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On internal solitary waves. II

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ABSTRACT

The asymptotic evolution of straight-crested internal solitary waves, from prescribed initial conditions, in that parametric domain in which quadratic nonlinearity, cubic nonlinearity, and dispersion are weak and of comparable significance is calculated through inverse-scattering theory. An initially rectangular displacement, which presumably serves as an example of any initial displacement of compact support, yields N solitary waves, where $N \geq 1$. The fastest of these waves may resemble a bore (which, by definition, implies a net change in surface level between upstream and downstream limits) but is evanescent in both the upstream and downstream limits. An initial change of surface level of the right strength with a rise facing in the direction of propagation is found to yield (asymptotically) a true bore, but the asymptotic solution for an opposite facing rise does not comprise such a component. This last difficulty suggests a problem of uniqueness for initial displacements that are not of compact support.

1. Introduction

I consider here the asymptotic evolution of straight-crested internal solitary waves, from prescribed initial conditions, in that parametric domain in which quadratic nonlinearity, cubic nonlinearity, and dispersion are weak and of comparable significance. Appropriate choices of the amplitude and horizontal length scales a and l then yield the Korteweg–deVries-like evolution equation [I(5.6)[1] with $\mathscr{A} \equiv \eta, \mathscr{U}_1 = 2$, and $\mathscr{U}_2 = 1$]

$$\eta_\tau = 24\eta(1-\eta)\eta_\xi + \eta_{\xi\xi\xi}, \tag{1.1}$$

where ξ is a dimensionless coordinate in a reference frame moving to the left with the velocity of the basic internal wave (of which there may be a discrete family with different vertical profiles—see I), τ is a dimensionless, slow time, and η is the dimensionless, vertical displacement of a reference surface (e.g., the interface in a two-layer model).

It is evident that (1.1) is invariant under the transformation

$$\eta_* = 1 - \eta. \tag{1.2}$$

However, the energy measure $\int_{-\infty}^{\infty} \eta_*^2 \, d\xi$ diverges if $\int_{-\infty}^{\infty} \eta^2 d\xi$ converges and conversely. The divergence of this integral implies that the corresponding solution cannot be uniformly valid over the entire domain $-\infty < \xi < \infty$, although it may represent a useful approximation in some more limited domain.

A family of solitary-wave solutions of (1.1) is given by [cf. I(3.15) after allowing for the present scaling]

$$\eta(\xi,\tau;\mu) \equiv \eta_\mu = 2\mu(1+\mu)^{-1} (\cosh^2\theta - \mu \sinh^2\theta)^{-1}, \tag{1.3a}$$

where

$$\theta = \kappa\xi + 4\kappa^3\tau + \lambda, \quad \kappa = 2\mu^{\frac{1}{2}}/(1+\mu), \tag{1.3b, c}$$

μ is the family parameter, and λ is a phase constant. The solution (1.3) is formally valid for all μ and λ, but η is non-trivial, real, and bounded for $-\infty < \theta < \infty$ only in the parametric domain $0 < \mu < 1$ $(0 < \kappa < 1)$ with λ real.

An isolated (in parameter space) solution of (1.1) is given by[2]

$$\eta(\xi, \tau; 1) \equiv \eta_1 = \tfrac{1}{2}(1 + \tanh\theta), \quad \theta = \xi + 4\tau + \lambda_1, \tag{1.4a, b}$$

[1] The prefix I refers to an equation in Miles (1979). The present notation follows that reference except as noted.

[2] This solution can be placed in the seemingly more general form $\eta = A(1+\tanh\theta)\{1 + (2A-1)\tanh\theta\}$, where $\eta = A$ at $\theta = 0$, and $0 < A < 1$; however, this form is equivalent to (1.4a) with θ therein replaced by $\theta - \tanh^{-1}(2A-1)$, which is equivalent to a shift in the phase.

