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On the mathematics of data assimilation

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ABSTRACT

The problem of convergence of a "forward-backward" assimilation is considered for the most general dynamical system. Using elementary techniques of stability theory, it is shown that the variation, over one assimilation cycle, of the difference between the assimilating model and the state to be reconstructed is, to the first order, determined by a perfectly defined amplification matrix. This leads to a straightforward criterion for convergence, depending on the eigenvalues of that matrix.

This convergence criterion has in effect already been shown in a previous article to be verified by the linearized meteorological equations. It is shown here to be verified by the non-linear shallow-water equations, in the case of successive observations of the geopotential field, at least when these observations are sufficiently close in time. Numerical experiments support the theoretical results, and they together lead to the following description of the effects of an assimilation of geopotential observations. The divergent part of the wind field is reconstructed more rapidly, because it is directly influenced by introductions of observations. The rotational part is reconstructed indirectly and more slowly, mostly through the effect of Coriolis acceleration. These results are independent of whether or not the flow to be reconstructed is geostrophic, and do not require the presence of any dissipative process in the assimilating model.

The effect of observing and/or modelling errors is considered, and small errors are shown not to modify the convergence properties of an assimilation.

Finally, a theoretical method is defined through which the amplification matrix over one assimilation cycle can be made equal to zero, thereby optimizing the convergence of the process.

1. Introduction

The theoretical and mathematical problems associated with assimilation of meteorological observations are rather complex and still incompletely understood. The basis for the development of assimilation techniques is still very empirical, and a systematic and rigorous study of many of the underlying problems has not been carried out. In a previous article (Talagrand, 1981), some of these problems have been considered on linearized versions of the meteorological equations. It was shown that energy conservation ensures for these equations the convergence of an assimilation of the simplest updating type, under the only, and obviously necessary, condition that the available observations are compatible with only one solution of the model equations. In the present article, similar problems are considered for the most general dynamical system, either linear or non-linear. Two examples will illustrate the approach taken here. As elementary and artificial as these examples are, they do contain the essential features of what is to follow.

The evolution equations of a one-dimensional harmonic oscillator can be written, in appropriate "non-dimensional" units:

$$\frac{dx}{dt} = y \tag{1.1a}$$

$$\frac{dy}{dt} = -x \tag{1.1b}$$

where x and y are two parameters which together define the state of the system at any time t (e.g. the

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position and velocity of an oscillatory point particle). Any initial conditions $x(t_0)$ and $y(t_0)$ at a given time t_0 define a unique solution to eqs. (1.1), viz

$$x(t) = x(t_0)\cos(t - t_0) + y(t_0)\sin(t - t_0) \quad (1.2a)$$

$$y(t) = -x(t_0)\sin(t - t_0) + y(t_0)\cos(t - t_0) \quad (1.2b)$$

Let us suppose that, for one particular solution $(x(t))$, $(y(t))$ of eqs. (1.1), we know from observations the values $x(t_0)$ and $x(t_1)$ of the parameter x at two different times t_0 and t_1, and that we want to determine from these values the complete state of the system, i.e. x and y, at a given time, t_0 for instance. The answer to this question is trivial. For $t = t_1$, eq. (1.2a) defines $y(t_0)$ provided $\sin(t_1 - t_0)$ is not 0. If $\sin(t_1 - t_0)$ is 0, then necessarily $x(t_1) = \pm x(t_0)$, the second observation is redundant, and $y(t_0)$ remains undetermined.

Let us now suppose that we wish to determine $y(t_0)$ by a forward-backward assimilation of the type already described in Talagrand (1981), and performed between instants t_0 and t_1. The assimilation process will start at time t_0 from the observed value $x(t_0)$ and from some arbitrary value $\tilde{y}_1(t_0)$. The latter can be written as

$$\tilde{y}_1(t_0) = y(t_0) + \Delta y_0$$

where Δy_0 is the unknown initial error on y. The integration of eqs. (1.1) will produce for the value of parameter y at time t_1

$$\begin{aligned}\tilde{y}_1(t_1) &= -x(t_0)\sin(t_1 - t_0) + \tilde{y}_1(t_0)\cos(t_1 - t_0)\\ &= y(t_1) + \Delta y_0\cos(t_1 - t_0)\end{aligned}$$

After replacement of the value produced for x by the forward integration with the observed value $x(t_1)$, the backward integration from t_1 to t_0 will produce for the value of y at time t_0

$$\begin{aligned}\tilde{y}_2(t_0) &= -x(t_1)\sin(t_0 - t_1) + \tilde{y}_1(t_1)\cos(t_0 - t_1)\\ &= y(t_0) + \Delta y_0\cos^2(t_1 - t_0)\end{aligned}$$

The error on $y(t_0)$ will thus be multiplied by $\cos^2(t_1 - t_0)$ at each assimilation cycle, and will tend to 0 as the assimilation will proceed, provided $|\cos(t_1 - t_0)| \neq 1$. The process will be convergent in the sense that it will progressively reconstitute the unknown value $y(t_0)$. This will be true except if $|\cos(t_1 - t_0)| = 1$ but then as seen above, $y(t_0)$ is not uniquely determined anyway.

If, instead of system (1.1), we now consider the system

$$\frac{dx}{dt} = y \quad (1.3a)$$

$$\frac{dy}{dt} = x \quad (1.3b)$$

the general solution of which, from initial conditions $(x(t_0), y(t_0))$ reads

$$\begin{aligned}x(t) &= x(t_0)\cosh(t - t_0) + y(t_0)\sinh(t - t_0)\\ y(t) &= x(t_0)\sinh(t - t_0) + y(t_0)\cosh(t - t_0)\end{aligned}$$

we readily see that the knowledge of $x(t_0)$ and $x(t_1)$ $(t_1 \neq t_0)$ uniquely determines the corresponding solution. The same reasoning as above shows, however, that the error on $y(t_0)$ will be multiplied by $\cosh^2(t_1 - t_0) > 1$ at each cycle of an assimilation, and will grow beyond any limit as the assimilation will proceed.

It thus appears that, depending on the particular system under consideration, an assimilation may, or may not, reconstitute the complete state of the system. An obvious question is then: What will be the situation with more complex systems of equations and especially with meteorological equations? This question will be considered in the present paper for systems with a finite number of degrees of freedom. This restriction is legitimate since assimilations and forecasts can in practice be performed only with models discretized to a finite number of parameters. To the first order, the difference between the assimilating model and the solution under observation will be shown to be multiplied at each cycle of a forward-backward assimilation by a perfectly defined matrix. The eigenvalues of this matrix determine the convergence or divergence of the process. This leads to a general convergence criterion.

Section 2 presents some basic notions of stability theory, which are used in Section 3 to derive the general convergence criterion for a forward-backward assimilation. This criterion is shown in Section 4 to be satisfied by the non-linear shallow-water equations, in the case of successive observations, close in time, of the complete mass field. The effect of observational and/or modelling error is considered in Section 5. Section 6 presents a theoretical possibility for accelerating the convergence of an assimilation, derived from the general criterion of Section 3. The results are discussed, and a number of conclusions drawn, in Section 7.

Most of the content of the present article is part of the author's doctoral thesis (Talagrand, 1977) which will be referred to as T77. The article (Talagrand, 1981) more specifically devoted to the linearized meteorological equations will be referred to as T81.

2. Some basic notions of stability theory

We will consider a dynamical system (e.g. a numerical model of the atmosphere, but no specific hypothesis is necessary at this stage) whose state at any time t is defined by the values of r independent scalar parameters. These parameters make up a vector Z and we will assume that the time evolution of the system is described by a set of r differential equations of the first order, which can be summed up as

$$\frac{dZ}{dt} = H(Z) \tag{2.1}$$

where H is an r-valued function of Z. We do not allow in (2.1) for a dependence of H with time, but this could be done without resulting in any modification of the results to be presented below.

We will furthermore assume that Z is made of two parts X and Y, with respective dimensions p and q ($p + q = r$). The first part X (e.g. the mass field of the atmosphere) is supposed to be observed at successive times, while the second part Y (e.g. the wind field) is to be reconstructed from the successive observations of X. Upon this separation, eq. (2.1) becomes

$$\frac{dX}{dt} = F[X, Y] \tag{2.2a}$$

$$\frac{dY}{dt} = G[X, Y] \tag{2.2b}$$

where F and G are respectively a p-valued and a q-valued function of X and Y. An arbitrary initial time t_0 being chosen, any initial conditions (X_0, Y_0) at time t_0 define a unique solution to eqs. (2.2). We will denote by $[X(X_0, Y_0; t), Y(X_0, Y_0; t)]$ the values assumed at time t by that solution.

