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SHORT CONTRIBUTION

On the rate of ice formation in water cooled by a more saline sublayer

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ABSTRACT

A less saline water mass (of salinity S_1) at its freezing temperature (T_{1f}) is superposed on a more saline water mass (of salinity S_2) of a temperature $T_2 < T_{1f}$. Both water masses have temperatures lower than their respective temperatures for maximal density, thus $0 \leqslant S_1 < S_2 < 24.7‰$. As heat diffuses much faster than salt there will be thermal convection in the two water masses. Heat is transferred down into the lower layer and ice forms in the upper layer. The present paper shows that in the actual parameter range ($R_\rho \gg 10$) the heat exchange between the layers is controlled by the molecular processes in the pycnocline and not by the properties of the induced thermal convection in the homogeneous layers. A theory that predicts some experimental results on the growth-rate of ice quite well is developed.

1. Introduction

It is well known that a fluid whose density is dependent on both salinity and temperature may be in a state of vertical convection although the fluid is stably stratified, see Turner (1973, chap. 8). The case of interest in this paper is when the temperature is destabilizing (the so-called diffusive regime). This normally occurs when fresher, colder water rests on saltier, warmer water. If the temperatures of the water masses involved are below the temperature for maximal density, however, the temperature is destabilizing when fresher, warmer water rests on saltier, colder water. Ice formation by cooling from below in such a system (with the upper water mass being of freezing temperature) was studied both theoretically and experimentally by McClimans, Steen and Kjeldgaard (1977, 1978) (Their report from 1977 is denoted by MSK in the following). In the case of MSK the temperature is destabilizing in both the upper and lower layers and the salinity is thus less than 24.7‰. The physical properties of sea water are described in many textbooks, e.g. in Dietrich (1963).

Martin and Kauffman (1974) studied a similar system. In order to simulate conditions occurring in the Arctic, they instead used fresh water in the upper layer and sea water ($S = 34$‰) in the lower layer. The temperature is then stabilizing in the lower layer and vertical convection occurs only in the upper layer.

Martin and Kauffman were interested in ice formation in the so called under-ice melt ponds (see their paper for both a description of the phenomenon and a review of the subject). Therefore they started their laboratory experiments with a layer of pure ice floating on top of the upper, fresh layer. The authors have given a careful description of the ice growth in their experiments and we will repeat the main features here. The ice growth took place in three phases. Phase (1): The rapid diffusion of heat relative to salt causes supercooling at the bottom of the upper layer. Supercooled water rises to the ice layer at the top where it nucleates into thin, vertical, interlocking ice crystals. Phase 2: When the ice sheets have grown to the interface supercooling ceases. In the interfacial region a lateral ice growth starts and this continues until a horizontal ice sheet forms.

Phase 3: The horizontal ice sheet increases in thickness and migrates upwards.

The inspiration for the study by MSK was the possible ice growth by cooling from below in fjords where cold river water may flow out over a brackish layer of near freezing temperature. Accordingly they started their experiments without an ice cover on top of the upper layer. The bulk of the upper layer was supercooled by heat losses to the lower layer. Ice formation was initiated by the addition of a few snow flakes to the upper layer and a horizontal ice sheet then developed. This phase, which by obvious reasons was absent in the Martin and Kauffman experiment, may be termed Phase 0 for instance.

The experiments by MSK were finished in Phase 1. At the end of the experiments the horizontal ice sheet at the top contained more than 90% of all ice generated and horizontal ice growth in the pycnocline region was never observed (Steen, personal communication). Thus the process of heat exchange between the layers was obviously the same throughout the experiments.

In the following we will consider two water masses $(S, T) = (S_1, T_1)$, (S_2, T_2). Both water masses have temperatures lower than their respective temperatures for maximal density, $T_m(S)$, so $S_1 < S_2 < 24.7‰$. The lighter, fresher upper water mass is supposed to be of freezing temperature and all its boundaries are thermally insulated except for the lower one through which heat can be exchanged with the underlying water mass. The water masses are separated by a pycnocline of thickness 2δ (see Fig. 1). For $T_1 > T_2$ heat is transferred to the lower layer and some water in the upper layer is thereby supercooled. Ice will then form in this layer and the latent heat of ice formation is released, keeping the temperature of the bulk of the upper layer at freezing temperature. The lower, otherwise insulated layer is heated by conduction from the upper layer. The lower layer is supposed to be either infinitely deep or cooled so that the bulk of this layer is kept at a constant temperature.