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where λ_1 is real. This solution evidently provides a model of a nondissipative, forward-facing bore for which η increases from 0 to 1 as θ increases from $-\infty$ to ∞. The transformation (1.2) yields

$$\eta_{1*} = \tfrac{1}{2}(1 - \tanh \theta), \tag{1.5}$$

which appears to provide a model of a rearward-facing bore. Note that both $\int_{-\infty}^{\infty} \eta_1^2 \, d\xi$ and $\int_{-\infty}^{\infty} \eta_{1*}^2 \, d\xi$ diverge.

My primary aim in the present paper is to explore the possible evolution of the solutions η_μ, η_1, and η_{1*} from an initial displacement η_0. In Section 2, I outline the asymptotic (as $\tau \uparrow \infty$) solution of the initial-value problem for (1.1). In Section 3, I consider an initially positive, rectangular displacement, which presumably is representative of any η_0 for which $V \equiv \int_{-\infty}^{\infty} \eta_0 \, d\xi$ is positive and finite. The resulting asymptotic solution comprises N terms of the form η_μ with $1 > \mu_1 > \mu_2 \cdots > \mu_N > 0$ and $N \geq 1$. The dominant term ($\mu = \mu_1$) may resemble the bore-like solution η_1 as $V \uparrow \infty$ with θ fixed, but it vanishes exponentially as $\theta \to \pm\infty$ with V fixed.

In Section 4, I consider the initial displacement $\eta_0 = \tfrac{1}{2}(1 + \tanh k\xi)$ and find that the asymptotic solution does comprise η_1 if $k > 0$ (so that the rise of η_0 faces in the direction of propagation), and η_0 and η then exhibit the same increase, i.e. 1, between $\xi = -\infty$ and ∞. I also obtain an asymptotic solution for $-1 < k < 0$ (so that the rise in η_0 faces away from the direction of propagation), but this solution vanishes at both $\xi = -\infty$ and $\xi = +\infty$ and therefore does not exhibit the same decrease as η_0, i.e. -1. The difficulty appears to be connected with the fact that $\eta = \eta_{1*} = \eta_0$ is an admissible solution of the initial-value problem if $k = 1$; it suggests a possible lack of uniqueness for the asymptotic solution if η_0 is not evanescent at $\xi = -\infty$.

2. Inverse-scattering solution

The evolution equation (1.1) is reduced to the Korteweg–deVries (KdV) equation [cf. I(5.7) with $\mathscr{B} \equiv \zeta$ after allowing for the present scaling]

$$\zeta_\tau = 12\zeta\zeta_\xi + \zeta_{\xi\xi\xi} \tag{2.1}$$

through the transformation [cf. I(5.8) with $\mathscr{A} \equiv \eta$ and $\mathscr{B} \equiv \zeta$]

$$\zeta = \eta_\xi + 2\eta(1 - \eta), \tag{2.2}$$

by virtue of which solutions of (1.1) may be obtained, at least in principle, through inverse-scattering theory [Gardner et al. (1974); Whitham (1974)]. But note that the substitution of η_{1*} from (1.5) into (2.2) yields $\zeta = 0$, in consequence of which the inverse of (2.2) may not be unique.

Let $\eta_0(\xi)$ and

$$\zeta_0(\xi) = \eta_0'(\xi) + 2\eta_0(1 - \eta_0) \tag{2.3}$$

be corresponding initial data for η and ζ; then the asymptotic solution of (2.1) is dominated by a discrete set of solitary waves (solitons):

$$\zeta \sim \sum_{n=1}^{N} \kappa_n^2 \operatorname{sech}^2(\kappa_n \xi + 4\kappa_n^3 \tau + \nu_n) \quad (\tau \uparrow \infty), \tag{2.4}$$

where $\kappa_1 > \kappa_2 > \ldots > \kappa_N > 0$ are the discrete eigenvalues of the Schroedinger equation

$$\psi''(\xi) + \{-\kappa^2 + 2\zeta_0(\xi)\}\psi(\xi) = 0 \quad (-\infty < \xi < \infty) \tag{2.5}$$

subject to the asymptotic boundary condition

$$\psi \sim e^{-\kappa\xi} \quad (\xi \uparrow \infty), \tag{2.6}$$

and $\nu_1, \nu_2, \ldots, \nu_N$ are constants (which are determined by the full solution of the Marchenko integral equation that is implied by the inverse-scattering algorithm). A necessary and sufficient condition for $N \geq 1$ is

$$\int_{-\infty}^{\infty} \zeta_0 \, d\xi \geq 0. \tag{2.7}$$

It also is necessary, for the existence of the solution of (2.5) and (2.6), that this integral be bounded [see Gardner et al. (1974) for further restrictions on the inverse-scattering algorithm], which, in turn, requires that either $\eta_0 \to 0$ or $\eta_0 \to 1$ as $\xi \to \pm\infty$.

