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The kinetic energy spectrum *vis-à-vis* a statistical model for geopotential

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ABSTRACT

The relationship between a statistical model for geopotential height and the geostrophic eddy kinetic energy spectrum is elaborated. Ensemble spatial variability of the isobaric field is represented with a continuous linear stochastic model in the natural coordinates of a rotating sphere. The corresponding bi-frequency spectral density for the height field is derived, and a linear filter technique is applied to obtain a description of its implications for the eddy kinetic energy spectrum. The direct correspondence between spatially autoregressive models and bi-frequency spectra, and the companion correspondence between these autoregressive models and lag-correlation functions are discussed with regard to their combined potential for elucidating properties of the frequency (wave number) power spectrum.

The development produces an analytic tool for further investigation of the dependence of geostrophic kinetic energy on latitudinal and longitudinal wave numbers. It is argued that the somewhat simplistic nature of the approach is compensated by the sensitivity of the representations of distinct frequency components and the parsimony of statistical analyses based on the result. Possible implications for the retention of forecast-relevant energy distributions in objective analyses are considered.

1. Introduction

The purpose of the present paper is two-fold. We wish to verify conformance of a proposed statistical model to known properties of atmospheric fields and we wish to introduce a general extension of this model as a mechanism for sensitive analysis of atmospheric ensemble properties. The purpose is achieved by employing the analytic equivalence of the spectrum of power and the correlation function, through Fourier transform, and a simple but elegant linear filter technique.

In both the short- and long-term forecast roles of statistical modelling of atmospheric phenomena, accuracy of produced forecasts will depend critically upon the extent to which model properties reproduce *de facto*, large scale, ensemble relationships. Any proposed model should match stochastic properties of meteorological fields in so far as they are known. Ideally, a good statistical model should also be flexible enough to allow expression of departures from hypothetical simplifying assumptions to appear in data analyses, while not encumbering the analyses with error contamination. Some traditional techniques involve the estimation of such a great number of parameter values, in relation to the data base available for their estimation, that the product is "noisy" and uncertain. The approach proposed here is parsimonious in the use of data; and, in its generalized form, the underlying model provides refinement to existing statistical analysis procedures.

Recent evidence presented by Stanford (1979) supports the expectation that atmospheric fields contain built-in anisotropy affecting the largest scales of motion. This is attributed to the fact that large-scale motions are primarily geostrophic in nature and thus sensitive to the planetary rotation axis. Using satellite radiance data and analysing global stratospheric temperature variance Stanford found that the effect of anisotropy on planetary scale fluctuations is important when observational results are compared with isotropic turbulence predictions. While his work deals with temperature

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fluctuations rather than kinetic energy, he points out that theoretical predictions (Charney, 1971) and several experimental results suggest similar behaviour for kinetic energy. As will be shown here, Stanford's findings are analogous to spectral properties of kinetic energy implicit in an autocorrelation representation for mid-tropospheric geopotential, described by the present author (Thiébaux, 1976, 1977). Thus his work provides further support to the argument for representing anisotropy in statistical fields *vis-à-vis* forecasting objectives.

Notwithstanding the evidence that anisotropy of stochastic mean fields is a significant feature of their spectral distributions, there has been little work concerned with distinguishing between zonal and meridional spectral properties. The present paper attempts to fill this gap in proposing a representation for an isobaric field for which orthogonal frequency components are written into a single bi-frequency spectral density function, and discusses its potential *vis-à-vis* investigation of the atmospheric eddy kinetic energy spectrum.

Stanford (1979) has confirmed that the anisotropy of a statistical field propagates into its spectral distribution and has cited supporting evidence of zonal analyses.

2. Background

In recent literature dealing with objective analysis, selections of representations for statistical relationships between variables of geophysical systems have largely been made on an *ad hoc* basis. Specifically, functional forms for lag-correlations have been chosen to approximate correlation data arrays without regard for their correspondence to stochastic models of underlying processes or for implicit mathematical properties. This paper derives the spectral density function for a general Mth order autoregressive model approximating the stochastic behaviour of a dynamical system, and examines its high frequency log/log decay rates. The results are extended to the atmospheric kinetic energy spectrum for geostrophic winds, with evaluations made for various combinations of frequencies.

The choice of functions to represent ensemble relationships, on a look-alike basis, has done an acceptable job in some instances. Buell (1972), Gandin (1963), and Rutherford (1972) have suggested several forms for correlation functions to characterize lag-distance statistics for isobaric temperatures and heights. Julian and Thiébaux (1975) studied properties of some of these with reference to special considerations of objective analysis, and Thiébaux (1975) examined performances of analyses using the same correlation models.

Significant problems encountered in the study of statistical properties of functions chosen on an *ad hoc* basis and in their extension to representations of relations in greater dimensions have derived from failures of the functions to exhibit tractable analytic behaviour. These failures have been impediments to the derivation of cross-correlations consistent with dynamical relationships between component variables of a multivariate system; and they have led to computationally complex determinations of corresponding spectral characteristics. More recently (Thiébaux, 1976, 1977) it has been shown that most of the difficulties can be avoided if covariation statistics are derived from a low-order, autoregressive, stochastic model for the underlying dynamical system. The spectral properties and analysis performances of a simple isotropic model were on a par with previously investigated models. Furthermore, extension of this model to a directional (or, anisotropic) model accounted for most of the heterogeneity of observed statistics, with corresponding improvement in objective analysis accuracy.