The rate of heat production by freezing in the upper layer is equal to the rate of heat transfer to the lower layer. In order to model the growth-rate of ice, we have to calculate the heat exchange between the two layers. As heat diffuses much more rapidly than salt, we know that there must, at least intermittently, exist unstable (with respect to

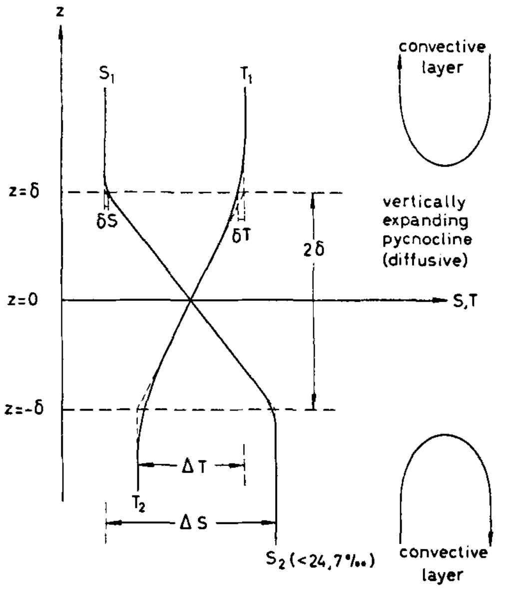

Fig. 1. Definition sketch for the parameters in and around the pycnocline.

an appropriate Rayleigh number) regions at the upper and lower ends of the pycnocline. Convective motions are in this way induced in the two layers. The exchange through the pycnocline is dependent upon the overall stratification, the thickness of the pycnocline and the diffusivities of salt and heat. It may be influenced in several ways by the convective motions in the two layers. These motions may for instance erode the pycnocline, thereby increasing the gradients. They may also generate breaking waves and turbulence in the pycnocline whereby the effective diffusivities for salt and heat are increased compared to the molecular diffusivities.

MSK constructed a model for the vertical heat flux that included only the convecting layers surrounding the pycnocline. The present paper, however, shows that the growing pycnocline controls the heat flux in the actual parameter range and presents a heat flux model that predicts the laboratory observations made by MSK. In contrast to the model proposed by MSK, the present model does not contain any constant that has to be determined from experimental data. However, an *in situ* observation of the halocline thickness is required for application.

2. Why the pycnocline determines the heat flux

Turner (1973) showed that the ratio between the density fluxes of salt and heat ($\beta F_S/\alpha F_T$) is unity when the density ratio $R_\rho = \beta\Delta S/\alpha\Delta T$ is equal to 1. For R_ρ greater than 1 the density flux ratio decreases to a constant value of 0.15 when $2 \le R_\rho \le 7$. α and β are defined by the following linear equation of state at constant pressure $\rho = \rho_0(1 + \alpha T + \beta S)$ where ρ_0 is some reference density. Linden (1974) suggested that the increase in the flux ratio for $R_\rho < 2$ is caused by the turbulent entrainment of the outer parts of the pycnocline, caused by convective motions in the more or less homogeneous layers outside the pycnocline.

For the case we are interested in, R_ρ is typically in the range $100 < R_\rho < 1000$, a regime somewhat discussed by Stern (1975). For sufficiently large values of R_ρ erosion of the pycnocline by convective motions is insignificant and the flux ratio through the ends of the pycnocline ($z = \pm\delta$) is, see Stern (1975, p. 205),

$$\left(\frac{\beta F_S}{\alpha F_T}\right)_{z=\pm\delta} = \left(\frac{k_S}{k_T}\right)^{1/2} \tag{1}$$

where k_S and k_T are the molecular diffusivities of salt and heat, respectively. If the fluxes through the pycnocline are caused by purely molecular processes, the density flux ratio is

$$\left(\frac{\beta F_S}{\alpha F_T}\right)_{z=0} = \frac{\beta k_S \dfrac{\Delta S - 2\delta S}{2\delta}}{\alpha k_T \dfrac{\Delta T - 2\delta T}{2\delta}} = \frac{\beta k_S(\Delta S - 2\delta S)}{\alpha k_T(\Delta T - 2\delta T)} \tag{2}$$

where ΔT and ΔS are the differences in bulk temperatures and salinities between the two water masses and 2δ is the thickness of the pycnocline. δT and δS are the temperature and salinity ranges in the outer part of the pycnocline where intermittent convection occurs. If the convection is nonpenetrative δT and δS are related by (see Stern, 1975)

$$\alpha\delta T = \beta\delta S. \tag{3}$$

The absolute magnitude of δT and δS depends on the appropriate critical Rayleigh number for the intermittent convection (c.f. Howards model described by e.g. Turner, 1973).

