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An inquiry on the nature of CISK. Part I

By MANKIN MAK, *Laboratory for Atmospheric Research, University of Illinois, Urbana, Illinois 61801, U.S.A.*

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ABSTRACT

It is shown that a puzzling conspicuous feature in Charney and Eliassen's (1964) result represents essentially an unstable internal gravity wave distorted by the use of the balance condition. This mode arises from the part of the heating representation associated with the vertical velocity per se at the interior level. This mode is fundamentally identical with the most unstable mode in a conventional Wave-CISK model. Such modes would not arise if the heating is parameterized in general solely in terms of the rotational component of the flow. The connection between this proposed treatment and the one using an additional frequency-dependent heating parameter will be discussed.

1. Introduction

Since the original works on CISK (Conditional Instability of the Second Kind) by Charney and Eliassen (1964; to be referred to as CE) and Ooyama (1964), there have been a large number of articles which attempted to either identify and correct the shortcomings of the original CISK or extend the concept of CISK to investigating wave disturbances (the so-called Wave-CISK theory). The highlights in the subsequent conceptual developments are briefly noted as follows. Considerable efforts were made to account for the absence of short-length cutoff in the result of CE. Chang and Williams (1974) attributed this shortcoming to the treatment of the lower boundary condition. CE effectively assumed a balance between the heating and the adiabatic cooling at the top of the Ekman layer. Chang and Williams showed that if this restriction is removed, the adiabatic cooling effect would be progressively greater at shorter length scale whereas the destabilizing effect of the heating only reaches a finite value. Consequently, there would be a short-length cutoff. The use of this treatment in a quasi-geostrophic wave model had yielded a similar effect (Chang, 1971). The large-scale cutoff is essentially the Rossby deformation radius. Davies and de Guzman (1979) recently showed that if the heating profile has a maximum at a low level, there can also be a short-length cutoff. But the structure of such unstable mode (see their Fig. 3) is substantially different from that of a typical hurricane depression (e.g., La Seur and Hawkins, 1963).

It has also been found that the CISK solution in general is sensitive to the vertical resolution of the model (Chang and Williams, 1974; Koss, 1976). This is closely related to its sensitivity to the heating profile, or more fundamentally to the parameterization scheme (Stark, 1976). Nevertheless, with the use of a given heating scheme a proper vertical staggering of the dependent variables would enable us to study the basic instability process even with a coarse resolution (Koss, 1976). Stevens et al. (1977) showed that if the damping effect due to the vertical transport of horizontal momentum by the cumulus clouds is incorporated into a model, the CISK solution would be much less sensitive to the heating profile.

An extension of the original CISK concept is the so-called Wave-CISK theory. Its basic premise is that the convergence property of the disturbance itself can induce the cumulus heating. In the early Wave-CISK formulations a specific assumption was made to relate the diabatic heating linearly

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proportional to the vertical velocity at a chosen low level (Hayashi, 1970; Lindzen, 1974). The serious defect of such formulation is the result that the shortest possible gravity waves are the distinctly most unstable modes, contrary to observations. Even when a closed scheme of parameterization is used, the same feature prevails although the growth rate is much smaller (Stark, 1976). The existing method proposed for the purpose of suppressing the unrealistic short gravity waves in a Wave-CISK model is to introduce an additional frequency-dependent heating parameter (Hayashi, 1971; Kuo, 1975; Davies, 1979). We will return to this issue later. Stevens, Lindzen and Shapiro (1977) demonstrated a basic inadequacy of an inviscid theory, namely the relative magnitudes of the various wave fields in such unstable mode are in poor agreement with the observed counterparts. They used a diagnostic approach to show that the damping effect of the vertical transport of horizontal momentum by the cumulus clouds very likely plays an important role.