The present paper considers the implications of the last mentioned statistical models of isobaric height for the eddy kinetic energy spectrum. Specifically a bi-frequency spectrum, written explicitly as a function of the (orthogonal) latitudinal and longitudinal wave numbers, is derived by applying the geostrophic relations to a general statistical autoregressive model for geopotential in latitude, longitude coordinates. The possibility of extending the equivalence of autocorrelation structure and spectral behaviour, for geopotential, to estimate energy distributions of large-scale turbulence is discussed. And its usefulness for confirmation of appropriate model choice (in the reverse) is considered.

The development will use the notation

$$U(\lambda, \phi) = -\frac{\kappa}{\sin \phi} \frac{\partial Z}{\partial \phi} \quad \text{and}$$

$$V(\lambda, \phi) = \frac{\kappa}{\sin\phi\cos\phi}\frac{\partial Z}{\partial\lambda} \tag{1}$$

to define the components of geostrophic wind velocity $W = U + iV$ and thus the eddy kinetic energy in terms of pressure-level height Z, via

$$E = WW^* = U^2 + V^2. \tag{2}$$

Extension of one-dimensional spectral representations and linear filter properties of time series to two-dimensional isobaric surfaces provides the means for a simple elaboration of geopotential model implications. Comparisons are made with results of geophysical theory and observations (Charney, 1971; Horn and Bryson, 1963; Wiin-Nielsen, 1967; Julian and Cline, 1974; and Stanford, 1979); and possibilities for more sensitive and parsimonious analysis of data, brought by these results are addressed.

3. Derivation of the energy spectrum and its large wave number properties

3.1. *Two-dimensional spectral representations and linear filters*

A time series spectral representation (see Koopmans, 1974, for example) may be generalized for a stochastic process evolving in two dimensions, say as

$$Z(\lambda, \phi) = \iint e^{i\mu\lambda + i\nu\phi}\,\xi(d\mu, d\nu), \tag{3}$$

to write the geopotential anomaly on an isobaric surface as a Fourier integral over continuous frequency space. Here (λ, ϕ) is the surface (longitude, latitude) index, and ξ is the random spectral measure defined on frequency space and generating stochastic Fourier coefficients of Z.

Since $Z(\lambda, \phi)$ is the deviation of geopotential from the mean field, it has ensemble mean zero. ξ is assumed to be zero-mean and uncorrelated over frequency space; and

$$\langle \xi(d\mu, d\nu)\xi^*(d\mu', d\nu')\rangle = \begin{cases} 0 \text{ unless } (\mu, \nu) = (\mu', \nu') \\ F(d\mu, d\nu) \text{ if } (\mu, \nu) = \\ (\mu', \nu') \end{cases} \tag{4}$$

defines the corresponding spectral distribution function F, provided $(d\mu, d\nu)$ is interpreted as an infinitesimal region centred at (μ, ν). Thus

$$F(A) = \iint_A f_Z(\mu, \nu)\, d\mu\, d\nu \tag{5}$$

for all frequency sets, where f_Z is the height field spectral density function (s.d.f.).

Any variable which can be represented as the output of a linear filter for which Z is the input, namely as

$$\iint B(\mu, \nu)\, e^{i\mu\lambda + i\nu\phi}\,\xi(d\mu, d\nu),$$

has corresponding spectral density

$$|B(\mu, \nu)|^2 f_Z(\mu, \nu).$$

Consequently, if both f_Z and the transfer function of the filter, B, are known the spectral distribution for the output variable is known as well.

3.2. *Derivation of the eddy kinetic energy spectrum and its large wave number log/log decay rate*

The isobaric wind velocity has been defined as

$$W(\lambda, \phi) = -\frac{\kappa}{\sin\phi}\frac{\partial Z(\lambda, \phi)}{\partial\phi} + i\frac{\kappa}{\sin\phi\cos\phi} \times \frac{\partial Z(\lambda, \phi)}{\partial\lambda} \tag{6}$$

which is a linear filter acting on Z for which the transfer function is

$$\begin{aligned} B(\mu, \nu) &= -\frac{\kappa}{\sin\phi}(i\nu) + i\frac{\kappa}{\sin\phi\cos\phi}(i\mu) \\ &= -\frac{\kappa}{\sin\phi}\left(\frac{\mu}{\cos\phi} + i\nu\right). \end{aligned} \tag{7}$$

Using the result of Section 3.1 the s.d.f. for atmospheric kinetic energy is obtained immediately, in terms of f_Z and the squared gain of the filter, as

$$f_W(\mu, \nu) = g(\mu, \nu) f_Z(\mu, \nu) \tag{8}$$

with

$$g(\mu, \nu) = B(\mu, \nu)B^*(\mu, \nu) = \frac{\kappa^2}{\sin^2\phi}\left(\frac{\mu^2}{\cos^2\phi} + \nu^2\right). \tag{9}$$

Since the publication of the Horn and Bryson 1963 paper $a \times n^{-b}$ has been accepted as a good approximation for the wave number spectrum for eddy kinetic energy, when the field is assumed to be isotropic, and considerable attention has been directed to determination of the value of b in the region of planetary wave numbers $n = 7$–16.