For the stationary case in which the pycnocline has a constant thickness, the following conditions must be fulfilled:

$$\left(\frac{F_S}{F_T}\right)_{z=\pm\delta} = \left(\frac{F_S}{F_T}\right)_{z=0}. \tag{4}$$

From eqs. (1), (2) and (4) we then get

$$\frac{\beta(\Delta S - 2\delta S)}{\alpha(\Delta T - 2\delta T)} = \left(\frac{k_T}{k_S}\right)^{1/2}. \tag{5}$$

If $R_\rho \gg 1$ we obtain by the help of eq. (3)

$$\frac{\beta(\Delta S - 2\delta S)}{\alpha(\Delta T - 2\delta T)} \gtrsim \frac{\beta\Delta S}{\alpha\Delta T}. \tag{6}$$

For a stationary case we require from (5) and (6) that

$$R_\rho \le \left(\frac{k_T}{k_S}\right)^{1/2}. \tag{7}$$

However, in the cases studied here $100 < R_\rho < 1000$ while for a heat-salt system $(k_T/k_S)^{1/2} \approx 10$ and hence *these cases can not be stationary.* The thickness of the pycnocline must increase with time.

Because of the high values of R_ρ we expect the diffusion of salt mainly to have an effect on the pycnocline thickness and we assume in the following that very little salt diffuses into the upper layer. We also assume that the weak convective motions in the layers do not erode the pycnocline. If the salt leakage into the upper layer is negligible we can in a very simple way calculate the evolution of the thickness of the pycnocline from its initial thickness. Assuming $\delta T/\Delta T \ll 1$ the heat flux from the upper to the lower layer is in this case completely determined by the thickness of the pycnocline and the temperature difference ΔT.

3. Evolution of the pycnocline thickness

We assume the convective motions in the upper and lower layers to be very weak. Then the vertical velocity shear (caused by the convective motions) over the pycnocline is small and a Richardson number for the pycnocline based upon this velocity shear, the density difference between the layers and the thickness of the pycnocline will be very large. If this is true we can be quite confident that salt and heat diffuse through the pycnocline solely by molecular processes (see Turner, 1973; Linden, 1974 and Stern, 1975). The appropriate diffusion equation for salt then is

$$\frac{\partial S}{\partial t} = \frac{\partial}{\partial z} k_S \frac{\partial S}{\partial z} . \tag{8}$$

The surface separating the upper layer from the nonconvecting part of the pycnocline is $z = \delta$ (with symmetry $z = -\delta$ is the coordinate for the lower boundary surface) and the thickness of the pycnocline is 2δ (see Fig. 1). In order to calculate the evolution of the pycnocline thickness we integrate eq. (8) from $z = 0$ to $z = \delta$. This gives

$$\int_0^{\delta} \frac{\partial S}{\partial t} dz = k_S \left(\frac{\partial S}{\partial z}\right)_{z=\delta} - k_S \left(\frac{\partial S}{\partial z}\right)_{z=0}$$

For $R_\rho \gg 10$, however, the salt flux into the upper layer is supposed negligible and thus we use the following boundary condition $k_S(\partial S/\partial z)_{z=\delta} = 0$. Assuming for simplicity a linear salinity distribution within the pycnocline (except near $z = \pm\delta$) we obtain

$$\frac{\partial}{\partial t} \bar{S}\delta - S_\delta \frac{\partial \delta}{\partial t} = k_S \frac{S_2 - S_1}{2\delta}$$

where $\bar{S} = (3S_1 + S_2)/4$ and $S_\delta = S_1$. Since $\partial/\partial t(S_1, S_2) = 0$, we obtain

$$\frac{S_2 - S_1}{4} \frac{\partial \delta}{\partial t} = k_S \frac{S_2 - S_1}{2\delta}$$