We will in the following be interested in solutions defined by "perturbed" initial conditions. Any "perturbations" $(\Delta X_0, \Delta Y_0)$ imposed on (X_0, Y_0) define a unique solution, which can be written as $[X(X_0, Y_0; t) + \Delta X(t), Y(X_0, Y_0; t) + \Delta Y(t)]$. The time evolution of the perturbations $(\Delta X(t), \Delta Y(t))$ is given by the *perturbation system* in the vicinity of the solution $[X(X_0, Y_0; t)\; Y(X_0, Y_0; t)]$

$$\frac{d\Delta X}{dt} = F[X(X_0, Y_0; t) + \Delta X(t), Y(X_0, Y_0, t) + \Delta Y(t)] - F[X(X_0, Y_0; t), Y(X_0, Y_0; t)]$$

$$\frac{d\Delta Y}{dt} = G[X(X_0, Y_0, t) + \Delta X(t), Y[X_0, Y_0; t) + \Delta Y(t)] - G[X(X_0, Y_0; t), Y(X_0, Y_0; t)] \tag{2.3b}$$

Linearizing eqs. (2.3) with respect to ΔX and ΔY leads to

$$\frac{d\delta X}{dt} = \frac{DF}{DX}(t)\, \delta X + \frac{DF}{DY}(t)\, \delta Y \tag{2.4a}$$

$$\frac{d\delta Y}{dt} = \frac{DG}{DX}(t)\, \delta X + \frac{DG}{DY}(t)\, \delta Y \tag{2.4b}$$

where the prefixes Δ have been replaced with δ's (in the following, we will systematically use Δ's for solutions of the exact perturbation system (2.3) and δ's for solutions of the linearized system (2.4)). System (2.4) is the *linearized perturbation system* in the vicinity of the solution $[X(X_0, Y_0; t), Y(X_0, Y_0; t)]$. In (2.4a), $DF/DX(t)$ is the jacobian matrix of the derivatives of F with respect to X, i.e. the $p \times p$ matrix whose entry in row i and column j is the partial derivative of the ith component of F with respect to the jth component of X. The argument t means that this observation is taken at the point $[X(X_0, Y_0; t), Y(X_0, Y_0; t)]$.

$DF/DY(t)$, $DG/DX(t)$, $DG/DY(t)$ are similarly defined jacobian matrices with respective dimensions $p \times q$, $q \times p$, $q \times q$.

Except if the basic system (2.2) is itself linear, there is one linearized perturbation system (2.4) for each solution $[X(X_0, Y_0; t), Y(X_0, Y_0; t)]$ of (2.2). When the solution $[X(X_0, Y_0; t), Y(X_0, Y_0; t)]$ is stationary with time, the linearized system (2.4), whose coefficients are then independent of time, is commonly used for studying the stability of the corresponding equilibrium.

Any initial conditions $(\delta X_0, \delta Y_0)$ at time t_0 define a unique solution to the linearized system (2.4). This solution can be written as

$$\delta X(t) = R_x^x(t, t_0)\, \delta X_0 + R_x^y(t, t_0)\, \delta Y_0 \tag{2.5a}$$

$$\delta Y(t) = R_y^x(t, t_0)\, \delta X_0 + R_y^y(t, t_0)\, \delta Y_0 \tag{2.5b}$$

where the four matrices $R_x^x(t, t_0)$, $R_x^y(t, t_0)$, $R_y^x(t, t_0)$, $R_y^y(t, t_0)$, which have respective dimensions $p \times p$, $p \times q$, $q \times p$, $q \times q$, make up together a matrix $R(t, t_0)$ with dimensions $r \times r$:

$$R(t, t_0) = \begin{pmatrix} R_x^x(t, t_0) & R_x^y(t, t_0) \\ R_y^x(t, t_0) & R_y^y(t, t_0) \end{pmatrix} \tag{2.6}$$

$R(t, t_0)$ is called the *resolvent matrix* of the linearized system (2.4) between the times t_0 and t.

We will be interested in the linearized system (2.4) for the following reason. For small initial perturbations $(\Delta X_0, \Delta Y_0)$, system (2.4) can be considered as describing, in first approximation, the time evolution of the resulting perturbation. This vague statement is made precise by the following theorem:

Theorem (P). *At any given time t, the difference between the solutions of the exact and linearized perturbation systems (2.3) and (2.4) defined by the same initial conditions $(\Delta X_0, \Delta Y_0)$ is an infinitesimal of higher order than $(\Delta X_0, \Delta Y_0)$.*

It results that the solution $(\Delta X(t), \Delta Y(t))$ of the exact perturbation system can be written as

$$\Delta X(t) = R_x^x(t, t_0)\, \Delta X_0 + R_x^y(t, t_0)\, \Delta Y_0 + o(\Delta X_0, \Delta Y_0) \tag{2.7a}$$

$$\Delta Y(t) = R_y^x(t, t_0)\, \Delta X_0 + R_y^y(t, t_0)\, \Delta Y_0 + o(\Delta X_0, \Delta Y_0) \tag{2.7b}$$

where, following a standard convention, $o(\Delta X_0, \Delta Y_0)$ denotes a function of ΔX_0 and ΔY_0 such that the ratio

$$\frac{|o(\Delta X_0, \Delta Y_0)|}{|\Delta X_0| + |\Delta Y_0|}$$

tends to 0 when ΔX_0 and ΔY_0 tend to 0.

Theorem (P) essentially means that the solution $[X(X_0, Y_0; t), Y(X_0, Y_0; t)]$ is at any time t a continuous and differentiable function of the initial conditions (X_0, Y_0) and that the partial derivatives of the components of $X(X_0, Y_0; t)$ and $Y(X_0, Y_0; t)$ with respect to the components of X_0 and Y_0 are the entries of the resolvent matrix (2.6).

Theorem (P) is classical and is the basis to many results of stability theory. Its proof can be found in many books dealing with the theory of ordinary differential equations, e.g. in Coddington and Levinson (1955).

We will use in the following a few additional properties of the resolvent matrix R. The choice of the initial time t_0 is arbitrary and, given a solution of eqs. (2.2), the corresponding resolvent matrix $R(t'', t')$ is defined for any two times t' and t'' at which this solution itself exists, whether $t'' \geqslant t'$ or $t'' < t'$. Since integrating system (2.4) from t' to t'', and then to a third time t''' produces the same result as integrating directly from t' to t''', the corresponding resolvent matrices must satisfy the following relationship

$$R(t''', t'')\, R(t'', t') = R(t''', t') \tag{2.8}$$

which, for $t''' = t'$, reduces to

$$R(t', t'')\, R(t'', t') = I \tag{2.9}$$

where I is the unit matrix of order r. Using decomposition (2.6), each of these two relationships can be readily decomposed into four relationships between the corresponding matrices R_x^x, R_x^y, R_y^x, R_y^y.

3. General convergence criterion for a forward-backward assimilation

We will formalize the assimilation problem by assuming that, for a particular solution $[X(t), Y(t)]$ of system (2.2), hereafter referred to as the *observed solution*, the exact values of the p parameters making up the vector X have been measured at N successive times $t_1, t_2, \ldots, t_N$. We will study under which conditions an assimilation performed on these observations will reconstruct the complete state of the system, i.e. *X and Y* at a given time, t_N for example.

The assimilation will be assumed to be performed according to the following simple procedure. The numerical model is integrated alternatively forward and backward in time over the time interval $[t_1, t_N]$. Whenever the model time reaches an observation time t_i, whether in a forward or in a backward integration, the p values predicted by the model for X are replaced with the observed values, while the q values predicted for Y are left unchanged. This procedure calls for an important remark. The nature of the parameters which make up X is imposed by the observations, but the nature of the parameters which make up Y is not imposed by the conditions of the problem. The nature of Y can be chosen arbitrarily under the only condition that X and Y together uniquely define the complete state of the system. For instance, in the case of an atmospheric model, with X representing, say, the mass field, one can choose for Y either the velocity field $\mathbb{V}$, or the momentum field $\rho\mathbb{V}$, or any other combination of the mass and velocity field which, together with the mass field, completely defines the state of the flow. These different choices are obviously not equivalent for the assimilation. The nature of X being given, there are infinitely many possible choices for the nature of Y. For this reason, the hypothesis that Y is not modified at an introduction time is much less restrictive than could *a priori* seem. We will come back to this point in Section 6 and will assume for the time being that one particular arbitrary choice has been made for Y.

We will furthermore assume, until Section 5, that the observations are perfectly accurate and that the assimilating model perfectly simulates the physical system under observation. This is the classical "identical twin" hypothesis.

3.1. *The amplification matrix*

We will choose the latest observation time t_N as the arbitrary origin of the successive forward-backward assimilation cycles. Setting $t_N - t_1 = T$, it will be convenient to introduce along each assimilation cycle an auxiliary time variable τ, defined module $2T$. This variable will increase from 0 to T in the backward phase of the cycle, and from T to $2T$ in the forward phase. To each of its values τ, there corresponds a unique value of the real time. Conversely, to any value t of the real time, $t_1 < t < t_N$, there correspond two values τ and τ' of the auxiliary time, such that $\tau + \tau' = 2T$, and belonging respectively to the backward and forward phases of the assimilation cycle. The instants in the cycle when observations of X are introduced into the model will be denoted τ_1 (coinciding with t_N), $\tau_2, \ldots, \tau_N$ (coinciding with t_1), $\tau_{N+1}, \ldots, \tau_{2(N-1)}$ (coinciding with t_{N-1}). We shall define $M = 2(N-1)$. M introductions of observations are performed in the course of one assimilation cycle.

For any two values τ and τ' of the auxiliary time variable, it will be convenient to denote by $R(\tau', \tau)$ the resolvent matrix (2.6) between the corresponding values of the real time. A similar notation will be used for the submatrices $R_x^x, R_x^y, \ldots$.