One conspicuous puzzling feature in the result of CE has attracted hardly any attention in the literature. It turns out to be related to the controversy about the dominance of short waves in the Wave-CISK formulation. The puzzling feature is that the growth rate in CE becomes infinitely large at a certain length scale when their model heating parameter μ exceeds the stratification parameter $1/\kappa$. Such a feature is also present in the results of Bates (1973) who called it Type B CISK. Shukla (1978) found a similarly puzzling result in his instability analysis of monsoon depression (see his Fig. 4). In CE the authors discredited the puzzling feature on the ground that the critical value of μ is somewhat larger than the expected atmospheric value (0.91 versus 0.8). This appears to be a weak argument. It is the purpose of this note to show that this puzzling feature has a simple explanation and has implications of fundamental interests. We will specifically show that the puzzling feature in CE is in essence related to the dominant unstable mode in a conventional Wave-CISK model except that it is manifested in a disguised fashion. We will do so by demonstrating that the Wave-CISK modes and the puzzling feature in CE arise from basically the same mathematical origin. Based on this finding, we may attempt to devise a method which can suppress the unrealistic modes without introducing a new parameter. The connection between the proposed method and the one requiring the use of an additional frequency-dependent heating parameter will be discussed. An application of this method will be given in a separate article.

The clue that leads to this analysis is the realization that the condensational heating formulations in CE and in the conventional Wave-CISK models are partly similar. Although the Ekman layer dynamics is invoked in CE to relate a part of the heating to the vorticity at the bottom level, the remaining part of the heating in CE is proportional to the p-velocity per se at the interior level. In the Wave-CISK models, the heating is usually assumed to be proportional to the p-velocity at a low level. But it should be noted that essentially similar results would be obtained when the heating is related to the p-velocity at all levels (Stark, 1976). The common element of relating the heating to the divergent component of the flow may be suspected of being responsible for the puzzling feature in CE as well as for the linear increase of the growth rate with wavenumber of the conventional Wave-CISK modes.

2. A simple model analysis

A kinematically simple model may be used to reexamine the nature of the puzzling feature in CE. We will consider a two-layer slab-symmetric model having the same physics as those incorporated in CE except that "unbalanced" disturbances will also be examined. As special cases we will be able to examine the specific effects of the heating formulation on the disturbances investigated in CE and in a Wave-CISK formulation. The governing equations in a p-coordinate for inviscid perturbations symmetric with respect to the x-axis in a stably stratified basic state at rest are

$$\frac{\partial u}{\partial t} = fv \tag{1}$$

$$\lambda \frac{\partial v}{\partial t} = -\frac{\partial \phi}{\partial y} - fu \tag{2}$$

$$0 = -\frac{\partial \phi}{\partial p} - \frac{RT}{p} \tag{3}$$

$$\frac{\partial T}{\partial t} = \frac{S}{R} p\omega + \frac{\dot{Q}}{c_p} \tag{4}$$

$$\frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0 \tag{5}$$

where $\lambda = 0$ (or 1) is a trace indicator for describing balanced (or unbalanced) disturbances. u, v, ω, ϕ and T are respectively the velocity components in the x-, y- and p-direction, the geopotential and the temperature. R is the gas constant, $\dot{Q}/c_p$ the condensational heating to be paramaterized and

$$S \equiv -\frac{R}{p}\left(\frac{\partial \bar{T}}{\partial p} - \frac{R\bar{T}}{c_p\, p}\right)$$

the static stability. Unconditional heating is used for simplicity. The normal mode solution in the form of $u(y, p, t) = U(p)\, e^{\gamma t + ily}$ etc. exists. Upon substituting this form of solution into eqs. (1) to (5), the resulting five equations for the amplitude functions can be reduced to two as

$$\gamma U = -\frac{f}{il}\frac{dW}{dp} \tag{6}$$

$$\frac{\gamma}{il}\frac{dU}{dp} + \lambda\frac{\gamma^2}{l^2 f}\frac{d^2 W}{dp^2} = \frac{S}{f}W + H \tag{7}$$