or

$$\frac{\delta}{\partial t} \delta^2 = 4k_S .$$

The solution to this equation is

$$\delta_t = \sqrt{\delta_0^2 + 4k_S t} \tag{9}$$

where δ_0 is the pycnocline thickness at $t = 0$. Note that the thickness of the pycnocline, δ_t, is independent of the bulk salinity difference, ΔS, as long as $R_\rho \gg 10$ (permitting us to assume that $k_S(\partial S/\partial z)_{z=\pm\delta} = 0$). This condition seems fulfilled for all possible combinations of S_2 and S_1 in the actual parameter range $0 \leq S_1 < S_2 < 24.7$, $T_1 = T_{1f}$ and $T_{2f} \leq T_2 < T_{1f}, T_{2m}$ where T_{1f}, T_{2f} are the freezing temperatures of the two water masses and T_{2m} is the temperature for maximal density of the lower water mass.

4. Determination of the growth-rate of ice

The heat transferred from the upper to the lower layer is released by the freezing process. The heat of fusion is L. The growth-rate of ice is

$$\dot{M} = \frac{1}{L} \frac{\rho_w}{\rho_{\text{ice}}} c_p k_T \frac{\Delta T}{2\delta} \tag{10}$$

where c_p is the specific heat of water of density ρ_w and ρ_{ice} is the density of ice. At time t we have the following ice thickness

$$M = \int_0^t \frac{\rho_w c_p k_T}{\rho_{\text{ice}} L} \frac{\Delta T}{2\delta} dt'$$

$$= \frac{\rho_w c_p k_T}{\rho_{\text{ice}} 4 \cdot L \cdot k_S} \Delta T[(\delta_0^2 + 4k_S t)^{1/2} - \delta_0]$$

where eq. (9) has been used. The mean growth-rate of ice during the period $(0, t)$ is

$$\bar{\dot{M}} = \frac{M}{t} = \frac{\rho_w c_p k_T}{\rho_{\text{ice}} \cdot 4 \cdot L k_S} \cdot \frac{\Delta T}{t} [(\delta_0^2 + 4k_S t)^{1/2} - \delta_0]$$

or

$$\bar{\dot{M}} = \frac{\rho_w c_p k_T}{\rho_{\text{ice}} \cdot L \cdot 4 \cdot k_S} \frac{\Delta T}{t} [\delta_t - (\delta_t^2 - 4k_S t)^{1/2}]. \tag{11}$$

We can thus calculate the mean growth-rate of ice when knowing the temperature difference between the layers, ΔT, the time during which the process acts, t, and the final thickness of the pycnocline, $2\delta_t$ (or equivalently the initial thickness $2\delta_0$).

5. Application of the theory to the experiments by MSK

We will use the parameter values given in Table 1. Equation (11) for the mean growth-rate (m/day) then is

$$\bar{\dot{M}} = 0.0064 \frac{\Delta T}{t} [\delta_t - (\delta_t^2 - 3 \cdot 10^{-9} \cdot t)^{1/2}]. \tag{11'}$$

Unfortunately MSK did not publish all the information needed for the present theory. The

Table 1. *Physical constants used in the numerical application of the growth-rate theory for ice*

$L = 3.35 \times 10^5\ \mathrm{Jkg^{-1}}$	$k_T = 1.4 \times 10^{-7}\ \mathrm{m^2\,s^{-1}}$
$c_p = 4200\ \mathrm{Jkg^{-1}\,k^{-1}}$	$k_S = 7.5 \times 10^{-10}\ \mathrm{m^2\,s^{-1}}$
$\rho_w/\rho_{ice} = 1.1$	

missing parts were, however, kindly given to the author of one of them (Steen, personal communication). In Table 2 the experimental results and our own calculations are given for their runs 2–11. (Run 12 was bad because of bottom freezing, in runs 13 and 14 not all necessary information was obtained and in runs 15 and 16 the salinity in the lower layer was too high for the theory to apply.)