Let us now consider the difference $(\Delta X, \Delta Y)$ between the assimilating model and the observed solution. Between two observation times, this difference varies according to the perturbation system (2.3) where the "unperturbed" solution $[X(X_0, Y_0; t), Y(X_0, Y_0; t)]$ is now the observed solution $[X(t), Y(t)]$. At an observation time τ_l, ΔX is set equal to 0, while ΔY remains unchanged. Let ΔY_n be the Y-difference at the end of the nth assimilation cycle. In the $(n + 1)$st cycle, starting at time τ_1, the model integration from τ_1 to τ_2 will produce a difference ΔX which will be then set equal to 0, and a difference $\Delta Y(\tau_2)$ which, according to (2.7b) is equal to

$$\Delta Y(\tau_2) = R_y^y(\tau_2, \tau_1)\, \Delta Y_n + o(\Delta Y_n)$$

Similarly, the subsequent integration of the model from τ_2 to τ_3 will produce a difference

$$\begin{aligned}\Delta Y(\tau_3) &= R_y^y(\tau_3, \tau_2)\, \Delta Y(\tau_2) + o(\Delta Y(\tau_2)) \\ &= R_y^y(\tau_3, \tau_2)\, R_y^y(\tau_2, \tau_1)\, \Delta Y_n + o(\Delta Y_n)\end{aligned}$$

This argument, carried out over the complete assimilation cycle, shows that the difference ΔY_{n+1} at the end of the $(n + 1)$st cycle will be

$$\Delta Y_{n+1} = A\, \Delta Y_n + o(\Delta Y_n) \tag{3.1}$$

where A is the matrix

$$A = R_y^y(\tau_1, \tau_M)\, R_y^y(\tau_M, \tau_{M-1}) \ldots R_y^y(\tau_2, \tau_1) \tag{3.2}$$

A is the product of M square matrices of order q each of which represents the effect of the assimilation over one interval (τ_l, τ_{l+1}). A will be called the *amplification matrix* of the difference ΔY over one assimilation cycle. It is entirely determined by the linearized perturbation system (2.4) in the vicinity of the observed solution and, more precisely, by only one, namely R_y^y, of the four submatrices which make up R (see eq. (2.6)). It must be noted that, contrary to R, R_y^y does not satisfy a "contracting" relationship of type (2.8), so that expression (3.2) cannot be written in a more concise form.

Now, according to a general result of matrix algebra, the relevant parameter for the behaviour of ΔY_n as n tends to infinity in the *spectral radius*

$\rho(A)$, which is by definition the largest modulus of the eigenvalues of A (see e.g. Varga (1962)):

—if $\rho(A)$ is strictly less than 1, ΔY_n will tend to 0 as n increases to infinity, provided the initial difference ΔY_0 is small enough so that the $o(\Delta Y_0)$ term in (3.1) is negligible compared with $A \Delta Y_0$;

—if $\rho(A)$ is larger than 1, the components of ΔY_n along eigenvector(s) corresponding to eigenvalue(s) with modulus larger than 1, will be amplified by the assimilation, and ΔY_n will not tend to 0;

—finally, if $\rho(A)$ is exactly 1, the component of ΔY_n along eigenvector(s) corresponding to eigenvalue(s) with modulus equal to 1, will be neither amplified nor reduced in the product $A\Delta Y_n$, and the behaviour of ΔY_n as n tends to infinity will depend on the higher order term $o(\Delta Y_n)$ in (3.1).

We see that $\rho(A) \leqslant 1$ is a necessary condition for convergence of an assimilation, and $\rho(A) < 1$ a sufficient condition, when convergence is defined as follows: there exists some number $\varepsilon > 0$ such that ΔY_n will tend to 0 as n tends to infinity provided the initial difference ΔY_0 is less than ε. It must be noted that this definition of convergence is not as strict as the one considered in T81, which did not impose any condition on the initial difference. It is so because we have allowed here for the basic eqs. (2.2) to be non-linear, and we must be content with a less stringent definition of convergence. But this limitation is of no real importance since an assimilation procedure is of no interest whatsoever if it does not converge at least in the sense defined here.

For the sake of simplicity, we have assumed that the "same" p parameters, making up the vector X, are observed at the successive times $t_1, t_2, \ldots, t_N$. This assumption is in effect not necessary and an amplification matrix of type (3.2) can be obtained through a similar derivation when the nature, or even the number, of the observed parameters varies with the observation time t_i.

3.2. *Examples*

Let us consider system (1.1), with the scalar x standing for the vector X ($p = 1$) and y standing for Y ($q = 1$). System (1.1) being linear and homogeneous, the linearized perturbation system (2.4) is identical with (1.1) for any "observed" solution. The resolvent matrix is given by eqs. (1.2):

$$R(t, t_0) = \begin{pmatrix} \cos(t - t_0) & \sin(t - t_0) \\ -\sin(t - t_0) & \cos(t - t_0) \end{pmatrix}$$

The matrix $R_y^y(t, t_0)$ reduces to the scalar $\cos(t - t_0)$, and the amplification matrix A to a product of factors of the form $\cos(t_{i\pm 1} - t_i)$. The spectral radius $\rho(A)$ will be less than 1, and an assimilation of observations of x will be convergent, except if $|\cos(t_{i+1} - t_i)| = 1$ for all i's, i.e. if $t_{i+1} - t_i = n_i \pi$, n_i integer. If the latter condition is satisfied, the successive observations of x are redundant, and the "observed" solution is not uniquely defined anyway. These results generalize those presented in the introduction.

In the case of system (1.3), whose resolvent matrix is

$$R(t, t_0) = \begin{pmatrix} \cosh(t - t_0) & \sinh(t - t_0) \\ \sinh(t - t_0) & \cosh(t - t_0) \end{pmatrix}$$

the amplification matrix A is the product of factors of the form $\cosh(t_{i\pm 1} - t_i)$. A will always be larger than 1, and an assimilation of observations of x will never converge. This too generalizes the results of the introduction.

Because of the linearity of systems (1.1) and (1.3), the two above examples are almost trivially simple. Various more complex non-linear systems are considered in T77.

3.3. *The case of successive observations close in time*

Except when the basic system (2.2) is linear, the explicit determination of the amplification matrix A and/or of its spectral radius is most often rather difficult, since this normally requires the explicit knowledge of the linearized perturbation system (2.4) and of its resolvent matrix. In few non-linear systems can these be explicitly determined for any solution $[X(t), Y(t)]$. However, in the case when the successive observations of X are close in time, the amplification matrix and the corresponding convergence criterion assume simple forms, which depend only on conditions local in time. We will now proceed to the study of this simpler case.

We will assume first that X has been observed at two successive times t_1 and t_2. The amplification matrix (3.2) is then

$$A = R_y^y(t_2, t_1)\, R_y^y(t_1, t_2)$$

where we do not use the auxiliary time τ any more.

The reversibility equation (2.9) implies, for $t' = t_2$ and $t'' = t_1$ and, taking into account decomposition (2.6)

$$R_y^x(t_2, t_1)\, R_x^y(t_1, t_2) + R_y^y(t_2, t_1)\, R_y^y(t_1, t_2) = I$$

where I is now the unit matrix of order q. This leads to

$$A = I - R_y^x(t_2, t_1)\, R_x^y(t_1, t_2) \tag{3.3}$$

For any solution of the linearized perturbation system (2.4), the developments of $\delta X(t + \Delta t)$ and $\delta Y(t + \Delta t)$ with respect to Δt read:

$$\delta X(t + \Delta t) = \delta X(t) + \Delta t \frac{DF}{DX}(t)\, \delta X(t) + \Delta t \frac{DF}{DY}(t)\, \delta Y(t) + o(\Delta t)$$

$$\delta Y(t + \Delta t) = \delta Y(t) + \Delta t \frac{DG}{DX}(t)\, \delta X(t) + \Delta t \frac{DG}{DY}(t)\, \delta Y(t) + o(\Delta t)$$

from which one obtains, by comparing with (2.5)

$$R_x^y(t + \Delta t, t) = \Delta t \frac{DF}{DY}(t) + o(\Delta t)$$

$$R_y^x(t + \Delta t, t) = \Delta t \frac{DG}{DX}(t) + o(\Delta t)$$

Setting $t_2 - t_1 = \Delta t$, and carrying these expressions into (3.3) leads to the following expression for A

$$A = I + \Delta t^2 \frac{DG}{DX}(t_1)\, \frac{DF}{DY} t_2 + o(\Delta t^2)$$

To order Δt, the jacobian matrices DG/DX and DF/DY can be taken indifferently at time t_1 or t_2. We shall simply write

$$A = I + \Delta t^2 \frac{DG}{DX} \frac{DF}{DY} + o(\Delta t^2) \tag{3.4}$$

with no more precision.