where $H(p) \equiv R\dot{Q}/c_p f p$ is the heating, $U(p)$ and $W(p)$ the amplitude functions of u and ω. Casting eqs. (6) and (7) into a finite-difference form for a two-layer model as in CE, we obtain

$$\gamma U_1 = -\frac{f}{il\Delta}(W_2 - W_0) \tag{8}$$

$$\gamma U_3 = -\frac{f}{il\Delta}(W_4 - W_2) \tag{9}$$

$$\frac{\gamma}{il\Delta}(U_3 - U_1) + \lambda\frac{\gamma^2}{fl^2\Delta^2}(W_4 - 2W_2 + W_0) = \frac{S}{f}W_2 + H_2 \tag{10}$$

where the subscript n ($n = 0, 1, 2, 3, 4$) refers to pressure levels $p_n = n\Delta/2$ with $\Delta = 500$ mb. For the moment, we use boundary conditions and a heating representation similar to those used in CE,

$$W_0 = 0 \tag{11}$$

$$W_4 = W_{\text{Ekman}} = ilKU_4 \tag{12}$$

$$H_2 = -\xi(W_4 + aW_2) \tag{13}$$

where $a = 2\bar{\rho}_2/\bar{\rho}_4$, $\bar{\rho}_n$ the density at the nth level, K the frictional parameter based on the Ekman layer dynamics ($K = \bar{\rho}_4\, g\sqrt{\nu/2f}\,\sin 2\alpha$, ν the eddy viscous coefficient, α the surface cross-isobar angle), ξ a parameter indicating the heating intensity. Equations (11), (12), (13) are the counterparts of eqs. (4.5), (2.4) and (4.3) in CE. The additional condition used in CE (their eq. (5.7)) for closing the set of equations is equivalent to a statement of no temperature perturbation at the bottom level of the model (Chang and Williams, 1974). This implies

$$U_4 = U_3 \tag{14}$$

Upon substituting eqs. (11), (12), (13), (14) into (8), (9) and (10) we obtain

$$\gamma U_1 = -\frac{f}{il\Delta}W_2 \tag{15}$$

$$\left(\gamma + \frac{fK}{\Delta}\right)U_3 = \frac{f}{il\Delta}W_2 \tag{16}$$

$$\frac{\gamma}{il\Delta}(U_3 - U_1) + \lambda\left(\frac{i\gamma^2 K}{lf\Delta^2}U_3 - \frac{2\gamma^2}{fl^2\Delta^2}W_2\right) = \frac{S}{f}W_2 - \xi(ilKU_3 + aW_2) \tag{17}$$

for the unknowns U_1, U_3 and W_2. Existence of non-trivial solution implies a characteristic equation of γ as

$$\lambda\left[\gamma^3 + \gamma^2\frac{fK}{2\Delta}\right] + \gamma\left[f^2 + \frac{l^2\Delta^2 f}{2}\left(\frac{S}{f} - a\xi\right)\right] + \frac{l^2 f^2\Delta K}{2}\left[\frac{f}{l^2\Delta^2} + \frac{S}{f} - (1+a)\xi\right] = 0 \tag{18}$$

3. Results

Special case 1

To compare the model with the CE formulation, we set $\lambda = 0$ (balanced dynamics) and $a = 1$. It follows from (18) that there is only one root of γ, namely

$$\gamma = \frac{l^2 f^2\Delta K\left(2\xi - \frac{f}{l^2\Delta^2} - \frac{S}{f}\right)}{2f^2 + l^2\Delta^2 f\left(\frac{S}{f} - \xi\right)} \tag{19}$$

This solution is the counterpart of the growth rate result in CE with basically identical properties. Instability would not be possible unless the heating parameter ξ is sufficiently large to overcome the stabilizing effects of rotation $(-f/l^2\,\Delta^2)$ and of the stratification $(-S/f)$ as indicated in the numerator of (19). On the other hand, if ξ exceeds S/f, the denominator of (19) would be zero at $l = l_*$ where

$$l_* = \sqrt{\frac{2f}{\Delta^2(\xi - S/f)}}$$

It follows that γ would be infinitely large at $l = l_*$. This is the counterpart of the puzzling feature in CE. So long as $S/2f < \xi < S/f$, the growth rate would be finite for all length scales corresponding to the major part of the result in CE.