(For a recent discussion and additional references see Stanford, 1979.) Consequently, interest has centred on the rate of change of the logarithm of the spectral density with the logarithm of wave number or, equivalently, *frequency* as considered here. We shall wish to examine this property in the present framework, namely, for a separable bi-frequency spectrum for geopotential in which the frequency components are uniquely defined with respect to the earth's axis of rotation.

Although we have defined a two-dimensional frequency space, the only frequency dependencies of practical significance are those of the distinct zonal and meridional frequencies, μ and ν. For economy of space we let ω index first one and then the other, and obtain

$$\frac{\partial \log f_W}{\partial \log \omega} = \frac{\partial}{\partial \log \omega} [\log g(\mu, \nu) + \log f_Z(\mu, \nu)]$$

$$= \frac{\omega}{g} \frac{\partial g}{\partial \omega} + \frac{\omega}{f_Z} \frac{\partial f_Z}{\partial \omega}$$

where the first term is

$$\frac{\partial \log g}{\partial \log \omega} = \omega \left[\frac{\kappa^2}{\sin^2 \phi \cos^2 \phi} (\mu^2 + \nu^2 \cos^2 \phi) \right]^{-1}$$

$$\times \frac{\kappa^2}{\sin^2 \phi \cos^2 \phi} \begin{cases} 2\mu & \omega = \mu \\ 2\nu \cos^2 \phi, & \omega = \nu \end{cases}$$

$$= \begin{cases} 2\mu^2/(\mu^2 + \nu^2 \cos^2 \phi), & \omega = \mu \\ 2\nu^2 \cos^2 \phi/(\mu^2 + \nu^2 \cos^2 \phi), & \omega = \nu \end{cases} \tag{11}$$

Apparently the rate of change of the logarithm of the squared gain with the logarithm of either of the frequencies depends on the relative value of the companion frequency. That is, the $\log \mu$ profile of the $\log g(\mu, \nu)$ surface depends on the location of the "slice" with respect to frequency ν, and vice versa. If the profile is regarded for fixed, finite ν, we see that as the μ frequency becomes large the limiting value of $\partial \log g/\partial \log \mu$ is 2. However, at the points along any $c\mu = \nu \cos \phi$ transect, the $\partial \log g/\partial \log \mu$ term is identically $2/(1 + c^2)$. Along this same transect, $\partial \log g/\partial \log \nu$ is $2c^2/(c^2 + 1)$. Since the second term in (10) describes behaviour of the geopotential spectral density function, f_Z, the wind energy spectrum has log/log derivative in either zonal or meridional frequency which exceeds that of the isobaric height field by a value between two and zero, with the increment depending on the ratio of frequency components.

The above may appear counter-intuitive at first sight. However, these results rest on the assumption that it is possible to examine the spectrum as a function of one frequency (only) while either fixing the other or manipulating it at will. Although such detail can be studied with the present model, data analyses generally have not admitted this flexibility; and the above results cannot be compared with data-generated-spectra reported in the literature. If we wish to compare model properties with spectral values computed from nearly constant-latitude data, for example, then we require the integral of f_W over the whole range of ν frequencies:

$$E_1(\mu) = \int f_W(\mu, \nu)\, d\nu$$

$$= K_1(\phi)\mu^2 \int f_Z(\mu, \nu)\, d\nu + K_2(\phi) \times \int \nu^2 f_Z(\mu, \nu)\, d\nu; \tag{12}$$

Its form will be determined by the form of the s.d.f. for geopotential; and it will be a latitude-dependent function of only the zonal frequency, μ.

4. Geopotential covariance models and their implications

4.1. *Statistical autoregressive models for a spatially coherent system*

In the context of this work, representing the ensemble behaviour of a field variable with a spatially autoregressive process signifies approximating, or modelling, the stochastic mean behaviour with a random process which comes within white noise of satisfying a linear differential equation (L.D.E.) in the dimensions of its spatial domain. The simplest such representation for which there is reasonable justification is the now familiar uni-dimensional red noise model. The latter is synonymous with a first-order autoregressive process in a single index variable which represents distance rather than time, and corresponds to a persistent-anomaly prediction, in depending on only the most proximal mean deviation. (Although the use of the red noise model is justifiable in some instances, we will want to include more detailed spatial structure.)

The *order* of an autoregressive process evolving in a single dimension is the order of the L.D.E., or the highest order derivative on which the current state can be said to depend. (Here, we restrict evidence of dependence to its appearance in the observed field, so that white noise departures include observation errors.) The order is reflected in the response, or weighting, function of the equivalent linear filter representation and in the autocorrelation structure derivable from it.

In representing an anisotropic field with an autoregressive process the L.D.E. is a partial differential equation in more than one independent variable, and *order* is an array corresponding to the array of continuous index variables: the jth order component is the highest derivative in the jth spatial coordinate.

For statistical autoregressive models of any dimension the propagations of model properties into lag-correlation structure provide for their evaluation, or estimation, via observed correlations. Not only may the dimensionality of statistical fields be studied with the mechanism of correlation or spectral matching, as done in the references discussed in Section 1, but evidence of their corresponding order may be assessed as well.