DG/DX is a $q \times p$ matrix, DF/DY a $p \times q$ matrix. Their product $DG/DX\; DF/DY$ is therefore a square matrix of order q, as necessary. Let us call $\lambda_j (j = 1, \ldots, q)$ its eigenvalues. The eigenvalues of A are

$$a_j = 1 + \lambda_j \Delta t^2 + o(\Delta t^2) \quad j = 1, \ldots, q \tag{3.5}$$

The convergence criterion derived above is that, in the complex plane, all the eigenvalues a_j lie inside the circle (C) centred at the origin, with radius unity. For Δt small enough, this condition is satisfied if all the eigenvalues λ_j have a strictly negative real part (Fig. 1)

$$\mathcal{R}(\lambda_j) < 0 \quad j = 1, \ldots, q \tag{3.6}$$

This leads to the following conclusions:

(i) if condition (3.6) is satisfied, the spectral radius of A will be strictly less than 1 for small Δt, and an assimilation will converge in the sense defined in subsection 3.1;

(ii) if $\mathcal{R}(\lambda_j) > 0$ for at least one j, the spectral radius of A will be larger than 1 for small Δt, and an assimilation will diverge;

(iii) if $\mathcal{R}(\lambda_j) \leqslant 0$ for all j's, with equality occurring for some j, the spectral radius of A will be less than, equal to or larger than 1, depending on the $o(\Delta t^2)$ term in (3.4). No general conclusion

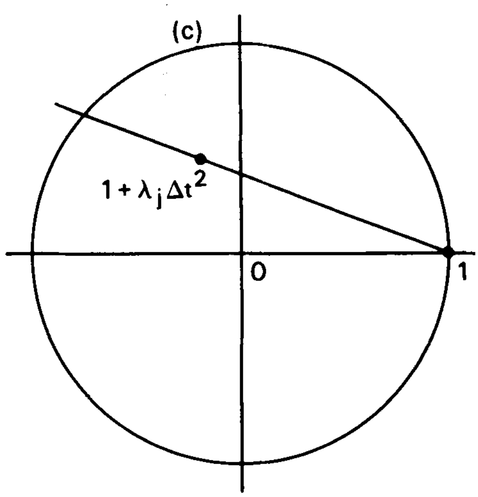

Fig. 1. Graphical illustration of condition (3.6). Four small Δt, the point $1 + \lambda_j \Delta t^2 + o(\Delta t^2)$ lies inside circle (C) if the real part $\mathcal{R}(\lambda_j)$ of λ_j is strictly negative, and outside (C) is $\mathcal{R}(\lambda_j)$ is strictly positive. When $\mathcal{R}(\lambda_j) = 0$, the point may lie inside, on or outside circle (C), depending on $o(\Delta t^2)$.

is then possible as to the convergence of an assimilation.

The matrix $DG/DX\ DF/DY$ is the product of two matrices, each of which represents the dependence of the time evolution of one of the two parts X or Y with respect to the other. The matrix $DG/DX\ DF/DY$ therefore represents in some sense the coupling between the time evolutions of X and Y. It will be called the *coupling matrix* between X and Y.

In the case when X has been observed at $N + 1$ successive times separated by a constant time interval Δt, it can be shown (e.g. by induction on N) that the amplification matrix is

$$A = I + N\Delta t^2 \frac{DG}{DX}\frac{DF}{DY} + o(\Delta t^2) \tag{3.7}$$

which leads to the same convergence criterion (3.6). Equation (3.7) provides the answer to the following question. Assuming condition (3.6) to be satisfied, is it more efficient to perform an assimilation on two observations of X separated by $N\Delta t$, or on $N + 1$ observations separated by Δt? Denoting by μ the smallest modulus of the real parts of the eigenvalues of $DG/DX\ DF/DY$, the spectral radius of A will be equal to $1 - \mu N^2 \Delta t^2 + o(\Delta t^2)$ in the first case, and to $1 - \mu N \Delta t^2 + o(\Delta t^2)$ in the second. For small Δt, the former value is smaller, which shows that it is more efficient to use two observations only.

The convergence criterion (3.6) can easily be illustrated with the two examples considered in the introduction. In the case of system (1.1), the matrix DG/DX reduces to the scalar

$$\frac{\partial}{\partial x}\frac{dy}{dt} = -1$$

and DF/DY to

$$\frac{\partial}{\partial y}\frac{dx}{dt} = 1$$

The coupling matrix reduces to -1, and satisfies condition (3.6), in agreement with the results already obtained. For system (1.3), the coupling matrix is $+1$, and does not satisfy (3.6), also in agreement with the results previously obtained.

Other examples relative to non-linear systems are treated in T77.

4. The meteorological equations

The simplest systems of meteorological equations to which the results of the foregoing section can be applied are the linearized versions of either the shallow-water equations or the multi-level primitive equations. Because of linearity, the amplification matrix (3.2) is then independent of the particular solution under observation, and is entirely determined by the space-time distribution of the observations. Moreover, there is no $o(\Delta Y_n)$ term in (3.1), so that the possible convergence of an assimilation is independent on any condition on the initial error ΔY_0. The equations linearized in the vicinity of a state of rest have been considered in T81, and it has been shown there that energy conservation ensures the convergence of an assimilation under the only, and obviously necessary, condition that the observations uniquely determine the observed solution. It can be mentioned at this point that, if the fully non-linear equations are used, and the observed solution is a state of rest, the linearized perturbation eqs. (2.4) are the equations considered in T81. The results which have been established there apply therefore in that particular case to the non-linear equations. Indeed, the numerical results presented inT81 have been obtained with the non-linear equations, the "observed" solution being a state of rest.

In the general case of the non-linear equations, two approaches are feasible: either determining the amplification matrix (3.2) for given, explicitly known solutions, or considering only criterion (3.6), which is local in time. We will take the latter approach, in the case of the shallow-water equations

$$\frac{\partial \varphi}{\partial t} + \nabla \cdot (\varphi \mathbb{V}) = 0 \tag{4.1a}$$

$$\frac{\partial \mathbb{V}}{\partial t} + (\mathbb{V} \cdot \nabla)\mathbb{V} + \nabla \varphi + \mathbf{n} \times f \mathbb{V} = 0 \tag{4.1b}$$

where φ is the free surface geopotential, and $\mathbb{V}$ the horizontal velocity, of a fluid covering a horizontal domain S; f is the Coriolis parameter, $\mathbf{n}$ a unit vertical vector positive upwards, and ∇ the two-dimensional del operator. We will assume that the geopotential φ has been observed over the entire domain S at successive times, and that updatings of observations of φ are performed without modifying the velocity. The geopotential field φ and the velocity field $\mathbb{V}$ therefore stand respectively for the vectors X and Y of Section 2, and eqs. (4.1a) and (4.1b) for eqs. (2.2a) and (2.2b). An important remark, already made in T81, must be made at this point. As φ appears in (4.1b) only through a gradient, an introduction of observations of geopotential will influence the first time derivative of the divergence $\nabla \cdot \mathbb{V}$, but the second derivative only of the vorticity $\mathbf{n} \cdot \nabla \times \mathbb{V}$. We can therefore expect an assimilation of observations of φ to have more influence on the divergent part of the wind field than on its rotational part.

In accordance with the developments of the previous section, eqs. (4.1) must be considered as being the equations of a numerical model, discretized to a finite number of parameters, either in the physical or in the phase space. However, for the sake of simplicity, and also because we are not interested at this stage in any particular discretization, we shall keep to notations usual for a continuum, and use a number of properties of the ordinary non-discretized del operator. The results to be established hereafter will therefore be valid only for discretizations in which all the necessary properties of the del operator are retained.

The velocity field $\mathbb{V}$ appearing linearly in eqs. (4.1a), the equivalent of the jacobian matrix DF/DY is now the linear operator L_1 which to any "perturbation" $\partial\mathbb{V}$ of the wind field associates the scalar field defined by

$$L_1(\partial\mathbb{V}) = -\nabla \cdot (\varphi \partial \mathbb{V})$$

Similarly, the geopotential φ appears linearly in eqs. (4.2b) and the equivalent of the jacobian matrix DG/DX is the linear operator L_2 which, to any perturbation $\partial\varphi$ of the geopotential field associates the vector field defined by

$$L_2(\partial\varphi) = -\nabla(\partial\varphi)$$

The equivalent of the coupling matrix DG/DX DF/DY is the linear operator obtained by composing L_1 with L_2, i.e. the operator C which, to any wind field perturbation $\partial\mathbb{V}$ associates the following vector field

$$C(\partial\mathbb{V}) = L_2[L_1(\partial\mathbb{V})] = \nabla[\nabla \cdot (\varphi\partial\mathbb{V})] \tag{4.2}$$

The amplification matrix (3.4) becomes

$$A = I + \Delta t^2 C + o(\Delta t^2) \tag{4.3}$$

The next step is to determine the signs of the real parts of the eigenvalues of C. To this end we will introduce the scalar product defined for any two wind field perturbations $\partial\mathbb{V}$ and $\partial\mathbb{V}'$ (with possibly complex components) by

$$\langle \partial\mathbb{V}, \partial\mathbb{V}' \rangle = \int_S \varphi\, \partial\mathbb{V} \cdot \partial\mathbb{V}'^*\, dS \tag{4.4}$$

where $\partial\mathbb{V}'^*$ is the complex conjugate of $\partial\mathbb{V}'$. Since the geopotential φ is positive everywhere on the domain S, (4.4) defines a scalar product. For any two perturbations $\partial\mathbb{V}$ and $\partial\mathbb{V}'$

$$\begin{aligned}\langle \partial\mathbb{V}, C(\partial\mathbb{V}') \rangle &= \int_S \varphi\, \partial\mathbb{V} \cdot \nabla[\nabla \cdot (\varphi\partial\mathbb{V}'^*)]\, dS \\ &= -\int_S \nabla \cdot (\varphi\partial\mathbb{V})\, \nabla \cdot (\varphi\partial\mathbb{V}'^*)\, dS\end{aligned} \tag{4.5}$$

The latter expression is symmetrical with respect to $\partial\mathbb{V}$ and $\partial\mathbb{V}'^*$. This means that the operator C is self-adjoint with respect to the scalar product (4.4). According to a basic result of linear algebra, the eigenvalues of C are therefore real. Let λ be one of these eigenvalues, and $\partial\mathbb{V}$ an eigenfunction corresponding to λ, so that $C(\partial\mathbb{V}) = \lambda\partial\mathbb{V}$. Setting $\partial\mathbb{V}' = \partial\mathbb{V}$ in (4.5) one obtains

$$\lambda\langle \partial\mathbb{V}, \partial\mathbb{V} \rangle = -\int_S [\nabla \cdot (\varphi\partial\mathbb{V})]^2\, dS$$

which shows that λ is negative except if

$$\nabla \cdot (\varphi\partial\mathbb{V}) = 0 \quad \text{everywhere on } S \tag{4.6}$$

It is obvious from (4.2) that any perturbation $\partial\mathbb{V}$ which satisfies this condition is an eigenfunction of C, associated with eigenvalue 0. All the eigenvalues of C are therefore negative, except one which is 0. This is the case (iii) of subsection 3.3 for which no direct conclusion is possible as to the convergence of an assimilation.