Now returning to the general case ($\lambda = 1$), we solve (18) for the three roots. Two of the roots obviously represent the inertio-gravity wave modes and the remaining root represents the inertio-rotational mode. It is found that all three roots are either real and negative or complex with a negative real part if 2ξ is smaller than S/f. Thus they are all stable. When 2ξ is larger than S/f, there is one unstable mode. Fig. 1 shows the variations of the growth rate of the unstable mode with the length scale $L \equiv 1/l$ for different values of heating $\xi f/S$. These computations are made using $f = 0.377 \times 10^{-4}\ \mathrm{s}^{-1}$ (15 degree latitude), $K = (\Delta/f)\ 1.72 \times 10^{-6}\ \mathrm{s}^{-1}$ (the same value used in CE) and $S = 1 \times 10^{-2}\ \mathrm{m}^2\,\mathrm{mb}^{-2}\,\mathrm{s}^{-2}$. The resemblance of this figure to CE's Fig. 1 is quite evident. The puzzling feature in CE is the counterpart of the subset of curves in our figure for $\xi f/S > 1$. For $\xi f/S < 1$ the growth rate increases towards an asymptotic value with increasingly small L. This mode represents the inertio-rotational mode. But for $\xi f/S > 1$, as exemplified by the curve labelled 1.1, the result is qualitatively different in the range of small length scales. Specifically, the growth rate invariably increases linearly with decreasing L. This feature is always present when we use any value of ξ greater than S/f. On the other hand, there appears to be analytic continuation of the variation of the growth rate with ξ in the range of large length scales. The implications of these results will become obvious when all three roots are examined together. Fig. 2 shows the real part of all three roots of γ at each L for the case $\xi f/S = 1.1$. A value of L indicated by L_* in the figure is particularly interesting. For $L > L_*$, two roots form a complex conjugate pair, having a negative real part. They represent stable inertio-gravity wave modes. Their frequency dependence on L is shown in Fig. 3. It is seen that the frequency increases asymptotically with L towards the value of f. The remaining root is an unstable inertio-rotational mode. But for $L < L_*$, all three roots are real and distinct. One of the inertio-gravity mode emerges as unstable. Its growth rate increases linearly without bound with decreasing values of L. The other inertio-gravity mode as well as the inertio-rotational mode are stable. This abrupt change of character of the unstable mode across L_* clearly has no physical basis, and is suspected of being a mathematical result of the model formulation. This is confirmed by the considerations of the following two special cases.

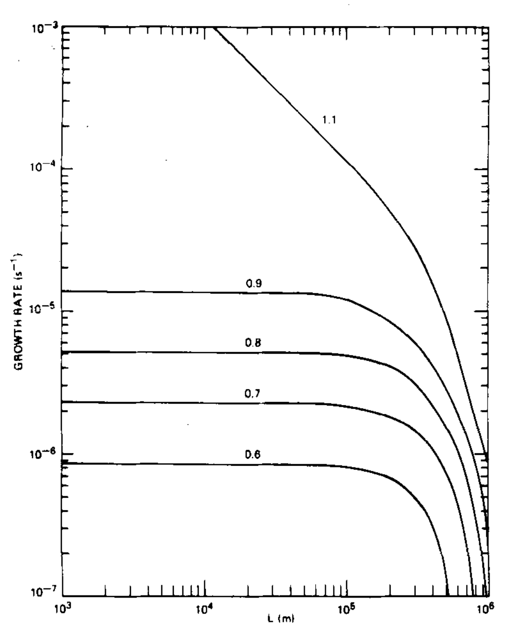

Fig. 1. Variations of the growth rate with length scale computed from eq. (18) for different relative heating intensity. Each curve is labelled with the value of $\xi/(S/f)$.