4.2. *Orthogonal factorization in statistical representations*

In work dealing with two-dimensional models for tropospheric height field correlation arrays it has been shown (Thiébaux, 1976, 1977) that the statistical relationship between isobaric heights at geographically separate locations is not purely a function of the distance between the locations, but is strongly dependent on their relative azimuth. A model which accounts for most of the heterogeneity of sample correlations, as seen in plots of correlation versus distance only, is derived by factorization of the response function of a linear filter representation for geopotential where the input is white noise. The factorization is consistent with the concept of orthogonal, zonal and meridional weighting of inputs, and in turn implies factorization of the autocovariance function (a.c.f.) of zonal lag $\sigma = \Delta\lambda$ and meridional lag $\tau = \Delta\phi$, as

$$C_Z(\sigma, \tau) = C_1(\sigma)C_2(\tau). \tag{13}$$

Since the a.c.f. and s.d.f. are inverse Fourier transforms of one another then it follows from the above that the spectral density function also factors:

$$\begin{aligned} f_Z(\mu, \nu) &= \int\int e^{-2\pi i(\mu\sigma+\nu\tau)}\, C_Z(\sigma, \tau)\, d\sigma\, d\tau \\ &= \left[\int e^{-2\pi i\mu\sigma}\, C_1(\sigma)\, d\sigma\right]\left[\int e^{-2\pi i\nu\tau}\, C_2(\tau)\, d\tau\right] \\ &= S_1(\mu)S_2(\nu), \end{aligned} \tag{14}$$

where S_1 and S_2 are Fourier transforms of covariance components C_1 and C_2, respectively. Hence from (8), (9) and (12),

$$\begin{aligned} f_W(\mu, \nu) &= \frac{\kappa}{\sin^2\phi}\left(\frac{\mu^2}{\cos^2\phi} + \nu^2\right) S_1(\mu)S_2(\nu) \\ &= [K_1(\phi)\mu^2 + K_2(\phi)\nu^2]S_1(\mu)S_2(\nu) \end{aligned} \tag{15}$$

and

$$\begin{aligned} E_1(\mu) &= \frac{\kappa}{\sin^2\phi}\left[\frac{1}{\cos^2\phi}\mu^2\int S_2(\nu)\, d\nu + \int \nu^2 S_2(\nu)\, d\nu\right] S_1(\mu) \\ &= [L_1(\phi)\mu^2 + L_2(\phi)]S_1(\mu) \end{aligned} \tag{16}$$

Deduction of the implications of (15) and (16) for the kinetic energy spectrum requires specification of the separate factors: equivalently of C_Z or f_Z. In the Thiébaux, 1976, 1977, references both a.c.f. components of (13) were represented as second-order autoregressive model (A.R.M.) covariances. A more general formulation for matching an autoregressive model in two physical coordinates to a frequency spectrum represents each component as pertaining to an Mth order autoregression on a single dependent variable, where M is unspecified. With this, the separate components can be uniquely parameterized, including the possibility that the orders are distinct.

4.3. *Modelling the frequency space components*

Since a continuous index, Mth order, A.R.M. represents a stochastic variable X as a white-noise departure from a linear combination of its first M derivatives:

$$\sum_{k=0}^{M} a_k \frac{d^k X}{dt^k} = \varepsilon(t) \tag{17}$$

this is notationally equivalent to writing the white-noise process $\{\varepsilon(t)\}$ as the output of a linear filter with input $\{X(t)\}$. Hence statistical properties of the X-process can be described by inverting the filter and writing them in terms of the constant noise variance, say η^2, and the a_k coefficients. The transfer function for the filter taking X to ε is the characteristic polynomial for the differential equation evaluated at $i\omega$:

$$B(\omega) = \sum_{k=0}^{M} a_k(i\omega)^k = P(\rho)|_{\rho=i\omega} \tag{18}$$

Consequently, if none of the roots of P lie on the imaginary axis then $|B(\omega)|^{-2}$ is bounded. It follows that the spectral density function for X is

$$S(\omega) = f_X(\omega) = \eta^2|B(\omega)|^{-2}. \tag{19}$$

Anticipating application of the general representation it is noted that

$$S(\omega) = \eta^2\left[\sum_0^{2M} A_k\omega^k\right]^{-1}$$

implies

$$\frac{dS}{d\omega} = \eta^2\left[-\sum_1^{2M} kA_k\omega^{k-1}\right]\left[\sum_0^{2M} A_k\omega^k\right]^{-2} \tag{20}$$

which in turn implies

$$\frac{\omega}{S}\frac{dS}{d\omega} = -\frac{\omega\sum_0^{2M} A_k\omega^k}{\eta^2}\frac{\eta^2\sum_1^{2M} kA_k\omega^{k-1}}{\left[\sum_0^{2M} A_k\omega^k\right]^2}$$

$$= -\sum_1^{2M} kA_k\omega^k/\sum_0^{2M} A_k\omega^k$$

$$= -\frac{2MA_{2M} + \text{terms in } \omega^{-m}}{A_{2M} + \text{terms in } \omega^{-m}} \tag{21}$$

for positive integer values of m. Thus it follows that

$$\lim_{\omega\to\infty}\left(\frac{\omega}{S}\frac{dS}{d\omega}\right) = -2M. \tag{22}$$

whatever M is.