Ignoring for the time being the $o(\Delta t^2)$ term in (4.3) we see that, for small Δt, the eigenvalues of the amplification matrix have moduli strictly less than 1, except one eigenvalue which is equal to 1, and is associated with perturbations verifying (4.6). Accordingly the wind field will be reconstituted in an assimilation, except for a residual difference $\partial\mathbb{V}_\infty$ verifying (4.6). Moreover

—we see from eq. (4.1a) that the knowledge of φ and $\partial\varphi/\partial t$ at a given time defines the wind field up to an additional vector field satisfying (4.6). An assimilation will therefore reconstitute all the information contained in $\partial\varphi/\partial t$. If more than two observations of φ are available, the information they contain can possibly influence the assimilation only through the $o(\Delta t^2)$ term.

—we see by taking the curl of (4.3) that the vorticity of the wind difference is not modified in an assimilation cycle. The residual difference $\partial \mathbb{V}_\infty$ will therefore be defined by the following two conditions: it satisfies (4.6) and its vorticity is *equal to the vorticity of the difference at the beginning of the assimilation.*

These results are true only if Δt is small enough so that no eigenvalue of the amplification matrix (4.3) is larger than 1. Still ignoring the $o(\Delta t^2)$ term, this condition reads

$$-1 < 1 + \lambda_M \Delta t^2$$

or

$$\Delta t < \sqrt{-\frac{2}{\lambda_M}} \tag{4.7}$$

where λ_M is the (negative) eigenvalue of C with the largest modulus. It is not difficult to see that condition (4.7) is basically the same as the Courant–Friedrichs–Lewy condition for stability of a numerical integration of eqs. (4.1). It leads for Δt to values of the same magnitude, typically $\Delta t < 15$ min for ordinary spatial resolutions.

The additional term $o(\Delta t^2)$ in (4.3) turns out to be too complex to be studied analytically, and numerical experiments have been performed in order to determine if, and how, it modifies the above results. These experiments, of the identical-twin type, were performed with a barotropic version of a spherical grid-point primitive equation model developed at Laboratoire de Météorologie Dynamique and described in detail in Sadourny (1975). (The same basic model was used for the numerical experiments that have been reported in T81.) The grid is a latitude-longitude staggered grid (of type C, in the terminology of Arakawa and Lamb, 1977). The spatial differencing schemes conserve the total mass, kinetic energy, potential energy and enstrophy of the flow. It was specifically checked that all the properties required for the validity of the above analytical developments were conserved by these schemes.

For the experiments described here, a rather low spatial resolution (40 grid-points along a latitude circle, 25 gridpoints between both poles along a meridian) was used, together with a discretization timestep $\Delta\tau = 12$ min. Moreover, two modifications were added to the original model for the specific purpose of the present experiments:

—in accordance with eqs. (4.1) all energy sources and sinks were removed;

—the object of the experiments being to assimilate, through a forward-backward procedure, observations separated by one timestep, it was necessary to use a one-level time differencing scheme which was exactly reversible with respect to time. The scheme originally used in the model was the leapfrog scheme which, being a two-level scheme, did not meet these requirements. A one-level exactly reversible scheme was therefore specially developed. This scheme is described in T77. It was specifically checked that the use of this scheme did not alter either of the eqs. (4.2) or (4.3).

The total number of parameters of a shallow-water equation model is three times the number of parameters defining the mass field. Two successive observations of the latter cannot consequently define the complete state of the flow. Accordingly, the experiments were performed with three successive observations of the complete mass field φ, separated by Δt. Equation (3.4) is then replaced by eq. (3.7), with $N = 2$. Condition (4.7) is replaced by the still stricter condition

$$\Delta t < \sqrt{-\frac{1}{\lambda_M}} \equiv \Delta t_C \tag{4.8}$$

For the spatial resolution used in the present experiments the value of Δt_C is about 12.6 min. Two different values were used for the time interval Δt between successive observations. The first one, equal to one model timestep

$$\Delta t = \Delta\tau = 12 \text{ min} \simeq 0.95\ \Delta t_C$$

satisfies condition (4.8). The second one, three times as large

$$\Delta t = 3\Delta\tau \simeq 2.85\ \Delta t_C$$

does not.

Fig. 2 shows the variations of the root-mean-square wind difference in the two experiments. The difference decreases in both cases and more rapidly, for the same number of assimilation cycles, with the larger value of Δt. This means that, when Δt reaches the limit value Δt_C, the $o(\Delta t^2)$ term is no more negligible, and indeed is such as to

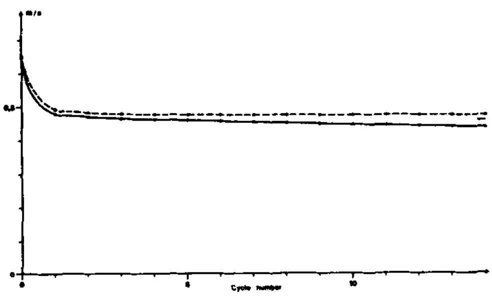

Fig. 2. Variations, as functions of the number of assimilation cycles, of the rms wind difference in two assimilations of mass observations. The upper curve corresponds to a time interval between observations $\Delta t = \Delta t_c$, and the lower curve to $\Delta t = 3\Delta t_c$.

decrease further the spectral radius of the amplification matrix. An additional fact of interest can be seen from Fig. 3, which shows the variations of the rms vorticity difference in the same two assimilations. As said above, the vorticity is not modified by the first two terms of development (3.7), and only the $o(\Delta t^2)$ term can account for a possible variation of the vorticity difference. Fig. 3 shows that this difference decreases for both values of Δt, and more rapidly for the larger value.

It thus appears that, at least in the case of the numerical model used here, the $o(\Delta t^2)$ term contributes to the convergence of an assimilation and, particularly, leads to the reconstitution of the rotational part of the wind field. The basic mathematical reason for this has not been rigorously established, but two facts strongly suggest that the role of the Coriolis acceleration is here fundamental. First, it has been shown in T81, that, in the case of the linearized equations, it is the Coriolis acceleration, which is then the only interaction between divergence and vorticity, which ensures the reconstitution of vorticity. The second fact is apparent from Fig. 4, which shows, as functions of latitude, the proportion of the initial rms difference remaining after a given time of assimilation on divergence and vorticity respectively (this figure refers to the assimilation performed with the larger value $\Delta t = 3\Delta\tau$). The rate of reduction of the divergence difference is rapid and exhibits no variation with latitude. The rate of reduction of the vorticity difference, on the contrary, is slow, particularly in low latitudes. This suggests that the reconstitution of vorticity depends to a large extent on the Coriolis parameter. It must be noted, however, that the vorticity difference is reduced at all latitudes, including at the equator. In T81, the same fact was observed only with non-linear equations, and was ascribed to an additional effect of advection. The same explanation probably applies also here.

Because of the particular conditions under which the results presented here have been obtained (small time interval between successive observations, no dissipation), they are not directly relevant to the practical problem of assimilation. But they do provide a clear description of the processes at play in an assimilation of mass observations. First, and because of the direct influence of data introduction upon the divergence field, the latter is reconstructed at a rapid rate. The vorticity field is also reconstructed, but indirectly and more slowly, through Coriolis acceleration and, to a lesser

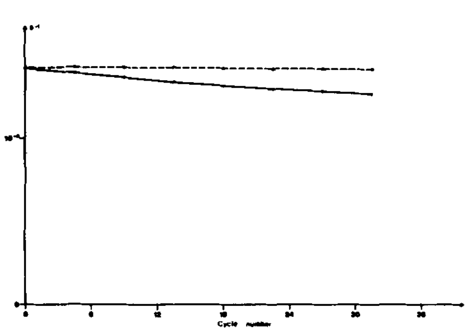

Fig. 3. Same as Fig. 2, for the variations of the rms vorticity difference.

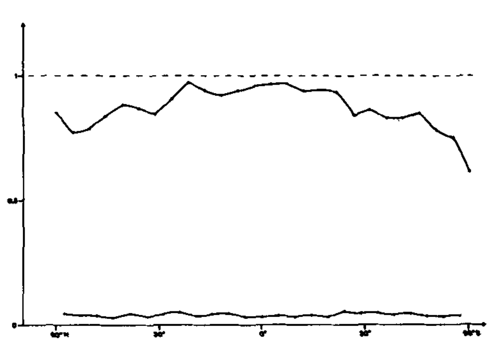

Fig. 4. Latitudinal distribution of the proportion of initial rms difference remaining after a given time of assimilation. Upper curve: vorticity rms difference. Lower curve: divergence rms difference.

extent, through advection. Except for the effect of advection, this qualitative description is identical to the one obtained in T81 for the linear equations.