It can be shown by direct substitution that a solution of (2.2) and (2.4) is given by (superposition is by virtue of spatial separation, not linearity)

$$\eta \sim \sum_{n=1}^{N} \eta(\xi, \tau; \mu_n) \quad (\tau \uparrow \infty), \tag{2.8a, b}$$

where $\eta(\xi,\tau;\mu_n)$ is given by (1.3) with

$$\lambda_n = \nu_n - \tfrac{1}{2}\ln\{(1 + \mu_n^{1/2})/(1 - \mu_n^{1/2})\} \tag{2.9}$$

if $\kappa_n < 1$. If $\kappa_1 = 1$ (κ_1 is, by definition, the largest eigenvalue, η_μ must be replaced by η_1 with $\lambda_1 = \nu_1$. If $\kappa_1 > 1$, η_μ is singular, and (2.8) is not an acceptable asymptotic solution. The limit $\kappa_1 \equiv 1 - \varepsilon \uparrow 1$ yields

$$\eta = (1 - 2\varepsilon)[1 + (1 - \tfrac{1}{4}\varepsilon)e^{-2\hat{\theta}} + \tfrac{1}{4}\varepsilon e^{2\hat{\theta}} + O(\varepsilon^2)]^{-1} \tag{2.10a}$$

$$\to \tfrac{1}{2}(1 + \tanh \hat{\theta}) = \eta_1(\hat{\theta}) \quad (\varepsilon \downarrow 0), \tag{2.10b}$$

where

$$\hat{\theta} = \kappa_1 \xi + 4\kappa_1^2 \tau + \nu_1. \tag{2.11}$$

This limit is not uniformly valid, and the limit $\hat{\theta} \uparrow \infty$ with ε fixed in (2.10a) yields $\eta \to 0$, in contrast to (2.10b), which yields $\eta \to 1$.

It appears (although I have not proved) that (2.8) is the required asymptotic solution if η_0 satisfies appropriate null conditions at $\xi = \pm\infty$, for then η also must satisfy these null conditions, which presumably render the inversion of (2.2) unique. But the fact that $\eta = \eta_*$ satisfies (2.2) with $\zeta = 0$ suggests that the inversion of (2.2) may not be unique in the absence of a null condition at $\xi = -\infty$.

3. Rectangular initial displacement

Consider the initial displacement

$$\eta_0(\xi) = \begin{matrix} 0 & (\xi > L) \\ A & (0 < \xi < L) \\ 0 & (\xi < 0) \end{matrix} \tag{3.1}$$

where $A > 0$. (No solitons evolve if $A < 0$.)

The substitution of (3.1) into (2.5) through (2.3) yields

$$\psi'' + \begin{Bmatrix} -\kappa^2 \\ k^2 \\ -\kappa^2 \end{Bmatrix} \psi = 0, \tag{3.2}$$

where

$$k^2 = 4A(1 - A) - \kappa^2 \tag{3.3}$$

and, here and in (3.5) below, the vertical ordering corresponds to (3.1). The solution of (3.2) subject to (2.6), a null condition at $\xi = -\infty$, continuity of ψ across $\xi = 0$ and $\xi = L$, and the jump conditions [obtained by integrating (2.5) across $\xi = 0$ and $\xi = L$]

$$\psi' \Big|_{L-}^{L+} - 2A\psi = 0 \quad (\xi = L),$$

$$\psi' \Big|_{0-}^{0+} + 2A\psi = 0 \quad (\xi = 0) \tag{3.4a, b}$$

is given by

$$\begin{aligned} &e^{-\kappa\xi} \\ \psi = \; &e^{-\kappa L}\{\cos k(\xi - L) - (\kappa + 2A)k^{-1} \\ &\times \sin k(\xi - L)\} \\ &e^{-\kappa L}\{\cos kL + (\kappa + 2A)k^{-1} \sin kL\}e^{\kappa\xi} \end{aligned} \tag{3.5}$$