4.4. *Implications for log/log energy spectrum decay rates*

In general it is assumed that the factorization of eq. (14) holds, and that the components correspond to orthogonal processes of possibly distinct orders M and N. Consequently, following eqs. (15) and (21)

$$f_W(\mu,\nu) = [K_1(\phi)\mu^2 + K_2(\phi)\nu^2]S_1(\mu)S_2(\nu)$$

$$= [K_1(\phi)\mu^2 + K^2(\phi)]/\sum_0^{2M} A_k\mu^k\sum_0^{2N} B_l\nu^l \tag{23}$$

and

$$\frac{d\log f_W}{d\log\mu} = \frac{\mu}{f_W}\frac{df_W}{d\mu} = \frac{2K_1(\phi)\mu^2}{K_1(\phi)\mu^2 + K_2(\phi)\nu^2}$$

$$-\sum_1^{2M} kA_k\mu^k/\sum_0^{2M} A_k\mu^k. \tag{24}$$

The high frequency (large wave number) log/log decay rates will vary with the region of frequency space in which the spectral density is viewed. Again following Section 3.2, if the complementary frequency is small the logarithmic rates of change with log frequency are

$$\frac{\mu}{f_W}\frac{\partial f_W}{\partial\mu} \approx 2 - 2M \quad \text{for large } \mu$$

and

$$\frac{\nu}{f_W}\frac{\partial f_W}{\partial\nu} \approx 2 - 2N \quad \text{for large } \nu.$$

However, if the spectral surface is viewed along the $\mu = \nu\cos\phi$ plane, then the above are reduced by one. Elsewhere in the frequency range, corresponding to at least one large wave number, the log/log decay rate with respect to the larger of μ and $\nu\cos\phi$ varies between these values.

From eq. (16) we obtain

$$E_1(\mu) = [L_1(\phi)\mu^2 + L_2(\phi)]/\sum_0^{2M} A_k\mu^k \tag{25}$$

which, together with (21), implies

$$\frac{d\log E_1}{d\log\mu} = \frac{2L_1(\phi)\mu^2}{L_1(\phi)\mu^2 + L_2(\phi)} - \sum_1^{2M} kA_k\mu^k/\sum_0^{2M} A_k\mu^k \tag{26}$$

Here we have integrated out the dependence on polar frequency ν so that for large μ the log/log decay rate is (unconditionally) close to $2 - 2M$.

To illustrate the behaviour of a two-dimensional atmospheric kinetic energy spectrum it will be assumed that both components of the geopotential s.d.f. match second-order A.R.M.s, with the parameterization of Thiébaux (1977). Fig. 1 describes the spectral components for the 500 mb height field, where S and wave number κ are plotted on log scales, for zonal and meridional components. Parameter values for the spectra were obtained by fitting the corresponding autocorrelation function to observed correlation values. With these values the KE spectrum is a surface described by the diagrams in Fig. 2. These show plots of the logarithm of the surface with wave numbers on log scales, and present the surface from opposing sides. Fig. 3 is the profile along the $\mu = \nu \cos \phi$ plane; and Fig. 4 shows the derivatives with respect to $\log \mu$ and $\log \nu$ along this transection. It may be noted that the large wave number asymptote is -3 in both cases, since $\mu = \nu \cos \phi$, although along any constant μ frequency section $\partial \log f_W / \partial \log \nu \to -2$ and vice versa. The derivatives vary between these two values, over the high frequency domain.

The marginal spectral density for wind energy, as a function of μ, may be derived explicitly in this special case. We take both components of the

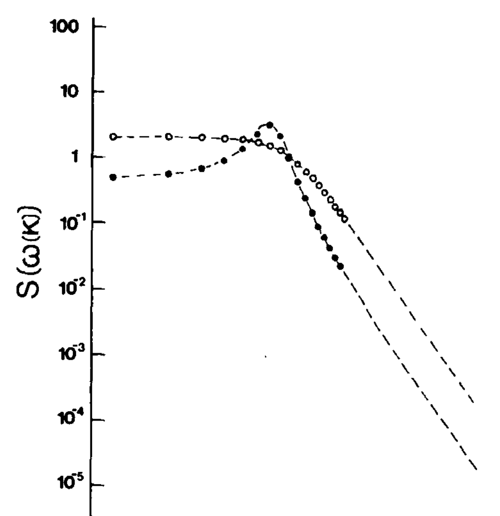

Fig. 1. Zonal and meridional spectral densities as functions of planetary wave numbers for the 500 mb winter geopotential field, from autoregressive models fitted to observed correlations of 1963–64 and 1964–65 North American radiosonde observations. The zonal s.d.f. $S_1(\mu(\kappa))$ is indicated by solid circles and the meridional s.d.f. $S_2(\nu(\kappa))$ by open circles.

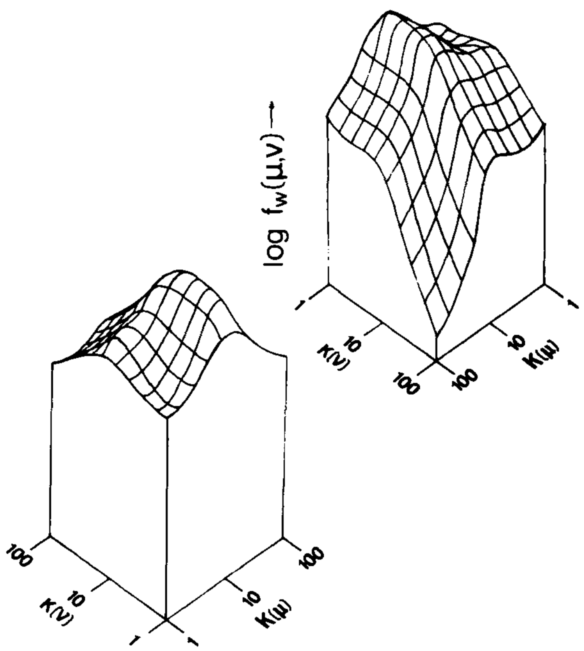

Fig. 2. The natural logarithm of the two-dimensional (spherical surface) spectral density for eddy kinetic energy plotted versus zonal and meridional planetary wave numbers, shown from opposing sides.

