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Symmetric instability of stratified geostrophic flow

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ABSTRACT

The linear stability of a stably stratified geostrophic current between parallel horizontal planes has been investigated theoretically. The disturbances considered are two-dimensional with axes aligned along the flow direction. Diffusive processes enter the stability problem through the Ekman number, E, and the Prandtl number Pr. The analysis is valid for general vertical stable stratification, characterized by $S = N^2/f^2$, where N is the Brunt-Väisälä frequency and f the (constant) Coriolis parameter. When E is small, the horizontal scale of the marginally stable disturbance is $\propto H(1 + \text{Pr}\ S)^{1/2}\ E^{1/3}$, where H is the distance between the bounding planes. Large values of E stabilize the system. For given viscosity, and $S = 0$, the system is destabilized by increasing Pr, while for $S \gg 1$ destabilization occurs if Pr $\neq 1$.

1. Introduction

Vortex rolls aligned along the mean flow are commonly observed in nature. When they occur near the ocean surface, they are often referred to as Langmuir circulations; see Faller (1971) for a comprehensive review of the subject. In the atmosphere the phenomenon frequently manifests itself as cloud streets (e.g. see Kuettner, 1971).

These circulations occur in a variety of situations and are probably not explained by one single physical mechanism. When the stratification is unstable, buoyancy-driven convection in the form of longitudinal rolls may account for the phenomenon (Kuo, 1963), while in neutral or stable conditions instability of the Ekman boundary layer flow may be responsible for such rolls (Faller, 1963, 1965; Brown, 1970). With particular reference to the upper ocean, Craik and Leibovich (1976) advance a theory for the origin of Langmuir circulations which involves a nonlinear interaction between surface waves and the frictional wind drift current.

Another type of roll-like structure may arise from the inertial instability of a rectilinear geostrophic current with respect to two-dimensional disturbances with axes aligned along the basic flow. This phenomenon is well known in meteorology, and is only a special case of the more general theory of symmetric instability of the baroclinic vortex (see the review by Eliassen and Kleinschmidt, 1957). When diffusion of momentum and heat are taken into account, destabilization of the vortex may occur when the Prandtl number is different from one (McIntyre, 1970), since then, angular momentum and heat diffuse at different rates. With particular reference to the ocean, Calman (1977) has performed laboratory experiments in a rotating salt stratified system, where angular momentum diffuses much faster than salt. He observed the formation of layers which may explain the occurrence of micro-structure (of order 1 m or less) in certain parts of the ocean.

In contrast to McIntyre (1970), who works in an infinite domain, we assume that the basic flow is confined between horizontal parallel planes in order to model a shallow atmosphere or ocean. This is similar to Kuo (1954), Lilly (1966), Faller and Kaylor (1969) and Gammelsrød (1975), for a vertically homogeneous fluid, and to Walton (1975), who considers a fluid with strong vertical stability in the limit of small Ekman numbers. The present paper investigates the linear stability problem for arbitrary vertical (stable) stratification, and the Ekman number does not necessarily have to be small.

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2. Model and basic flow

We consider the flow of a rotating viscous fluid bounded by two parallel horizontal planes at a distance H. A Cartesian coordinate system (x, y, z) with unit vectors $(\vec{i}, \vec{j}, \vec{k})$ is defined such that the z-axis is vertical and positive upwards, and the upper plane is situated at $z = H$. In the present paper emphasis will be put on basic principles, and accordingly we simplify the problem as much as possible while retaining the ability to describe the fundamental physical mechanisms involved. In the light of this, the Boussinesq approximation will be assumed. The governing equations may then be written

$$\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v} + f\vec{k}\times\vec{v} = -\nabla\pi + \nu\nabla^2\vec{v} - g\sigma\vec{k} \quad (1)$$

$$\frac{\partial \sigma}{\partial t} + \vec{v}\cdot\nabla\sigma = \kappa\nabla^2\sigma \quad (2)$$

$$\nabla\cdot\vec{v} = 0 \quad (3)$$

where $\vec{v}$ $(=u, v, w)$ is the velocity vector, f the constant Coriolis parameter, π the pressure per reference density ρ_0, σ a dimensionless density $(=\rho/\rho_0)$, g the acceleration of gravity, ν the kinematic viscosity and κ the thermal diffusivity.

The above formulation is probably most applicable to the ocean, although an alternative formulation for air (perfect gas) qualitatively leads to the same stability problem.