Special case 2

The heating is parameterized solely in terms of the rotational component of the flow at one level, such as due to the process of surface frictional convergence. Thus, we replace (13) by

$$H_2 = -\hat{\xi} W_4 = -\hat{\xi} i l K U_4 \tag{20}$$

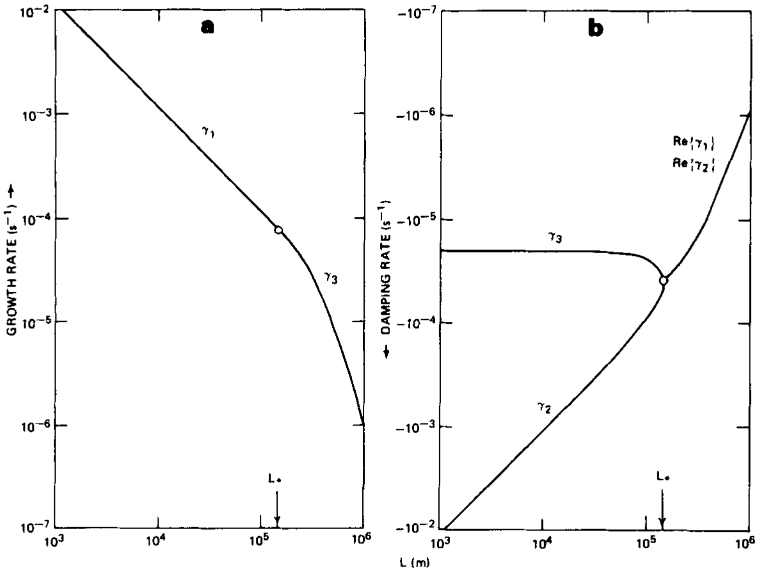

Fig. 2. Variations of the real part of the three eigenvalues of (18) with length scale. (γ_1, γ_2 refer to the inertio-gravity modes; γ_3 the inertio-rotational mode.)

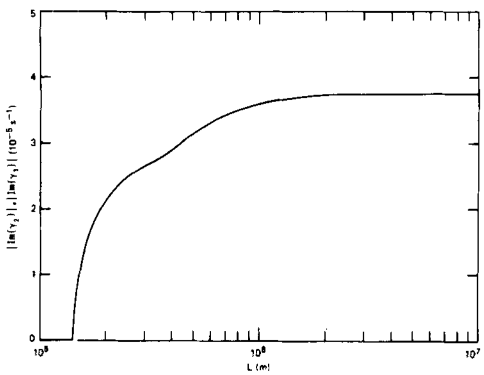

Fig. 3. Variations of the frequency of the inertio-gravity modes with length scale.

The corresponding characteristic equation would be

$$\gamma^3 + \gamma^2 \frac{fK}{2\Delta} + \gamma\left(f^2 + \frac{l^2\Delta^2 S}{2}\right) + \frac{l^2 f^2 \Delta K}{2} \times \left(\frac{f}{l^2\Delta^2} + \frac{S}{f} - \hat{\xi}\right) = 0 \tag{21}$$

The roots of γ satisfying (21) reveal that the two inertio-gravity wave modes are stable for all values of l and ξ, and that the inertio-rotational mode can, however, become unstable if $\hat{\xi}$ is greater than S/f. The growth rate of this mode is finite for all values of L. Furthermore, this solution is essentially unchanged when the "balance" assumption is invoked. It basically corresponds to the result obtained by Ooyama (1964) and Ogura (1964). The results show that what stabilizes the inertio-gravity wave modes is not the use of the balance condition but is rather associated with the representation of the diabatic heating.