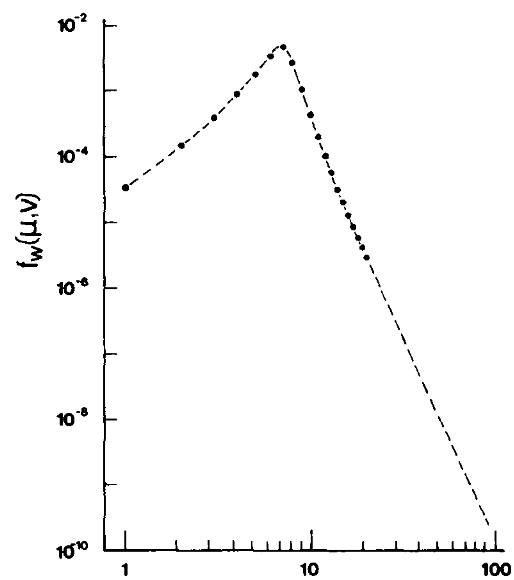

Fig. 3. Log/log plot of the spectral density function for 500 mb winter eddy kinetic energy, along the $\mu = \nu \cos \phi$ transect.

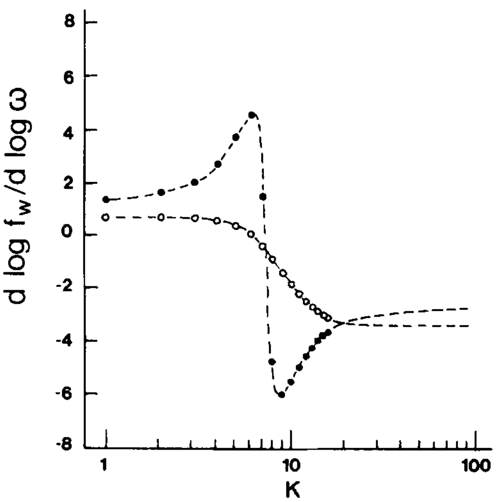

Fig. 4. Log/log spectral decay rate as a function of planetary wave number, for zonal and meridional frequencies, along the $\mu = \nu \cos \phi$ transect. The curve marked with solid circles corresponds to the zonal frequency μ derivative, and the open circles to the meridional frequency ν derivative.

bi-frequency spectrum to correspond to an a.c.f. of the general second-order form

$$R(\tau) = \left[\cos(a|\tau|) + \frac{c}{a}\sin(a|\tau|)\right] e^{-c|\tau|} \tag{27}$$

and determine the constants of eq. (16) by evaluating

$$\int S_2(\nu)\, d\nu \quad \text{and} \quad \int \nu^2 S_2(\nu)\, d\nu.$$

For the latter purpose we employ the mathematical device of defining the Fourier transform of the a.c.f. for both positive and negative frequencies, write

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)\, e^{2\pi i \omega \tau}\, d\tau, \quad -\infty < \omega < \infty, \tag{28}$$

and use properties of the Dirac delta function:

$$\delta(\tau) = \int_{-\infty}^{\infty} e^{2\pi i \omega \tau}\, d\omega$$

$$\text{with} \quad \delta''(\tau) = -4\pi^2 \int_{-\infty}^{\infty} \omega^2 e^{2\pi i \omega \tau}\, d\omega.$$

Accordingly

$$\int_{-\infty}^{\infty} S(\omega)\, d\omega = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} R(\tau)\, e^{2\pi i \omega \tau}\, d\tau \quad d\omega\right]$$

$$= \int_{-\infty}^{\infty} R(\tau)\, \delta(\tau)\, d\tau = R(0) = 1 \tag{29}$$

$$\int_{-\infty}^{\infty} \omega^2 S(\omega)\, d\omega$$

$$= \int_{-\infty}^{\infty} \omega^2 \left[\int_{-\infty}^{\infty} R(\tau)\, e^{2\pi i \omega \tau}\, d\tau\right] d\omega$$

$$= \int_{-\infty}^{\infty} R(\tau) \frac{\delta''(\tau)}{(-4\pi^2)}\, d\tau$$

$$= -\frac{1}{4\pi^2}\left[R(\tau)\delta'(\tau)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} R'(\tau)\delta'(\tau)\, d\tau\right]$$

$$= -\frac{1}{4\pi^2}\left[0 - R'(\tau)\delta(\tau)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} R''(\tau)\delta(\tau)\, d\tau\right]$$

$$= -\frac{1}{4\pi^2}[0 - R''(0)] = \frac{c^2 + a^2}{4\pi^2} \tag{30}$$

since

$$R'(\tau) = -\frac{c^2 + a^2}{a} \sin(a|\tau|)\, e^{-c|\tau|} \tag{31}$$

and

$$R''(\tau) = (c^2 + a^2)\left[-\cos(a|\tau|) + \frac{c}{a}\sin(a|\tau|)\right] e^{-c|\tau|} \tag{32}$$