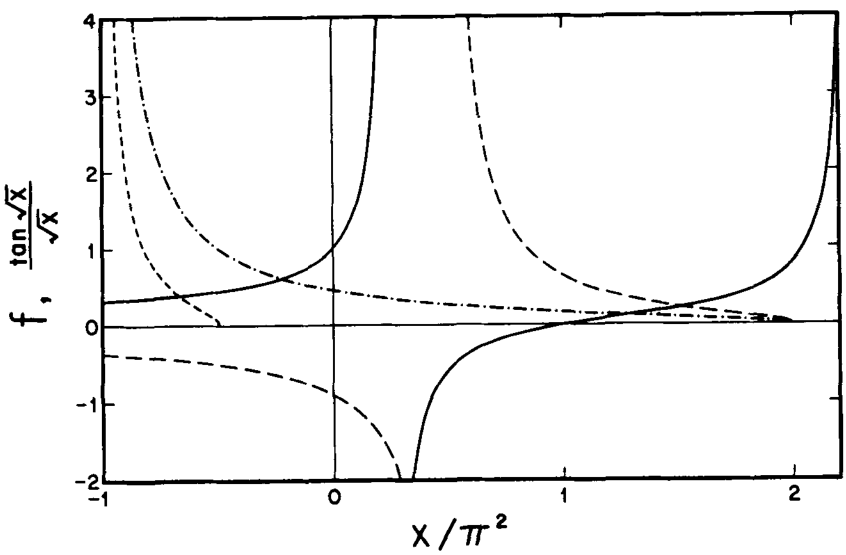

Fig. 1. The graphical solution of (3.7): $x^{-1/2} \tan x^{1/2}$ (——) vs $f(x)$ for: (a) $x_1 = 2\pi^2$ and $x_2 = -\pi^2$ (—·—), (b) $x_1 = 2\pi^2$ and $x_2 = \frac{1}{2}\pi^2$ (— — —), (c) $x_1 = \frac{1}{2}\pi^2$ and $x_2 = -\pi^2$ (- - -).

where k satisfies the eigenvalue equation

$$\tan kL = k\{2A(2A-1)+k^2\}^{-1} \times \{4A(1-A)-k^2\}^{1/2}. \quad (3.6)$$

The qualitative disposition of the roots of (3.6) may be inferred from the graphical solution of (see Fig. 1)

$$x^{-1/2}\tan x^{1/2} = (x-x_2)^{-1}(x_1-x)^{1/2} \equiv f(x), \quad (3.7)$$

where

$$x=(kL)^2, \quad x_1 = 4A(1-A)L^2, \quad x_2 = 2A(1-2A)L^2. \quad (3.8a, b, c)$$

Only the real roots of (3.7), which lie in $x < x_1$, are significant here in consequence of the requirement $\kappa^2 = (x_1 - x)/L^2 > 0$ [see (3.3)]. The number of these real roots is

$$N = 1 + [x_1/\pi^2], \quad (3.9)$$

where $[x_1/\pi^2] \equiv$ the integral part of x_1/π^2. There is at most one negative root [note that $x^{-1/2}\tan x^{1/2} = (-x)^{-1/2}\tanh(-x)^{1/2}$], for the existence of which $x_2 < 0$ and $x_1 < x_2^2$ are necessary and sufficient. The numerical determination of the roots is straightforward in principle, but it is more instructive to consider limiting cases analytically.