We assume that the basic density distribution Σ is linear in the y- and z-direction, and does not vary with x, i.e.

$$\Sigma = -\beta y - \gamma z \quad (4)$$

Since stable vertical stratification is most commonly found in the ocean and the atmosphere, we take $\gamma > 0$. We also choose $\beta > 0$, so that the y-axis points toward lighter fluid.

Assuming a stationary velocity distribution of the form

$$\vec{v} = U(z)\vec{i} \quad (5)$$

we obtain from (1) that

$$U = -g\beta z/f \quad (6)$$

which is often referred to as the thermal wind. In order to avoid an Ekman layer at $z = H$, we let the upper boundary move with the thermal wind velocity $-g\beta H/f$, while the lower is stationary. Then (4) and (5) are exact solutions of the basic equations.

3. Perturbation analysis

We investigate the stability of the basic state by introducing small perturbations into the solution. In particular we consider disturbances which are independent of the x-coordinate. These are often referred to as symmetric disturbances in meteorology, since the basic motion may be considered as a baroclinic vortex with infinite radius. When $\partial/\partial x = 0$, we may define a streamfunction ψ such that the perturbation velocity can be written

$$\vec{v} = (u, \psi_z, -\psi_y) \quad (7)$$

where subscripts denote partial differentiation. It proves convenient to introduce nondimensional variables into the problem. This is achieved by taking

$$H, \quad 1/f, \quad Hf, \quad H^2f^2, \quad Hf^2/g \quad (8)$$

as units of length, time, velocity, pressure and density, respectively. Eliminating the pressure, the linearized perturbation equations may be written

$$u_t = -B\psi_y + \psi_z + E\nabla_1^2 u \quad (9)$$

$$\nabla_1^2\psi_t = -u_z + \sigma_y + E\nabla_1^4\psi \quad (10)$$

$$\sigma_t = B\psi_z - S\psi_y + \mathrm{Pr}^{-1}E\nabla_1^2\sigma \quad (11)$$

where $\nabla_1^2 = \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the two-dimensional Laplacian.

The dimensionless parameters governing the stability of the system are

$B = g\beta/f^2$ the dimensionless thermal wind shear $(=|U_z|/f)$ which is a measure of the baroclinicity

$S = g\gamma/f^2$ the stratification parameter $(=N^2/f^2)$ where N is the Brunt-Väisälä frequency

$E = \nu/fH^2$ the Ekman number

$\mathrm{Pr} = \nu/\kappa$ the Prandtl number (12)

It often proves convenient to combine B and S to yield a Richardson number Ri. Since $B = |U_z|/f$ and $S = N^2/f^2$ we obtain

$$\mathrm{Ri} = S/B^2 \quad (13)$$

We take the horizontal boundaries to be impermeable. Hence the normal velocity must vanish there, i.e.

$$\psi = 0, \quad z = 0, 1 \tag{14}$$

Concerning the stresses and the density variations, some extreme cases are usually considered. At a rigid boundary a no-slip condition must be applied, i.e. $u = \psi_z = 0$, while for a stress-free boundary we have $u_z = \psi_{zz} = 0$. If the boundary is a perfect conductor of heat, then $\sigma = 0$ there, while at an insulating boundary $\sigma_z = 0$.

By assuming disturbances for the form $\exp(ily + \omega t)$ where l is a real wavenumber in the y-direction and ω the complex growth rate, the equations are

$$\omega u = -ilB\psi + D\psi + E(D^2 - l^2)\, u$$

$$\omega(D^2 - l^2)\, \psi = -Du + il\sigma + E(D^2 - l^2)^2\, \psi$$

$$\omega\sigma = BD\psi - ilS\psi + \mathrm{Pr}^{-1} E(D^2 - l^2)\, \sigma \tag{15}$$

where $D = d/dz$.

By specifying appropriate boundary conditions, the system of eqs. (15) can be solved numerically by Galerkin's method to yield the eigenvalues ω for any given set B, E, S, Pr and l. The procedure is straightforward, but rather lengthy, and requires a large amount of computer time. (See for example Weber (1978) for a related problem.)