Now we may write

$$E_1(\mu) = [K_1(\phi)\mu^2 + K_3(\phi)] S_1(\mu) \tag{33}$$

where

$$K_1(\phi) = \kappa^2/\sin^2\phi \cos^2\phi, \quad K_2(\phi) = \kappa^2/\sin^2\phi,$$

$$K_3(\phi) = K_2(\phi)(c_2^2 + a_2^2)/4\pi^2$$

and, by making the Fourier transform of R_1 directly,

$$S_1(\mu) = 4c_1(c_1^2 + a_1^2)/[(c_1^2 + a_1^2)^2 + 8(c_1^2 - a_1^2)\pi^2\mu^2 + 16\pi^4\mu^4]. \tag{34}$$

Finally, the log/log decay rate of the energy spectrum averaged with respect to the polar frequency component is

$$\frac{\mu}{E_1}\frac{dE_1}{d\mu}$$
$$=\frac{\mu}{E_1}\left\{2K_1(\phi)\mu S_1(\mu)+[K_1(\phi)\mu^2+K_3(\phi)]\frac{dS_1}{d\mu}\right\}$$
$$=\frac{\mu}{K_1(\phi)\mu^2+K_3(\phi)}\left\{2\mu K_1(\phi)-\frac{[K_1(\phi)\mu^2+K_3(\phi)][16(c_1^2-a_1^2)\pi^2\mu+64\pi^4\mu^3]}{(c_1^2+a_1^2)^2+8(c_1^2-a_1^2)\pi^2\mu^2+16\pi^4\mu^4}\right\}$$
$$=2\mu^2\{K_1(\phi)[(c_1^2+a_1^2)-16\pi^4\mu^4] + 2K_2(\phi)(c_2^2+a_2^2)[(a_1^2-c_1^2)-4\pi^2\mu^2]\}/D \quad (35)$$

where

$$D=\left[\mu^2K_1(\phi)+\frac{c_2^2+a_2^2}{4\pi^2}K_2(\phi)\right][(c_1^2+a_1^2) + 8(c_1^2-a_1^2)\pi^2\mu^2+16\pi^4\mu^4]. \quad (36)$$

Clearly

$$\frac{\mu}{E_1}\frac{dE_1}{d\mu}\to -2,$$

so that it should be approximately -2 for large values of the zonal frequency. The frequency (wave no.) dependencies of $L(\mu, \nu) = d\log f_W/d\log\mu$ and $L^*(\mu) = d\log E_1/d\log\mu$ are described by Table 1 for an array of zonal wave nos. The log/log decay rate of the bi-frequency spectral surface has been evaluated for two sets of meridional frequencies: one corresponding to polar wave nos. 5, 10, 15, and 20, and the other to meridional frequencies in fixed proportions to zonal frequencies of $\frac{1}{2}$, 1 and 2.

5. Comparisons with estimated and theoretical spectral behaviour

Wiin-Nielsen (1967) reported several investigations of energy spectra for large-scale geostrophic flow in the atmosphere, and showed a $\log S$:$\log\omega$ relationship with a slope of -2.8 in the wave number range $8 \le n \le 15$. This is in good agreement with the work of Horn and Bryson (1963); and it compares well with Charney's theory (1971) which predicts that the spectrum varies with the inverse third power of the wave

Table 1. *Log/log decay rates for the bi-spectral surface and the meridionally averaged spectral density, with zonal frequency μ*

Zonal wave No., N_1	μ	$L_W(\mu, \nu(N_2))$ $N_2=5$	$N_2=10$	$N_2=15$	$L_W(\mu,\nu)$ $\nu=\mu/2$	$\nu=\mu$	$\nu=2\mu$	$L_W^*(\mu)$
1	0.035	0.35	0.15	0.11	1.85	1.40	0.74	0.10
2	0.071	1.07	0.57	0.43	2.07	1.63	0.96	0.39
3	0.106	1.89	1.24	0.99	2.49	2.05	1.38	0.80
4	0.141	2.84	2.18	1.85	3.18	2.74	2.07	1.26
5	0.177	4.03	3.43	3.05	4.21	3.77	3.10	1.59
6	0.212	4.99	4.46	4.06	5.06	4.62	3.95	0.71
7	0.247	1.95	1.50	1.10	1.95	1.51	0.84	−4.54
8	0.283	−4.20	−4.59	−4.96	−4.25	−4.69	−5.36	−8.82
9	0.318	−5.42	−5.75	−6.10	−5.50	−5.94	−6.61	−7.68
10	0.353	−4.93	−5.22	−5.54	−5.04	−5.48	−6.15	−6.14
11	0.389	−4.36	−4.60	−4.89	−4.48	−4.93	−5.59	−5.09
12	0.424	−3.90	−4.11	−4.38	−4.04	−4.49	−5.15	−4.40
13	0.459	−3.57	−3.75	−3.99	−3.72	−4.16	−4.83	−3.92
14	0.495	−3.31	−3.47	−3.69	−3.47	−3.91	−4.58	−3.58
15	0.530	−3.11	−3.26	−3.46	−3.28	−3.72	−4.39	−3.32
16	0.565	−2.96	−3.09	−3.27	−3.13	−3.58	−4.21	−3.13
17	0.601	−2.83	−2.95	−3.12	−3.01	−3.46	−4.12	−2.98
18	0.636	−2.73	−2.84	−2.99	−2.92	−3.36	−4.03	−2.85
19	0.671	−2.65	−2.74	−2.88	−2.84	−3.28	−3.95	−2.75
20	0.707	−2.58	−2.67	−2.79	−2.77	−3.21	−3.88	−2.67
30	1.060	−2.24	−2.28	−2.35	−2.45	−2.90	−3.56	−2.28