(i) $A \uparrow \infty$

The points x_1 and x_2 are both negative and converge on $x = (-2AL)^2$, as also does the single root ($N = 1$ for $A > 1$), as $A \uparrow \infty$. A perturbation solution of (3.3), (3.7) and (3.8) yields

$$\kappa_1 \sim \tanh(2AL) \quad (A \uparrow \infty). \quad (3.10)$$

If AL is held fixed in this limit, $\eta_0 \sim AL\delta(x)$; however, (3.10) is asymptotically valid for all L. If $AL \uparrow \infty$, $\kappa_1 \uparrow 1$, and the limiting form of η is given by (2.10).

(ii) $L \uparrow \infty$

If AL is held fixed in this limit (so that $A \downarrow 0$), both x_1 and x_2 are large and positive, and the roots of (3.7) tend to $(n\pi)^2$, $n = 1, 2, \ldots, N$, and to an additional root that is contiguous to x_2. This remains true for $0 < A < \frac{1}{2}$, but if $A > \frac{1}{2}$ the dominant root ($n = 1$) is negative. A perturbation solution then yields

$$\kappa_1 \sim 1 - \tfrac{1}{2}(2-A^{-1})^2 e^{-2(2A-1)L}, \quad (2A-1)L \uparrow \infty, \quad (3.11)$$

and η_1 is given by (2.10). Note that η_1 is the complete asymptotic solution ($N = 1$) if $A > 1$.

4. A singular example

Consider the bore-like displacement

$$\eta_0 = \tfrac{1}{2}(1+\tanh k\xi) \quad (k > 0, -\infty < \xi < \infty), \quad (4.1)$$

the substitution of which into (2.3) yields

$$\zeta_0 = \tfrac{1}{2}(1+k)\,\text{sech}^2\, k\xi. \quad (4.2)$$

The solution of the eigenvalue problem posed by (2.5), (2.6) and (4.2) is given by Landau and Lifshitz (1958) and yields

$$\kappa_n = 1-(n-1)k \quad (k>0, n = 1, 2, \ldots, N), \quad (4.3)$$

where N is the largest integer for which $\kappa_N > 0$. Note that: (i) $\kappa_1 = 1$, so that the asymptotic solution obtained by substituting (4.3) into (2.8) comprises η_1 and (like η_0) has the limiting value 1 at $\xi = \infty$; (ii) $N = 1$ and (2.8) reduces to $\eta \sim \eta_1$ for $k \geq 1$; (iii) $\eta_0 \to H(\xi)$ (Heaviside's step function) and $\zeta_0 \to \delta(\xi)$ for $k \uparrow \infty$.

Now suppose that $-1 < k < 0$ in (4.1) and (4.2). The solution of the eigenvalue problem then yields

$$\kappa_n = 1 + nk \quad (-1 < k < 0, n = 1, 2, \ldots N), \quad (4.4)$$

all of the components of (2.8) vanish as $\xi \to \pm\infty$, and hence η (unlike η_0) vanishes at $\xi = -\infty$. There are no eigenvalues ($N = 0$), and (2.8) is empty, if $k < -1$. But $\eta = \eta_0 = \eta_{1*}$ if $k = -1$, and this suggests that the failure of η to reduce to η_0 at $\xi = -\infty$ is associated with the aforementioned non-uniqueness of the solution of (2.2).

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О ВНУТРЕННИХ УЕДИНЕННЫХ ВОЛНАХ. II

Методом обратной задачи теории рассеяния найдено асимптотическое поведение одномерных внутренних уединенных волн, рождающихся из данных начальных условий, в той области параметров, где квадратичная или кубичная нелинейности и дисперсия имеют одинаковый порядок малости. Из начального смещения прямоугольной формы, которое аппроксимирует любое начальное смещение с компактным носителем, получается N уединенных волн с $N \geqslant 1$. Самая быстрая из этих волн напоминает бор (который, по определению, представляет собой перепад между уровнями поверхности вверх и вниз по течению), и исчезает на бесконечности вверх и вниз по течению. Получено, что из начального изменения поверхности уровня, при котором подъем обращен в сторону распространения волны, асимптотически получается настоящий бор, а в противоположном случае асимптотическое решение бора не содержит. Эта последняя трудность требует решения проблемы единственности для начальных данных с некомпактным носителем.