To obtain some information about the qualitative behaviour of the problem, we shall introduce the simplification of replacing the vertical viscous and diffusive terms in the momentum and heat equations by a wave mode assumption, i.e.

$$\frac{\partial^2}{\partial z^2}(\vec{v}, \sigma) = -m^2(\vec{v}, \sigma) \tag{16}$$

where m in effect corresponds to a vertical wavenumber. The above assumption reduces the order of the equations, and in consequence no vorticity is generated at the boundaries. Accordingly, the assumption makes no sense in a problem where this production is essential for the generation of instability, as in plane Poiseuille flow. However, in problems where the instability mechanism basically relies on the conversion of energy from the basic state, diffusion of heat and vorticity from the boundaries into the fluid have little dynamical significance. The main effect of diffusion is a tendency to even out temperature and vorticity differences within the fluid, while the horizontal impermeable boundaries in effect set an upper limit for the vertical scale of motion. Hence (16) should be an acceptable qualitative approximation in problems where instability occurs as a result of conversion of potential energy (e.g. buoyancy-driven convection), or as here, when energy is extracted from the shear of the basic flow. With this simplification eqs. (15) reduce to

$$\Omega_2(\Omega_1^2 + 1)\, D^2\psi - ilB(\Omega_1 + \Omega_2)\, D\psi - \Omega_1 l^2(\Omega_1\Omega_2 + S)\,\psi = 0 \tag{17}$$

where

$$\Omega_1 = \omega + E(l^2 + m^2)$$

$$\Omega_2 = \omega + \mathrm{Pr}^{-1} E(l^2 + m^2)$$

The boundary condition (14) at the lower plane is satisfied by a solution of the form $\psi \propto \exp(ir_1 z) - \exp(ir_2 z)$, where

$$\left.\begin{matrix} r_1 \\ r_2 \end{matrix}\right\} = \frac{1}{2a}\,[b \pm (b^2 - 4ac)^{1/2}] \tag{18}$$

Here

$$a = \Omega_2(\Omega_1^2 + 1)$$

$$b = B(\Omega_1 + \Omega_2)$$

$$c = \Omega_1(\Omega_1\Omega_2 + S)$$

The boundary condition at the upper plane requires that

$$r_1 - r_2 = 2n\pi, \quad n = 1, 2, 3, \ldots \tag{19}$$

which may be rearranged to yield an expression for the growth rate. However, McIntyre (1970) found for an infinite domain that, although oscillatory instability did occur for $\mathrm{Pr} \neq 1$, the most unstable disturbance was monotonic. This was also shown by Walton (1975), for large S and small E, to apply for a model of finite vertical extent. This suggests that we should concentrate on monotonic disturbances which seem to be those most likely encountered in a physical problem. Then the transition to instability occurs via $\omega = 0$. With $n = 1$, which can be shown to be the most "dangerous" mode, (19) reduces to

$$B = \frac{2}{1 + \mathrm{Pr}}\left[\mathrm{Pr}\, S + \frac{\pi^2}{l^2} + \left(\mathrm{Pr}\, S + 1 + \frac{2\pi^2}{l^2}\right) E^2 k^2 + \left(1 + \frac{\pi^2}{l^2}\right) E^4 k^4\right]^{1/2} \tag{20}$$

where $k \equiv l^2 + m^2$ is a sort of squared overall wavenumber.

The evaluation of the coefficient m defined by (16) is somewhat arbitrary, but essentially m must correspond to an effective vertical wavenumber, i.e. it must be of the same order of magnitude as r_1, r_2 from (18). As Lilly (1966), we shall assume a root-mean-square value, i.e.

$$m^2 = \tfrac{1}{2}(r_1^2 + r_2^2) \qquad (21)$$

For the familiar Rayleigh-Bénard convection problem, which is an example where instability arises as a result of conversion of energy from the basic flow, application of the procedure outlined here, with use of (16) and (21), results in a critical Rayleigh number which equals the exact one for stress-free and perfectly conducting boundaries. This is also a qualitatively correct result for other types of boundary conditions. Analogously, we expect the present procedure to yield a qualitatively correct answer also for the symmetric instability problem.[1]

By insertion from (18) into (21) we obtain

$$E^2 k^3 - 2E^2(l^2 + \pi^2)\, k^2 + k - 2\pi^2 - l^2(1 + \mathrm{Pr}\, S) = 0 \qquad (22)$$

To find k from this equation is an elementary task and we leave out the details. Its value is then substituted back into (20). For fixed values of Pr, S and E, we compute the minimum value of B when l varies. This minimum is denoted by B_c, and the corresponding wavelength by l_c. Since k from (22) is a somewhat complicated function of l, this is most easily done be evaluating the left-hand side of (20) by a computer for sufficiently small steps in l and then determine the minimum by an interpolation procedure. However, for sufficiently small values of E, explicit expressions may be obtained for B_c and l_c as demonstrated in the following section.