number. Julian and Cline (1974) reported estimates of zonal wave number spectra for 500 mb wind velocities which decay logarithmically in the range $7 \leq n \leq 18$, with exponents between -2.1 and -2.8. Stanford's (1979) estimates of latitudinal wave number spectra of stratospheric temperature put the slope near -2.7 for the upper stratosphere and -4.1 for the lower stratosphere, in the same wave number range.

The range of values for the exponential decay rate over the inertial subrange undoubtedly reflects the variability of the wave number energy distribution with time of year as well as with pressure level, and discrepancies between spectral characteristics of temperature and eddy kinetic energy. Plots of observed spectra of kinetic energy for four isobaric levels (1000, 850, 500, 300 mb) presented by Hollingsworth et al. (1979) concur with the results of Stanford in suggesting that level may have a strong influence. The large wave number log/log decay rates of Hollingsworth et al. for the troposphere varied between a value near -2 at 1000 mb to slightly more than -3 at 300 mb.

The two-dimensional model spectral representation derived in preceding sections and evaluated with parameters obtained from 500 mb geopotential correlation modelling, is consistent with the tropospheric spectral estimates for wind data discussed above. In its more general form the model is obviously adaptable to regimes whose wave number dependencies differ statistically, as they appear in the stratosphere, for example.

6. Discussion of results and possible implications

The distinct representation of zonal and meridional wave numbers, at once, in a bi-frequency spectral density presents a mechanism for investigation of the combined effects of orthogonal frequency components on observed spectral behaviour. The evidence of its success provides preliminary confirmation of the underlying model as a valid representation of ensemble behaviour of geopotential and of geostrophic winds, in the mid-troposphere. And the ease with which it is generalized recommends its potential as a broadly applicable device for sensitive analysis of large-scale energy distributions. Thus it may offer a direct and relatively simple and parsimonious statistical analysis scheme in the natural coordinates of these isobaric phenomena, when used in conjunction with the parallel geopotential lag-correlation representation.

The flexibility of the dimensional model developed here may permit determining whether there is confirming physical evidence that large-scale motions along and across the axis of Earth's rotation have statistically distinct energy spectra. Reciprocally, knowledge of the energy spectrum may be useful in the selection of a lag-correlation model, as the basis for an objective analysis scheme or a statistical prediction algorithm. Results of a recent assessment project (Thiébaux, 1980) suggest that the use of a suboptimal model may be smoothing out too much of the differential field. This is in agreement with, and may in part explain, results of studies comparing kinetic energy budgets of observed and forecast fields (Bettge et al., 1976; Ward et al., 1977). They indicate that general circulation models of the atmosphere suffer from systematic loss of kinetic energy which is most pronounced during the initialization process and the early stages of a forecast cycle. The close correspondence of the *KE* spectrum, for the second-order autoregressive model, to theory and observations, as well as the considerably better fit to observed correlations lead to the hypothesis that its use in the objective analysis of data assimilation could retain features of energy distribution essential to forecast accuracy.

The proposed methods permit matching and estimation with a minimum of error contamination. Because there are few parameter values which must be estimated and because their estimation does not require prior interpolation of the data, the estimates are relatively stable and uncontaminated by extraneous influences.

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СОПОСТАВЛЕНИЕ СПЕКТРА КИНЕТИЧЕСКОЙ ЭНЕРГИИ СО СТАТИСТИЧЕСКОЙ МОДЕЛЬЮ ДЛЯ ГЕОПОТЕНЦИАЛА

Изучается связь между статистической моделью для поля высот геопотенциала и спектром вихревой кинетической энергии геострофического движения. Ансамбль пространственных флуктуаций поля высот изобарических поверхностей представлен с помощью непрерывной линейной стохастической модели в естественных координатах вращающейся сферы. Выводится соответствующая двухчастотная спектральная плотность для поля высот и для получения выводов о виде спектра вихревой кинетической энергии используется техника линейной фильтрации. Обсуждается прямое соответствие между пространственными авторегрессионными моделями и двухчастотными спектрами, и аналогичное соответствие между этими авторегрессионными моделями и пространственными корреляционными функциями в целях использования этих величин для выяснения свойств частотного (пространственного) спектра мощности.

Этот подход дает аналитический аппарат для дальнейшего исследования зависимости геострофической кинетической энергии от широтных и долготных волновых чисел. Приводятся аргументы в пользу того, что несколько упрощенный характер подхода компенсируется чувствительностью представлений выбранных частотных компонент и экономичностью статистического анализа, основанного на этих результатах. Рассматриваются возможные следствия по использованию в объективном анализе распределений энергии, имеющих отношение к прогнозу.