In the experiments there are two main errors. Heat from the upper layer is conducted through the walls, we call this contribution to the ice growth $\dot{M}_\infty$. With the figures given in MSK we find that $\dot{M}_\infty$ (*m*/day) is approximately given by

$$\dot{M}_\infty = 0.005\,\overline{T}_\infty \tag{12}$$

where $\overline{T}_\infty$ is the temperature difference between the upper layer and the surroundings (given by MSK). Heat will also be conducted vertically through the plexiglass walls. We call this contribution $\dot{M}_{wall}$ and it is approximately given by

$$\dot{M}_{wall} = 0.3\,\overline{\dot{M}}. \tag{13}$$

We can correct the values of the observed ice growth-rate, $\dot{M}_{obs}$ (given by MSK), and

$$\dot{M}_{corr} = \dot{M}_{obs} - \dot{M}_\infty - \dot{M}_{wall}. \tag{14}$$

The values of $\dot{M}_{corr}$ are given in Table 2.

As can be seen from Table 2 the correlation between the observed, $\dot{M}_{corr}$, and calculated, $\overline{\dot{M}}$, ice growth-rates is rather good. Further evidence in favour of the theory is the fact that the pycnocline thickness increased during the course of the experiments. According to Steen (personal communication) the thickness of the pycnocline at the start of an experiment was typically 0.03–0.04 m.

6. Discussion

In the development of the theory it was assumed that the convective motions in the actual parameter range neither erode the pycnocline nor create turbulence in the pycnocline. The rather good predictions of the theory justify *a posterori* these assumptions.

The theory in this note is somewhat related to the one given by Martin and Kauffman (1974). The main difference is that they treated a system with sea water ($S = 34‰$) in the lower layer and thus temperature was stabilizing in that layer. In both theories the rate of heat transfer from the upper to the lower layer is controlled by the pycnocline. As shown in section 2 of this paper the thickness and time evolution of the pycnocline must be expected to be important in all cases where $R_\rho > 10$ for salt-heat systems. The theory given by MSK, however, does not take into account the thickness of the pycnocline. Accordingly there is no time dependence in their calculations of the growth-rate of ice.

Table 2. *Observed and calculated quantities in the MSK-experiments*

Run	δ_t	t	ΔT	$\overline{\dot{M}}$	$\dot{M}_{corr}$
(no)	(m)	(s)	(k)	(m/day)	(m/day)
2	4.5×10^{-2}	2.25×10^5	1.04	0.21×10^{-2}	0.199×10^{-2}
4	4.0×10^{-2}	2.34×10^5	0.62	0.15×10^{-2}	0.143×10^{-2}
5	4.5×10^{-2}	1.57×10^5	0.80	0.16×10^{-2}	0.135×10^{-2}
6	4.5×10^{-2}	1.78×10^5	0.82	0.16×10^{-2}	0.148×10^{-2}
7	4.5×10^{-2}	1.62×10^5	0.63	0.12×10^{-2}	0.122×10^{-2}
8	4.5×10^{-2}	2.52×10^5	0.88	0.18×10^{-2}	0.172×10^{-2}
9	4.0×10^{-2}	2.21×10^5	0.52	0.12×10^{-2}	0.115×10^{-2}
10	3.5×10^{-2}	1.58×10^5	0.34	0.09×10^{-2}	0.097×10^{-2}
11	3.5×10^{-2}	1.48×10^5	0.23	0.06×10^{-2}	0.094×10^{-2}

When applying a theory for double-diffusive heat exchange to natural systems, such as fjords, one has to properly consider the shear effect on the thickness of the pycnocline (Stigebrandt et al., 1976). The dynamically determined thickness of the pycnocline, 2δ, is given by

$$2\delta \geq Ri_{\text{crit}} \frac{u^2}{g \frac{\Delta\rho}{\rho}}$$

where u is the velocity difference and $\Delta\rho$ is the density difference between the two layers and Ri_{crit} is the critical Richardson number of the system (on the order of 1).

The rate of ice growth by cooling from below is typically 0.002 m/day, see Table 2. The relative importance of this process in fjords and coastal waters is in general certainly not large, as the typical growth-rate of ice from heat losses to the atmosphere is typically 0.02 m/day. Freezing from below may, however, occur even during periods with mild weather in the winter. When the conditions are favourable, freezing from below may thus occur for a longer time than freezing from above.

Finally it is tempting to speculate a little about the pattern of the vertical ice sheets. Is the pattern determined by the initial mode of thermal convection in the upper layer? This would imply very narrow convection cells with aspect ratios of about 1:40 in the Martin and Kauffman experiment. Or is the pattern determined by some stochastic pattern in the initial ice-cover at the top? These and other questions may perhaps be answered by direct observations of the convective cells in a new set of laboratory experiments.

7. Acknowledgement

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