4. Results and discussion

4.1. *The nondiffusive case*

When $E = 0$ and $\mathrm{Pr}^{-1} E = 0$ in (17) we are back to the well-known nondiffusive case. The growth rate is then easily found to be

$$\omega = \pm \left[\frac{\left\{ -(S + 1 + 2n^2\pi^2/l^2) \pm [(S-1)^2 + 4(1 + n^2\pi^2/l^2)\, B^2]^{1/2} \right\}}{2(1 + n^2\pi^2/l^2)} \right]^{1/2} \qquad (23)$$

It is seen that for the upper sign of the inequality

$$B \lessgtr (S + n^2\pi^2/l^2)^{1/2} \qquad (24)$$

we have two pairs of purely imaginary roots, corresponding to two sets of progressive inertia/gravity waves. When the lower sign applies, the one pair of roots become real, and hence exponential growth first occur when

$$B > S^{1/2} \quad \text{or} \quad \mathrm{Ri} < 1 \quad \text{for } l = \infty \qquad (25)$$

which is Stone's (1966) result. But as seen from (23), when $B = (S + n^2\pi^2/l^2)^{1/2}$, we have a double root, $\omega^2 = 0$, which leads to a linear growth in time. The meridional stream function may then be written (taken $n = 1$) as

$$\psi = (A_0 + A_1 t) \sin \pi z \cos l\,(y + Bz) \equiv (A_0 + A_1 t)\, \varphi(y, z) \qquad (26)$$

where A_0 is an initial value, and we have let the real part represent the physical solution. Hence, for given wavelength and stratification, the phase speed of the slowest waves decreases towards zero, when B increases, before they can extract energy from the basic flow. Accordingly, the situation characterized by Ri $= 1$ must also be considered as unstable (although weaker than exponentially) when diffusive effects are not present. This point seems to have been overlooked in previous investigations.

From (9) and (11) we then obtain for u and σ on the critical curve

$$u = u_0 + (A_0 t + \tfrac{1}{2}A_1 t^2)(-B\varphi_y + \varphi_z) \qquad (27)$$

$$\sigma = \sigma_0 + (A_0 t + \tfrac{1}{2}A_1 t^2)(B\varphi_z - S\varphi_y) \qquad (28)$$

where u_0, σ_0 are initial values and φ is defined by (26).

4.2. *Inclusion of diffusive processes*

When E is small, but nonzero, several interesting results can be obtained explicitly from (20). Since an inviscid flow first becomes unstable to disturbances with zero wavelength, it is reasonable to

[1] During the final preparation of this paper the author became aware of the Ph.D. thesis by Emanuel (1978) who has done some numerical computations by a variational method. Comparison with his results shows that the agreement is very good, also quantitatively.

assume that the effect of viscosity sets the length scale in the diffusive case, i.e. we assume

$$l = l_0 E^{-\alpha} \tag{29}$$

as a leading term, where $\alpha > 0$ and $E \ll 1$ (see also Walton, 1975). Two different cases will be considered. First we assume $S = 0$, that is a vertically homogenous fluid. Then, in the inviscid case, instability will commence for any $B > 0$. This suggests that for $E \neq 0$, but small, we should take

$$B = B_0 E^{\beta} \tag{30}$$

as a leading term, where $\beta > 0$. By putting $S = 0$ in (20) and (22) and expressing all the variables in powers of E, balance to lowest order requires

$$\alpha = \beta = 1/3 \tag{31}$$

and

$$B_0^2 = \frac{4}{(1+\mathrm{Pr})^2}\left(\frac{\pi^2}{l_0^2} + l_0^4\right) \tag{32}$$

Minimizing B with respect to l_0, we obtain the critical value

$$B_c = \frac{2 \cdot 3^{1/2}}{1+\mathrm{Pr}}\left(\frac{\pi^2}{2}\right)^{1/3} E^{1/3} + O(E) \tag{33}$$

for

$$l_c = \left(\frac{\pi^2}{2}\right)^{1/6} E^{-1/3} + O(E^{1/3}) \tag{34}$$

The assessment of the order of the next terms in the expansion follows straight away from the analysis. In particular we note that increasing values of Pr destabilize the flow for given E. This is obvious, since when κ is small (Pr large) the density of an individual particle is nearly conserved, and hence meridional vorticity is generated nearly as in the nondiffusive case where instability commences for any $|B| > 0$. On the other hand, when κ is large (Pr small) the perturbation density σ nearly vanishes and the production of meridional vorticity (proportional to $-u_z + \sigma_y$ in nondimensional notation) is reduced and reaches a minimum when $\mathrm{Pr} = 0$, that is when σ vanishes identically. We further note from (34) that in this case the horizontal scale of the most unstable disturbance is determined by viscosity alone.

