac{t}{x}\right\}\{1 + \pi^2 x^2\}^{-1} \\ &-\frac{1}{2}\left\{\cos 3\pi t - 3\pi x \sin 3\pi t - \exp\frac{t}{x}\right\} \\ &\times \{1 + 9\pi^2 x^2\}^{-1}, \end{aligned} \tag{5b}$$

while the solution for the on-shore flow is deduced from eq. (1b)

$$\begin{aligned} U(x,t) = -V_t(x,t) = &-\frac{1}{2x}\exp\frac{t}{x} \\ &+ \left\{\pi \sin \pi t + x\pi^2 \cos \pi t + \frac{1}{x}\exp\frac{t}{x}\right\} \\ &\times \left\{1 + \pi^2 x^2\right\}^{-1} \\ &-\frac{1}{2}\left\{3\pi \sin 3\pi t + 9x\pi^2 \cos 3\pi t + \frac{1}{x}\exp\frac{t}{x}\right\} \\ &\times \left\{1 + 9\pi^2 x^2\right\}^{-1}. \end{aligned} \tag{5c}$$

The on-shore flow is not equal to zero at the coast-line ($x = 0$), but the total on-shore transport (integrated over the depth) is zero at $x = 0$. This condition is of course taken into account in formulating the boundary condition (2a). Although the wind-stress is zero at time $t = 0$, the non-uniform behaviour in the limit $x \to -\infty$, $t \to \infty$ is a part of the solution, so the results must be interpreted with some care. The transient flow in the region $t/x \ll 1$ may thus not be described correctly by the expressions (5b) and (5c).

The solutions given by (5b) and (5c) are depicted in Fig. 2, and should be compared with the corresponding numerical results given by Johnson and Manja (1980). They solved the problem on a β-plane, starting the integration at a latitude y_0 and then stepping northward. The wind-stress was chosen so as to provide a smooth start at y_0, while the wind-stress was independent of latitude far from y_0. Although we cannot expect that eqs. (5b) and (5c) are in exact agreement with the results presented by Johnson and Manja (1980), the present study reveals that the simple model considered yields the gross features of the transient motion in the shelf region. If we compare our Fig. 2 with Fig. 4 in the paper by Johnson and Manja (1980), we see that the on-shore currents are in fairly good agreement, but this is not the case for the long-shore currents. The discrepancies must be attributed to the omission of the β-effect, which prevents Rossby waves carrying energy off-shore, and the loss of variation along the coast. The treatment of the singular point in the numerical calculations of Johnson and Manja (1980) may also be due to this difference. Nevertheless, it is believed that in some cases the present model can be used to obtain information of the shelf flow without too much effort. This is, in a way, the

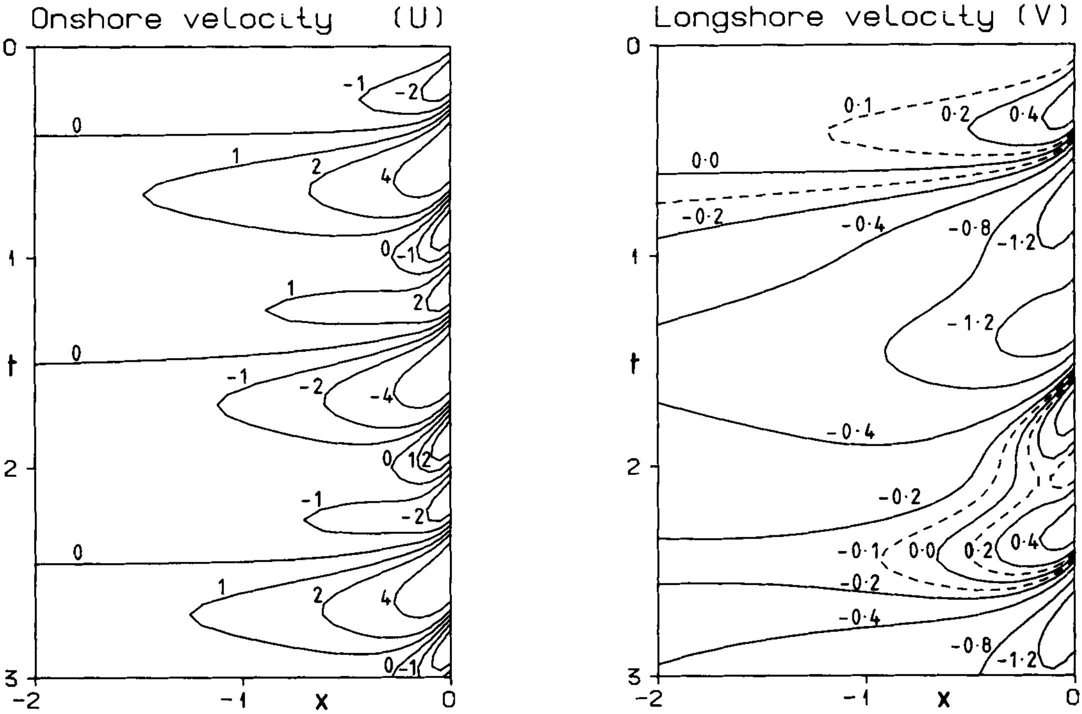

Fig. 2. Contours of (dimensionless) on-shore current U and long-shore current V in x-t space. The (dimensionless) wind-stress is given by eq. (5a), while $\alpha = 1$ and $f = 1$.

purpose of the present study, i.e. to study certain simplified models in order to gain important qualitative insight into the resulting transient flow in the shelf region.

4. Final remarks

In this note a simple solution of the response in a coastal region (due to a uniform wind-stress directed along the coast-line) is given in closed form. This solution can give important information as to the origin of the non-uniform behaviour of the space-time flow, even in the general situation with variable conditions along the coast-line. The preliminary results given by Birchfield and Lunde (1978) indicate that the singularity of the governing equations (at $x = 0$) is the source of the non-uniform behaviour of the transient solution. Besides, a closed form of the solution can also provide a useful check of the numerical solution given by Johnson and Manja (1980). It is believed that the case studied in the present paper yields a reasonable explanation of the subject under consideration. Nevertheless, more work should be encouraged, particularly that directed towards the various time scales of the transient motion in the coastal region.

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