It is noteworthy that the results presented here are independent on any specific features of the solution to be reconstituted and, in particular, on whether or not this solution of geostrophic. Nor do they require the presence in the assimilating model of any dissipative process, intended for instance at restoring the geostrophic balance after it has been disrupted by the introduction of observations. These facts are obvious for the theoretical results, which have been derived without any particular hypothesis about the "observed" solution, and without assuming the presence of any dissipative process. As for the numerical results, they have been obtained from an "observed" solution which had been produced by numerical integration of the inviscid eqs. (4.1). For some reason, this integration produced a rather large amount of gravity waves. This in no way prevented the reconstitution of both the divergent and rotational part of the wind field. The fact that neither geostrophicity, nor a process of geostrophic adjustment, is basically necessary for the success of an assimilation, has already been noted in T81 in the case of the linearized equations. We will come back to this point in the conclusion.

We have considered the case of successive observations of the complete mass field. A similar study can be made in the case of successive observations of the complete wind field (see T77). This study leads to the conclusion that a forward-backward assimilation performed on successive observations, close in time, of the wind field will reconstitute the complete mass field, up to an additive constant. The latter restriction corresponds to the fact that the total mass of the fluid can vary neither in a model integration nor in an introduction of wind observations. The total mass of the observed solution cannot therefore be reconstituted in an assimilation of wind observations. Neglecting the $o(\Delta t^2)$ term in the expression for the relevant amplification matrix leads for the time interval Δt between successive observations of the wind field to a limit value which turns out to be the same as for observations of the mass field (eq. (4.7)). As in the case of observations of the mass field, numerical experiments show (see T77) that an assimilation still converges for values of Δt larger than the limit (4.7).

5. The effect of observing and modelling errors

We have exclusively considered so far the "identical twin" case, in which observations are supposed to be exactly compatible with one model solution. This is not so in reality because of observing and modelling errors. We will now extend the formalism of Section 3 to the case when such observing and/or modelling errors are present.

First, one particular solution of the model (which need not be defined more precisely at this stage) is chosen *as a reference solution*. Then, for each observed datum, an "error" is defined as the difference between the datum itself and the corresponding value for the reference solution just chosen. The vector made up of all the errors thus defined will be noted E. With the notations of Section 3, the dimension of E is equal to the product Np of the number N of observation times by the dimension p of the observed vector X.

Denoting again by ΔY_n the Y-difference between the model state and the reference solution at the end of the nth assimilation cycle, a derivation similar to that of Section 3 leads to the following relationship between ΔY_n and ΔY_{n+1}

$$\Delta Y_{n+1} = A\Delta Y_n + BE + o(\Delta Y_n, E) \tag{5.1}$$

where A and B are matrices with respective dimensions $q \times q$ and $q \times Np$. A is the same matrix as in (3.1) since (5.1) must reduce to (3.1) when $E = 0$. The matrix B, like the matrix A, is entirely determined by the resolvent matrix of the linearized perturbation system (2.4) in the vicinity of the reference solution.

Now the question is: as the number n of assimilation cycles increases to infinity, and E remains constant, how will ΔY_n behave? The following result is proved in T77. If the spectral radius of A is strictly less than 1, and if E and ΔY_0 are small enough, ΔY_n will tend to a limit as n will tend to infinity. However, this limit will not in general be 0. This result means that, if the observed values remain close enough to one particular model solution, and if the spectral radius of the amplification matrix corresponding to that solution is strictly less than 1, then the assimilation will converge to a limit. However, the corresponding model states at times $t_1, t_2 \ldots t_N$ will not in general lie on one model solution.

The proof given in T77 is too general to provide information of specific interest for the meteorological problem. It does not in particular allow any practical estimate of how "small" E must be for an assimilation to remain convergent. It is probably mostly through numerical experimentation that precise information can be obtained on this point.

6. A theoretical method for accelerating convergence

There is no reason why the simple updating procedure considered so far, in which the vector Y is not modified when the vector X is updated, should be the most efficient one, except maybe if, according to the remark made in Section 3, the nature of Y has been appropriately chosen. In the case when X represents the mass field of the atmosphere, and Y the wind field, this procedure leads to a slow reconstitution of the rotational part of the wind field. More generally, it ignores an important source of information, namely, the values predicted for X, which are discarded at each introduction without being used in any way. The following simple example will show that this information can be useful.

Let us consider the linearized perturbation system (2.4), and an assimilation started at time t_1 from an unknown Y difference $\delta Y(t_1)$. At the next introduction time t_2, the X and Y differences are respectively $\delta X(t_2) = R^y_x \delta Y(t_1)$ and $\delta Y(t_2) = R^y_y \delta Y(t_1)$ (eqs. (2.5)). The difference $\delta X(t_2)$ is known since it is the difference between the predicted and observed values for X. If the matrix R^y_x is inversible in the sense that the knowledge of $\delta X(t_2)$ uniquely defines $\delta Y(t_1)$ (this is possible only if $p \geqslant q$), the latter can be written as $\delta Y(t_1) = (R^y_x)^{-1} \delta X(t_2)$ where $(R^y_x)^{-1}$ is a perfectly defined $q \times p$ matrix. The difference $\delta Y(t_2)$ is then equal to $R^y_y(R^y_x)^{-1} \delta X(t_2)$ and we see that by adding to the value predicted for Y the following correction

$$\delta' Y = -R^y_y(R^y_x)^{-1} \delta X(t_2) \tag{6.1}$$

the Y-difference is reduced to 0, i.e. the assimilation has reconstructed the complete state of the system.

The matrix R^y_x will of course not always be inversible, but this example suggests that applying on Y at each introduction time a correction

$$\Delta' Y = D \Delta X \tag{6.2}$$

where ΔX is the known X-difference, and D is an appropriate $q \times p$ matrix, can accelerate the convergence of an assimilation process, or render it convergent if it is otherwise divergent. Many assimilation techniques have indeed already been defined, some of which are in operational use, which belong to the general scheme (6.2). The "optimal analysis" when performed with the model forecast as "first-guess" as is most often the case (see e.g. Lorenc et al., 1977) is one of them. The matrix D, which is then determined on the basis of statistical considerations, is in that case equal to $-QS^{-1}$, where Q is the $q \times p$ matrix of the covariances of ΔY and ΔX, and S is the $p \times p$ matrix of the variances-covariances of ΔX. Two schemes, intended at restoring the geostrophic balance disrupted by the introduction of observations, have been defined by Kistler and McPherson (1975) and by Daley and Puri (1980) (scheme E). These schemes are particular cases of (6.2) and have been shown to accelerate the convergence. For still another example of (6.2), see Tadjbakhsh (1969).

We will here consider the following question. How is the amplification matrix of (3.2) modified by a correction of type (6.2)? We will assume that the correction matrix D can vary with the introduction time, and will again use the auxilliary time variable τ of Section 3. It is easily seen from (2.7b) and (6.2) that the difference $\Delta Y(\tau_{i+1})$ after introduction of observations at time τ_{i+1} depends on the difference after introduction of observations at time τ_i through the following relationship

$$\Delta Y(\tau_{i+1}) = R^y_y(\tau_{i+1}, \tau_i)\, \Delta Y(\tau_i) + D(\tau_{i+1})\, R^y_x(\tau_{i+1}, \tau_i)\, \Delta Y(\tau_i) + o(\Delta Y(\tau_i))$$

The amplification matrix over one complete assimilation cycle consequently becomes

$$A' = P(\tau_1, \tau_M)\, P(\tau_M, \tau_{M-1}) \ldots P(\tau_2, \tau_1) \tag{6.3}$$

where for any i $(i = 1, 2 \ldots, M)$

$$P(\tau_{i+1}, \tau_i) = R^y_y(\tau_{i+1}, \tau_i) + D(\tau_{i+1})\, R^y_x(\tau_{i+1}, \tau_i) \tag{6.4}$$

A correction of type (6.2) will be useful if the resulting spectral radius $\rho(A')$ is smaller than the original spectral radius $\rho(A)$ of (3.2) (and of course if $\rho(A')$ is also smaller than 1 in the case $\rho(A)$ was not). A particularly interesting case is when the matrix A' can be made equal to 0. In such a case, the decrease of the Y-difference as the assimilation proceeds will be faster than exponential. The following theorem holds.

Theorem (T): *The matrix A' can be made equal to 0 by an appropriate choice of the correction matrices $D(\tau_l)$ if, and only if, the following condition (C) is satisfied*

(C): *the only solution of the linearized perturbation system (2.4) which satisfies the condition $\delta X(t_i) = 0$ at all observation times t_i ($i = 1, \ldots, N$) is the null solution $\delta X(t) \equiv \delta Y(t) \equiv 0$.*

Condition (C) essentially means that, in the approximation defined by the linearized system (2.4), the available observations $X(t_i)$ uniquely define the observed solution.

The proof of theorem (T), which resorts only to basic notions of linear algebra, is given in the Appendix. It turns out that a complete assimilation cycle is not necessary to make the matrix A' equal to 0, but that the product of the matrices P of (6.4) over either of the two phases (forward or backward) of a cycle can be made null by an appropriate choice of the correction matrices $D(\tau_l)$. Also, it is not necessary for theorem (T) to hold that the same parameters be observed at the successive observation times, but the nature and even the numbers of the observed parameters can vary with time. Moreover, the set of correction matrices which make A' equal to zero is in general not unique.