When the vertical stratification is moderate or large, i.e. $S \gtrsim O(1)$, we still expect (29) to represent the proper length scale, but l_0 will now depend on S. Introducing the Richardson number, $\mathrm{Ri} = S/B^2$ and assuming that $R \sim O(1)$, we expand Ri in powers of E:

$$\mathrm{Ri} = R_0 + R_1 E^{\beta} + O(E^{2\beta}) \tag{35}$$

By inserting (29) and (35) into (20), we now find that balance requires

$$\alpha = 1/3$$
$$\beta = 2/3 \tag{36}$$

Equating equal powers in E yields

$$R_0 = \frac{(1+\mathrm{Pr})^2}{4\,\mathrm{Pr}}$$
$$R_1 = -\frac{(1+\mathrm{Pr})^2}{4\,\mathrm{Pr}^2 S}\left(\frac{\pi^2}{l_0^2} + (\mathrm{Pr}\,S + 1)^3 l_0^4\right) \tag{37}$$

By maximizing Ri with respect to l_0, we finally obtain

$$\mathrm{Ri}^c = \frac{(1+\mathrm{Pr})^2}{4\,\mathrm{Pr}} \times \left[1 - 3(1 + \mathrm{Pr}^{-1} S^{-1})\left(\frac{\pi^2}{2}\right)^{2/3} E^{2/3} + O(E^{4/3})\right] \tag{38}$$

$$l_c = (\mathrm{Pr}\,S + 1)^{-1/2}\left(\frac{\pi^2}{2}\right)^{1/6} E^{-1/3} + O(E^{1/3}) \tag{39}$$

By letting $S \gg 1$ and rescaling the horizontal length such that $l' = Bl$, (38) and (39) are seen to be identical to the results obtained by Walton (1975). Again the stabilizing effect of viscosity is noted, while for ν fixed, the system is destablized if $\mathrm{Pr} \neq 1$, analogous to the findings of McIntyre (1970). As expected, we see from (39) that strong vertical stratification (S large) gives a tendency towards cells with large horizontal extent. This tendency, however, is opposed when the heat diffusion coefficient is large (Pr small), since then a particle more easily adjusts its temperature to that of the surroundings. Hence the constraining effect of vertical stratification is less felt.

For larger values of E, the series expansions do not converge. In principle we can find B_c and l_c from (20) and (22) for general values of the parameters by minimizing B with respect to l for fixed values of Pr, S and E. This is rather laborious, however, and it proves much quicker to obtain B_c (or Ri^c) by numerical interpolation.

When $S = 0$, it is easily seen from (20) and (22) that only two independent parameters occur in the problem. One is the Ekman number E and the other may be defined as

$$F = (1 + \mathrm{Pr})\, B/2 \qquad (40)$$

In Fig. 1 we have displayed the critical baroclinicity parameter B_c as a function of E for small and moderate values of S and $\mathrm{Pr} = 1$. The curve for $S = 0$ represents as explained above, the parameter $F_c = (1 + \mathrm{Pr})B_c/2$ and is accordingly valid for all Pr. The dashed curve represents formula (33), which is seen to be a good approximation only for very small E. The open circles are values for $S = 0$ from Emanuel (1978) for stress-free boundaries, and the agreement is seen to be quite good. The analytical results of Kuo (1954) for neutral vertical stratification (not displayed in the figure) are very close to Emanuel's and ours. Unfortunately neither Kuo nor Emanuel report results for E smaller than 0.001. The figure clearly exhibits, for given Pr, the stabilizing effect of vertical stratification and viscous diffusion, while as discussed before for small E and $S = 0$, the effect of increasing the Prandtl number is to destabilize the system.

In Fig. 2 we have plotted the critical Richardson number as a function of E for $S \gtrsim O(1)$ and again chosen $\mathrm{Pr} = 1$. This value is not an unreasonable order of magnitude estimate for many problems in the earth's atmosphere and oceans involving turbulent heat transfer. For $\mathrm{Pr} = 1$ it can also easily be shown that the instability is monotonic. The dashed curves represent the analytical result (38). The formula is seen to be valid for larger E the larger S is, though it should never be extended beyond $E = 0.01$. Again the stabilizing effect of viscous diffusion is obvious.