Special case 3

For the sake of argument, let us now parameterize the heating solely in terms of the divergent component of the flow at one level, say W_2. We therefore replace (13) by

$$H_2 = -\hat{\xi} W_2 \tag{22}$$

We may as well leave out the effect of surface friction since it is not allowed in this case to have a direct feedback effect on the heating. Thus, we also set $K = 0$ and hence $W_4 = 0$. It follows that the characteristic equation would become simply

$$\frac{2\gamma^2}{l^2 f\Delta^2} + \frac{2f}{l^2\Delta^2} + \frac{S}{f} - \hat{\xi} = 0 \tag{23}$$

This indicates that only the two inertio-gravity wave modes can be excited. When $\hat{\xi}$ is greater than S/f and when L is sufficiently small, both roots have real values, one being positive and the other negative. Both the growth rate of the unstable mode and the decay rate of the stable mode increase linearly with decreasing length scale. These solutions are clearly the counterparts of the two modes labelled γ_1 and γ_2 in the range of small length scale in Fig. 2a. They essentially correspond to the conventional Wave-CISK modes.

4. Concluding remarks

In light of the results above, we may infer that the puzzling feature in CE and the Wave-CISK modes are essentially identical. They both arise from the use of a condensational heating that is directly related to the divergent component of the flow. It is pertinent to recall now the findings in two previous investigations (Stark, 1976; Lilly, 1960). Stark (1976) found that no matter whether the heating in a Wave-CISK model is related to the vertical velocity at only one level or at all levels, essentially similar unstable modes are obtained. Specifically the shortest possible gravity waves are most unstable. Lilly (1960) in his analysis of conditional instability set the heating to be proportional to the vertical velocity at each level. That treatment is equivalent to a special case of the Wave-CISK formulation, namely one that uses a uniform heating profile. The major result is also that the shortest gravity waves are most unstable. We may infer that the puzzling feature in CE, the most unstable mode in a conventional Wave-CISK model, and that in Lilly (1960) are all fundamentally alike. This realization suggests that if the heating is parameterized solely in terms of the rotational component of the flow, only the inertio-rotational mode would become unstable. The resulting stability property of such unstable mode is compatible with the basic philosophy of CISK. We are presently aware of only one mechanism that can associate the rotational component of a flow with the moisture convergence, namely the Ekman layer dynamics. We do not know enough about the complex scale coupling processes to conceptually account for such an association in general. An application of the proposed method will therefore have to be justified a posteriori. One such application in conjunction with the role of cumulus momentum transport in CISK will be reported in a separate article.

The connection between the treatment of heating advocated here and the one used in the modified Wave-CISK models is apparent (Hayashi, 1971; Kuo, 1975; Davies, 1979). In those formulations, the undesirable Wave-CISK modes are stabilized by introducing a frequency-dependent heating parameter. It was interpreted as a time lag between the heating and the moisture convergence by Davies (1979). The proposed treatment here effectively introduces in the heating an intrinsic time scale of the order of the reciprocal of the Coriolis parameter. This is so because the rotational component of the flow is strongly influenced by the rotation of the earth. Thus, it is not surprising to find similar effects. The advantage of the proposed method is that no additional parameter needs be introduced. We effectively propose to let the internally determined time scale of the rotational component of the flow dictate the time scale of the heating. Under this assumption, the vorticity dynamics of a large-scale disturbance modulates the moist convection implicitly through the secondary circulation. The moist convection in turn has a feedback effect on the vorticity dynamics itself.

5. Acknowledgement

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ИЗУЧЕНИЕ МЕХАНИЗМА CISK. ЧАСТЬ I

Показано, что загадочная особенность в результатах Чарни и Элиассена (1964) обусловлена только наличием неустойчивой внутренней гравитационной волны, искаженной в результате использования условия баланса. Такая мода возникает из той части выражения для нагрева, которая связана с наличием вертикальной скорости на внутреннем уровне. По существу эта мода есть наиболее неустойчивая мода в обычной модели волн CISK. Подобные моды не будут возникать, если процесс нагрева параметризовать только в слагаемых вихревого компонента потока. Обсуждается связь между предлагаемый трактовкой и другой, где используется дополнительный параметр, учитывающий нагрев в зависимости от масштаба волновых возмущений.