Theorem (T) can easily be illustrated with the linearized shallow-water equations on an f-plane. For wave-vector $\mathbb{k}$, these equations read

$$\frac{d\varphi}{dt} + \Phi_0 \mathscr{D} = 0 \tag{6.5a}$$

$$\frac{d\mathscr{D}}{dt} - \mathbb{k}^2 \varphi - f\zeta = 0 \tag{6.5b}$$

$$\frac{d\zeta}{dt} + f\mathscr{D} = 0 \tag{6.5c}$$

where φ is the deviation of the geopotential from its mean value Φ_0 and $\mathscr{D}$ and ζ are respectively the divergence and vorticity of the velocity field. The geopotential φ standing for X, and the velocity field $(\mathscr{D}, \zeta)$ for Y, the matrices R_x^y and R_y^y between two observation times t and $t + \Delta\tau$ have been determined in T81. They are (formula (4.2) of T81)

$$R_x^y = -\frac{\Phi_0}{\alpha}(\sin\beta \quad \gamma(1-\cos\beta))$$

$$R_y^y = \begin{pmatrix} \cos\beta & \gamma\sin\beta \\ -\gamma\sin\beta & 1-\gamma^2(1-\cos\beta) \end{pmatrix}$$

where

$$\alpha^2 = f^2 + \mathbb{k}^2\Phi_0, \quad \gamma = \frac{f}{\alpha}, \quad \beta = \alpha\Delta\tau$$

Given N successive observations of φ, separated by $\Delta\tau$, it has been shown in T81 that these observations uniquely define the corresponding solution of (6.5), i.e. they satisfy the above condition (C), if, and only if, the following conditions are simultaneously verified

$N \geqslant 3$
$\beta \neq l\pi, \quad l$ integer
$\gamma \neq 0$
$\gamma \neq 1$

It has also been shown in T81 that, under these conditions, the spectral radius of the amplification matrix corresponding to an assimilation performed without correction of type (6.2), is strictly less than 1. Theorem (T) tells us that, under these same conditions, there exist correction matrices D, with dimensions 2×1, which make the matrix A' of (6.3) equal to 0. Since condition (C) requires only $N \geqslant 3$, two such correction matrices must be sufficient. There must therefore exist two 2×1 matrices D_1 and D_2 such that

$$(R_y^y + D_2 R_x^y)(R_y^y + D_1 R_x^y) = 0$$

An easy calculation shows that one solution for this equation is

$$D_1 = D_2 = \frac{\alpha}{2\Phi_0} \begin{pmatrix} \dfrac{1+2\cos\beta}{\sin\beta} \\ \dfrac{1-2\gamma^2(1-\cos\beta)}{\gamma(1-\cos\beta)} \end{pmatrix}$$

This is always defined, except for values of Φ_0, β, γ for which condition (C) is not satisfied anyway.

Theorem (T) provides a theoretical basis for optimizing the convergence of an assimilation.

However, the explicit computation of a set of "optimal" correction matrices $D(\tau_l)$ in an operational assimilation raises a number of difficulties, the most basic of which is the following. The optimal matrices depend in the linearized perturbation system (2.4). The latter, in the case when the basic eqs. (2.2) are non-linear, depends in turn on the observed solution, which is precisely what is being looked for. It is therefore certainly impossible to determine the correction matrices which make the amplification matrix A' exactly equal to 0. But, since an assimilation performed with meteorological equations already converges with no correction of type (6.2) at all, it is reasonable to assume that its convergence can be accelerated by optimal matrices corresponding, not to the solution which is actually observed, but to some already known solution which can be considered as being some approximation of the observed solution. Studies are presently being carried out in order to assess the practicality of this approach.

One last remark about the general correction scheme (6.2) is of interest. We have mentioned in Section 3 that, given observations of given parameters making up the vector X, the choice of the complementary vector Y is arbitrary under the only condition that X and Y together completely define the state of the system under observation. More precisely, if Y is a possible choice, any q-vector W will also be a possible choice if W and X together completely define Y, i.e. if Y is a perfectly defined function R of W and X

$$Y = R(W, X)$$

An obvious question is then: How will the convergence properties of an assimilation be modified if the assimilation is performed in such a way that it is W, and not Y, which is left unmodified when observations of X are introduced?

Let us denote by ΔW and ΔX respectively the W and X differences, before an introduction of observations, between the model and the reference solutions. The corresponding Y difference is

$$\Delta Y_1 = \frac{DR}{DW} \Delta W + \frac{DR}{DX} \Delta X + o(\Delta W, \Delta X)$$

where the jacobian matrices DR/DW and DR/DW are taken on the reference solution. Introducing the observations of X without modifying W amounts to setting ΔX equal to 0, ΔW not being modified. The Y difference becomes

$$\Delta Y_2 = \frac{DR}{DW} \Delta W + o(\Delta W)$$

The change in Y resulting from the introduction of observations is therefore

$$\Delta' Y = \Delta Y_2 - \Delta Y_1 = -\frac{DR}{DX} \Delta X + o(\Delta W, \Delta X)$$

Except for the $o(\Delta W, \Delta X)$ term (irrelevant as long as we are concerned with the amplification matrix only), this expression is of type (6.2), the correction matrix D being equal to $-DR/DX$.

To first order with respect to ΔX, a change of representation from Y to W is therefore equivalent to a correction of type (6.2). It is easy to show that, conversely, any correction of type (6.2) can be interpreted as a change of representation from Y to some appropriate W.

7. Conclusion

A general criterion for convergence of a forward-backward assimilation has been derived. Although the proof has been obtained under somewhat simplified and idealized conditions, the general principles involved can be used in more complex situations, as it has in effect been done in Sections 5 and 6. Also, the hypothesis made that the assimilation is performed according to a forward-backward procedure is not fundamentally necessary. This hypothesis has been made because a forward-backward assimilation is an exactly iterative process, which lends itself more easily to a rigorous mathematical treatment. The case of a purely forward assimilation, in which the model is constantly integrated forward in time, and new data constantly fed into it, is mathematically more complex, since it is necessary to consider the possible asymptotic properties of the model solutions as time goes to infinity. But the general principles presented here remain valid and, in all cases, the convergence of an assimilation process will depend on an appropriate amplification matrix, which itself depends on the linearized perturbation system in the vicinity of the solution to be reconstructed.

In spite of its generality, the convergence criterion established in this article can be explicitly used in a number of specific cases. Examples have already been considered in T81 for the linearized meteorological equations, and another example treated here for the non-linear shallow-water equations. At the price of some additional analytical and/or theoretical work, the approach taken in these two articles can certainly be extended to other cases, for example to equations linearized in the vicinity of a zonal flow.

In the case of an assimilation of mass observations, performed without direct modification of the wind field, the theoretical and numerical results obtained in this article lead to a simple description of the effects of the assimilation: the divergent component of the wind field is reconstructed more rapidly, through direct influence from the mass field. This fact has already been mentioned by Rutherford and Asselin (1972). The rotational part is reconstructed more slowly, through the effect of Coriolis acceleration and, to a lesser extent, through the effect of advection. These results are summarized by the curves of Fig. 4. They generalize the results obtained in T81 in the particular case of the linearized equations.

One conclusion common to T81 and to the present paper is that the convergence of an assimilation does not require the presence in the model of a process capable of re-establishing geostrophic balance whenever the latter is disrupted. This fact is of great theoretical interest since, up to now, it has always been assumed more or less implicitly that it is because of geostrophic adjustment that an assimilation can converge at all. Our results show that it is not so, with the consequence that it is not necessarily by trying and modifying the properties of the model's geostrophic adjustment that an assimilation process will be made more efficient. This of course does not mean that we must ignore the fact that the solution to be reconstructed is in geostrophic balance, but simply means that processes other than geostrophic adjustment play a basic role in an assimilation and must be taken into account.

Another conclusion arising from our results is that the divergent part of the wind field can relatively easily be reconstructed from the observed history of the mass field. In all present assimilation procedures, the final value of the divergence is determined mostly by the initialization step, which is intended at suppressing unrealistic gravity waves. But it is not known to which accuracy the real value of the divergence is reconstructed by the initialization. In view of the fact that the divergence, in spite of its relatively small magnitude, is dynamically important, it would be of interest to define a method for determining its value with all the accuracy allowed by the observations. The results presented in this paper, together with those presented in T81, provide the theoretical basis of such a method.

Now, before the ideas developed in this paper can be used with full profit, more efficient numerical methods are required, and especially methods capable of reconstituting more rapidly the rotational part of the wind field, when only mass observations are available. This can be done on the basis of the theorem presented in Section 6, which describes a theoretical possibility for extracting the information contained in successive observations distributed in time. It is worth mentioning, however, that methods based on this theorem will allow to extract only what can be called the *dynamical* information, which can be obtained from the evolution equations of the system. They will ignore any kind of *statistical* information that can be obtained from known statistical properties of the fields to be reconstructed. Indeed, such statistical information is commonly used in present assimilation procedures, under the form, for instance, of "structure functions". Once it has been established to which extent the dynamical information can be useful for assimilation, it will presumably be necessary to define a method for an appropriate combined use of both types of information.

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9. Appendix

Proof of theorem (T)

The amplification matrix A' (eq. 6.3) can be made equal to 0 by an appropriate choice of the correction matrices $D(\tau_i)$ if, and only if, the following condition (C) is satisfied.

(C) The only solution of the linearized perturbation system (2.4) such that $\delta X(t_i) = 0$ at all observation times t_i ($i = 1, \ldots, N$) is the null solution $\delta X(t) \equiv \delta Y(t) \equiv 0$.