When $S \gg 1$ one is justified in making the hydrostatic approximation. By rescaling the previous horizontal wavenumber l as

$$l = l' \frac{B}{S}\left(\frac{1 + \mathrm{Pr}}{\mathrm{Pr}}\right) \qquad (41)$$

it is easily shown that the stationary problem again is governed by only two independent parameters; one is E, as before, while the other may be defined as

$$G = \frac{4\,\mathrm{Pr}}{(1 + \mathrm{Pr})^2}\,\mathrm{Ri} \qquad (42)$$

(see also Emanuel, 1978). This is also obvious from solutions (38) and (39), which in the case $S \gg 1$ reduce to

$$G_c = 1 - 3\left(\frac{\pi^2}{2}\right)^{2/3} E^{2/3}$$
$$l'_c = \frac{1}{2}\left(\frac{\pi^2}{2}\right)^{1/6} E^{-1/3} \qquad (43)$$

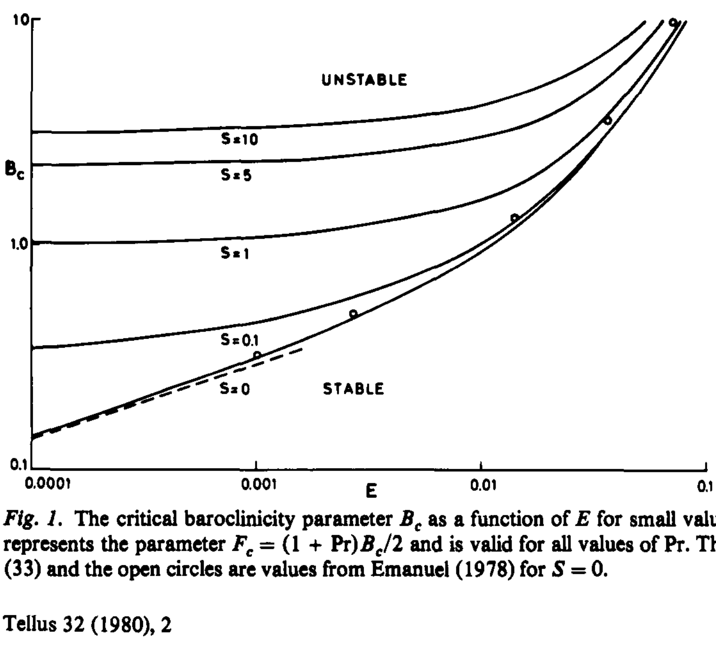

Fig. 1. The critical baroclinicity parameter B_c as a function of E for small values of S and $\mathrm{Pr} = 1$. The curve $S = 0$ represents the parameter $F_c = (1 + \mathrm{Pr})B_c/2$ and is valid for all values of Pr. The dashed curve is the analytical result (33) and the open circles are values from Emanuel (1978) for $S = 0$.

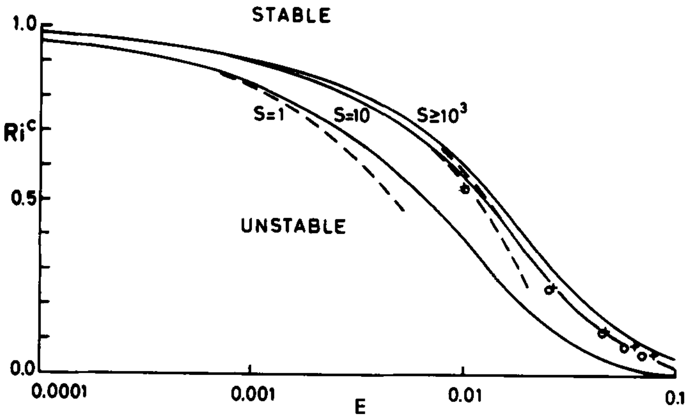

Fig. 2. The critical Richardson number Ri^c as a function of E for moderate and large values of S and $Pr = 1$. The dashed curves represent the analytical result (38). The open circles and crosses are values for free and rigid boundaries, respectively, from Emanuel (1978) for $S \gg 1$.

For $Pr = 1$ we have $G = Ri$ from (42), and hence the curve for $S \gg 1$ in Fig. 2 can be directly compared with Emanuel's results for the hydrostatic case. We have plotted some of his results for free boundaries (open circles) and rigid boundaries (crosses) within the Ekman number region he works in, and the agreement with the present analysis is seen to be quite satsifactory.

In Fig. 3 we have plotted the critical wavenumber corresponding to the situation depicted in Fig. 2. We have chosen to plot l_c times $(S + 1)^{1/2}$ as a function of E, and we see that the analytical results (39) (dashed curve) are valid for all S when E is small. In particular it is valid for larger E the larger S is, but not beyond $E \sim 0.01$. For comparison we have plotted some of Emanuel's results for $S = 0$ and free boundaries (open circles).

In Fig. 4 we have compared our nondimensional critical wavelength L (solid curve), defined by $2\pi/l'$ from (41) with Emanuel's result for $S \gg 1$ and free boundaries. Again the agreement is not too bad.

Unfortunately, Calman (1977) in his experiments with large Prandtl, or more correctly Schmidt numbers, does not give sufficient information about the values of his parameters to make possible direct comparison with his results.

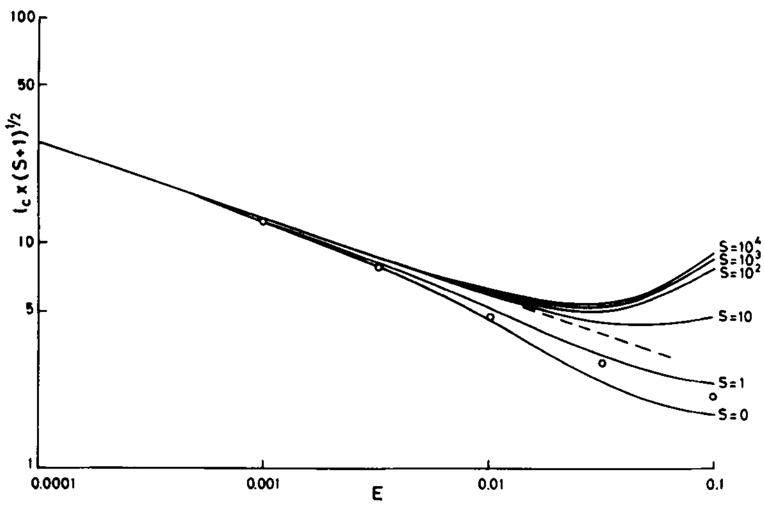

Fig. 3. The critical wavenumber l_c times $(S + 1)^{1/2}$ vs. E for a wide range of S and $Pr = 1$. The dashed curve is the analytical result (39). Open circles are from Emanuel (1978) for $S = 0$ and free boundaries.

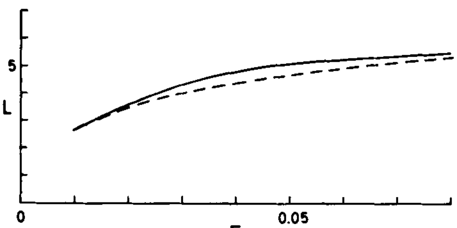

Fig. 4. Dimensionless critical wavelength $L = (2\pi/l_c)(2/\mathrm{Ri}^{1/2}S^{1/2})$ for $S \gg 1$. Solid curve: present result; dashed curve: Emanuel (1978) for free boundaries.

Even though a Prandtl number of order unity in some cases does provide a reasonable approximation for a turbulent ocean or atmosphere, there are cases where this is definitely not so. For example, in other planetary atmospheres or in stellar material, where the effective conductivity may be greatly enhanced by radiation, the Prandtl number may become quite small.

Let us for simplicity consider the limit $\mathrm{Pr} \to 0$. Then, as also discussed in connection with solutions (38) and (39), the exchange of heat between a particle and its surroundings takes place instantaneously. Hence a particle never "feels" the stratification, and the buoyancy effect drops out of the problem, or from (11): E finite, $\mathrm{Pr} \to 0$ implies that $\sigma \to 0$. This is in fact the situation considered by Gammelsrød (1975), but with a different choice for m in (16). The relevant stability boundary for this case, i.e. B_c, is obtained from Fig. 1 as twice the value of B_c for $S = 0$ and $\mathrm{Pr} = 1$. It can also be shown that the most unstable disturbance in this case is monotonic.

To apply the present analysis to the earth's atmosphere or oceans, appropriate values of the nondimensional parameters involved must be ascertained. Assuming for the atmosphere that $H =$ 1–10 km, $\nu = 10^5$ cm²/s, $f = 10^{-4}$ s^{-1}, $N = 10^{-2}$ s^{-1} we obtain

$$S = 10^4, \quad E = 10^{-3}\text{–}10^{-1} \tag{44}$$

while for the ocean we take $H = 100$–1000 m, $\nu = 10^2$ cm²/s, $f = 10^{-4}$ s^{-1}, $N = 10^{-3}$–10^{-2} s^{-1} which leads to

$$S = 10^2\text{–}10^4, \quad E = 10^{-4}\text{–}10^{-2} \tag{45}$$

It is thus seen that S and E are roughly comparable in the atmosphere and the ocean.

It is interesting to note from the present results that for many "normal" situations the criticial wavelengths becomes quite large. Thus for $E = 10^{-2}$, $S = 10^4$ and $\mathrm{Pr} = 1$, which are not unreasonable values for the ocean and the atmosphere, we find a critical value

$$\mathrm{Ri}^c = 0.6 \quad \text{for} \quad l_c = 0.062 \tag{46}$$

which corresponds to a critical wavelength $\lambda_c = 2\pi/l_c = 100$, or dimensionally $\lambda_* = 100$ H. The magnitude of the critical shear in this example is 0.013 s^{-1}.