Proof

A. Let us assume first that condition (C) is not satisfied. There exist values ($\delta X(t_1) = 0$, $\delta Y(t_1) \neq 0$) such that the corresponding solution of (2.4) verifies $\delta X(t_i) = 0$ at all observation times t_i. Let us consider an assimilation cycle started from such values. The integration from t_1 to t_2 will produce $\delta X(t_2) = 0$, i.e. the values predicted for X will be identical with the observed values. For any matrix $D(t_2)$, the corresponding correction (6.2) on Y will be 0. This reasoning, repeated successively for all observation times shows that the differences (δX, δY) will always follow the same solution of (2.4) along the cycle, and will assume at the end of the cycle their original values ($\delta X(t_1) = 0$, $\delta Y(t_1) \neq 0$). Therefore

$$A' \, \delta Y(t_1) = \delta Y(t_1)$$

which shows that A' cannot be 0.

B. Let us now assume that condition (C) is satisfied. We will first prove the following lemma.

Lemma. *Let A and B be two matrices, with respective dimensions $q \times q$ and $p \times q$, such that any row-vector of A belongs to the space generated by the row-vectors of B. There exists a $q \times p$ matrix G such that*

$$A = GB \tag{A.1}$$

To prove this lemma, we note that for any i ($i = 1, \ldots q$), the ith row-vector A_i of A can be expressed in one way at least as a linear combination of the row-vectors of B. Let G_i be a row-vector of dimension p whose components are the coefficients of such a linear combination. Then

$$A_i = G_i B$$

The $q \times p$ matrix G whose row-vectors are the vectors G_i thus defined satisfies condition (A.1), which proves the lemma.

Let us now define ($i = 1, \ldots, N-1$)

$$S_i = R_y^y(t_{i+1}, t_i)$$

$$T_i = R_x^y(t_{i+1}, t_i)$$

and consider a solution of the linearized perturbation system (2.4), corresponding to the initial conditions ($\delta X(t_1)$, $\delta Y(t_1)$). It is easy to see, by induction on i, that the values ($\delta X(t_i)$, $\delta Y(t_i)$) assumed by that solution at time t_i ($i \geqslant 2$) can be written as

$$\delta X(t_i) = L_i(\delta X(t_1), \ldots, \delta X(t_{i-1})) + T_{i-1} S_{i-2} \ldots S_1 \, \delta Y(t_1)$$

$$\delta Y(t_i) = M_i(\delta X(t_1), \ldots, \delta X(t_{i-1})) + S_{i-1} \ldots S_1 \, \delta Y(t_1)$$

where L_i and M_i are linear combinations, with matrix coefficients, of $\delta X(t_1), \ldots, \delta X(t_{i-1})$.

Condition (C) requires that if $\delta X(t_1) = \delta X(t_2) = \cdots = \delta X(t_N) = 0$, then necessarily $\delta Y(t_1) = 0$. This means that it is possible to extract from the $(N-1)p \times q$ matrix

$$\begin{pmatrix} T_1 \\ T_2 S_1 \\ \cdot \\ \cdot \\ \cdot \\ T_{N-1} S_{N-2} \ldots S_1 \end{pmatrix}$$

a non-zero determinant of order q, or equivalently that the space generated by the $(N-1)p$ row-vectors of this matrix has dimension q.

Let us call $\mathscr{E}_1$ the space generated by the row-vectors of matrix T_1 ($\mathscr{E}_1$ may consist of only the null vector if all the entries of T_1 are 0). Let us then call $\mathscr{E}_2$ a subspace of the space generated by the row-vectors of matrix $T_2 S_1$ such that any vector of the space generated by the row-vectors of

$$\begin{pmatrix} T_1 \\ T_2 S_1 \end{pmatrix}$$

can be uniquely decomposed as the sum of a vector of $\mathscr{E}_1$ and a vector $\mathscr{E}_2$. We can thus define recursively a sequence of spaces $\mathscr{E}_1, \mathscr{E}_2, \ldots, \mathscr{E}_{N-1}$ such that, for any i ($i = 1, \ldots, N-1$) $\mathscr{E}_i$ is a subspace of the space generated by the row-vectors of the matrix $T_i S_{i-1} \ldots S_1$ and such that any q-vector Y can be uniquely decomposed as a sum

$$Y = Y_1 + Y_2 + \cdots + Y_{N+1}$$

where Y_i belongs to $\mathscr{E}_i$ for all i's. Some of the spaces $\mathscr{E}_i$ may consist of only the null vector.

Let us denote E_i the $q \times q$ matrix which to any row vector Y with dimension q, associates its projection $Y_i = YE_i$ on the space $\mathscr{E}_i$. The matrices E_i verify the following condition

$$E_1 + E_2 + \cdots + E_{N+1} = I \qquad \text{(A.2a)}$$

where I is the identity matrix of order q, and multiply according to the following rules

$$E_i E_j = 0 \quad \text{if } i \neq j \qquad \text{(A.2b)}$$

$$E_i^2 = E_i \quad i = 1, \ldots, N-1 \qquad \text{(A.2c)}$$

We want to define correction matrices $D_1, D_2 \ldots$ such that the amplification matrix (6.3) is 0. It is sufficient for that to define $N - 1$ matrices $D_1, D_2, \ldots, D_{N-1}$ such that the amplification matrix over the forward phase of an assimilation cycle is 0.

$$(S_{N-1} + D_{N-1} T_{N-1}) \ldots (S_1 + D_1 T_1) = 0 \qquad \text{(A.3)}$$

For any i and any j ($i = 1, \ldots, N-1; j = 1, \ldots, q$) the jth row-vector of the $q \times q$ matrix $S_i \ldots S_1 E_i$ belongs to the space $\mathscr{E}_i$, since it is the projection on that space of the q-vector $Q_j S_i \ldots S_1$, where Q_j is the row-vector of dimension q whose jth component is 1, and whose other components are 0. Moreover the row-vectors of the $p \times q$ matrix $T_i S_{i-1} \ldots S_1$ generate a space of which $\mathscr{E}_i$ is by definition a subspace. $S_i \ldots S_1 E_i$ standing for A, and $T_i S_{i-1} \ldots S_1$ for B, the lemma proved above is applicable, and there exists a $q \times p$ matrix D_i such that

$$-S_i \ldots S_1 E_i = D_i T_i S_{i-1} \ldots S_1$$

We are now going to prove that the matrices D_i thus defined satisfy condition (A.3). Using eqs. (A.2) we successively find

$$S_1 + D_1 T_1 = S_1 - S_1 E_1 = S_1 \left(\sum_{j=2}^{N-1} E_j \right)$$

then

$$\begin{aligned}
(S_2 + D_2 T_2)(S_1 + D_1 T_1) &= (S_2 + D_2 T_2) S_1 \left(\sum_{j=2}^{N-1} E_j \right) \\
&= (S_2 S_1 + D_2 T_2 S_1) \left(\sum_{j=2}^{N-1} E_j \right) \\
&= (S_2 S_1 - S_2 S_1 E_2) \left(\sum_{j=2}^{N-1} E_j \right) \\
&= S_2 S_1 \left(\sum_{j=3}^{N-1} E_j \right)
\end{aligned}$$

and, by induction on i

$$(S_i + D_i T_i) \ldots (S_1 + D_1 T_1) = S_i \ldots S_1 \left(\sum_{j=i+1}^{N-1} E_j \right)$$

This leads finally to

$$\begin{aligned}
&(S_{N-1} + D_{N-1} T_{N-1}) \ldots (S_1 + D_1 T_1) = \\
&= (S_{N-1} + D_{N-1} T_{N-1}) S_{N-2} \ldots S_1 E_{N-1} \\
&= (S_{N-1} S_{N-2} \ldots S_1 \\
&\quad + D_{N-1} T_{N-1} S_{N-2} \ldots S_1) \; E_{N-1} = 0
\end{aligned}$$

which shows that condition (A.3) is satisfied, and completes the proof of theorem (T).

It is worth mentioning that, since the choice of the subspaces $\mathscr{E}_i$ is in general not unique, neither is that of the matrices D_i. Also, it is not difficult to check that the above proof can be extended without any basic change to the case when the dimension of the observed vector X (and, consequently, of the complementary vector Y) varies with the observation time t_i.

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О МАТЕМАТИКЕ АССИМИЛЯЦИИ ДАННЫХ

Проблема сходимости ассимиляции "вперед-назад" рассматривается для наиболее общей динамической системы. С использованием элементарной техники теории устойчивости показано, что на одном цикле ассимиляции вариация разности между ассимилирующей моделью и состоянием, которое надо реконструировать, с точностью первого порядка определяется идеально определенной матрицей усиления. Это ведет к прямому критерию сходимости, зависящему от собственных значений этой матрицы.

Этот критерий сходимости, как было показано в предыдущей статье, должен проверяться линеаризованными метеорологическими уравнениями. Здесь показано, что он должен проверяться нелинейными уравнениями теории мелкой воды в случае последовательных наблюдений поля геопотенциала, по крайней мере, если эти наблюдения достаточно близки во времени. Численные эксперименты поддерживают теоретические результаты и вместе они ведут к следующему описанию эффектов ассимиляции данных наблюдений геопотенциала. Дивергентная часть поля ветра реконструируется быстрее, потому что на нее прямо влияет введение данных наблюдений. Соленоидальная часть реконструируется косвенным образом и медленнее, главным образом, через эффект ускорения Кориолиса. Эти результаты не зависят от того, является ли реконструируемое поле геострофическим, или нет, и не требует присутствия каких-либо диссипативных процессов в модели ассимиляции.

Рассматривается роль ошибок наблюдения и или моделирования и показано, что малые ошибки не меняют свойств сходимости процесса ассимиляции. Наконец, предложен теоретический метод, с помощью которого матрица усиления на одном цикле ассимиляции может быть сделана равной О, оптимизируя тем самым процесс сходимости.