It can be shown that the cell structure in the meridional plane when $\mathrm{Pr} = 1$ does not differ much from that of the inviscid case. The most unstable horizontal wavelength is of course finite (and rather large in this example), but the cells are tilted towards heavier fluid and the tilt lines nearly coincide with the isophycnals of the basic flow.

5. Final remarks

Since shear of the same order of magnitude as the Brunt-Väisälä frequency is not uncommonly found in the ocean or the atmosphere, the generation of roll vortices by the symmetric instability mechanism discussed in this paper should be theoretically possible. The pertinent question is: Have cell structures originating from this mechanism been observed in nature, and if not, where should one look to find them?

Concerning the explanation of the Langmuir circulations observed in wind-driven shear flow in a shallow mixed layer of the upper ocean, symmetric instability may be of importance (Gammelsrød, 1975), although the Rossby number in the case he refers to probably is too large for the Coriolis effect to play a dominant part (Leibovich and Radhakrishnan, 1977). It is more likely that the present type of instability is associated with larger and deeper current systems. However, due to their large wavelengths (typically ten to a hundred times the depth), the cells may be difficult to observe.

Cloud streets are frequently observed in the atmosphere when cold Artic air flows over open warm sea. The cloud base is typically situated at a height of 1 km, just below the inversion layer. In such cases thermal convection in shear has often been offered as the most obvious explanation of the phenomenon. However, in these cases the Rayleigh number may of order 10^{12}–10^{15}, which is very much larger than the critical Rayleigh number corresponding to the onset of convection ($\sim 10^3$).

Accordingly the convective motions are turbulent with a characteristic scale much less than the height, H, of the inversion layer, and the heat and momentum transport may be considered as eddy processes. This may give rise to larger secondary convection cells, as demonstrated by Foster (1972), where he develops the idea of a hierarchy of convection leading to an increasingly larger scale. However, due to the large vertical eddy transfer associated with the primary convection, the mean vertical density stratification will be approximately neutral in the main part of the layer, i.e. $S = 0$ in our analysis. When we have a nonzero shear, then the symmetric instability mechanism discussed in the present paper may be important. Kuettner (1971) reports a vertical shear of the order of 10^{-2} s^{-1} in typical cloud street situations. This implies a baroclinicity parameter B of order 10^2. Taking $\kappa = \nu = 10^5$ cm^2/s, $f = 10^{-4}$ s^{-1} and $H = 1000$ m, i.e. Pr = 1, $E = 0.1$, the present analysis yields $B_c = 15.1$ so that symmetric instability may well be possible. The dimensional critical wavelength λ_* is found to be 3.7 H, which is in excellent agreement with the observed spacing of cloud streets (Kuettner, 1971). For $B = 100$, the growth rate in this example is 4.4 $\times 10^{-4}$ s^{-1}, corresponding to an e-folding time of about 38 min.

Cloud streets have also been observed at much higher levels (cirrus streets), and often associated with the jet stream (Doswell and Schaefer, 1976). In such cases, where the generation process occurs above the Ekman layer in a stably stratified atmosphere, the instability mechanism discussed here may be relevant.

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СИММЕТРИЧЕСКАЯ НЕУСТОЙЧИВОСТЬ СТРАТИФИЦИРОВАННОГО ГЕОСТРОФИЧЕСКОГО ТЕЧЕНИЯ

Теоретически исследуется линейная устойчивость устойчиво стратифицированного геострофического течения между параллельными горизонтальными плоскостями. Рассматриваются двумерные возмущения с осями по направлению течения. Диффузионные процессы входят в задачу устойчивости через число Экмана E и число Прандтля Pr. Анализ проводится для общего случая устойчивой вертикальной стратификации, характеризуемой параметром $S = N^2/f^2$ где N-частота Брента-Вяйсиля, а f-параметр Кориолиса (постоянный). Когда E мало, горизонтальный масштаб возмущения на границе устойчивости составляет $\propto \mathrm{H}(1+\mathrm{Pr}\,S)^{1/2}E^{1/3}$, где H-расстояние между граничными плоскостями. Бопвшие значения E стабилизирует систему. При заданной вязкости и $S=0$ система дестабилизируется с ростом Pr, тогда как при $S \gg 1$ дестабилизация имеет место, если $\mathrm{Pr} \neq 1$.