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Internal solitary waves in a linearly stratified fluid

By PATRICK D. WEIDMAN, *Department of Aerospace Engineering, University of Southern California, Los Angeles, California 90007, U.S.A.*

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ABSTRACT

The mode shapes for internal solitary waves propagating in a linearly stratified fluid of finite depth are determined in the limit of weak stratification. The solutions obtained for both a fixed upper boundary and a free surface are compared with known results for exponential stratification. The profound changes which occur depending on whether the upper surface is fixed or free are derived from the strong influence of the free surface when the stratification is weak. Evidence is presented supporting the conjecture that the free surface can no longer effect qualitative differences in the modal structure when the fluid is heavily stratified.

1. Introduction

The study of finite amplitude internal waves of permanent form can be divided into two classes according to whether the density variation occurs in a fluid of finite or infinite depth. In the first case, the streamline deviations are the "sech^2" type, like the classical free surface solitary wave. Solitary waves in deep water, on the other hand, take the form of a Lorentzian distribution (Ono, 1975) and were first studied by Benjamin (1967) and Davis and Acrivos (1967).

The initial work on solitary waves in fluids of finite depth, which is the subject of this paper, is evidently due to Keulegan (1953) who considered a two-fluid system. Peters and Stoker (1960) and later Long (1965) found solitary waves in a continuously stratified fluid for the case of exponential stratification. The papers by Benjamin (1966) and Benney (1966) are particularly important because they suggest that internal solitary waves are possible for *arbitrary* (stable) stratification. Both authors used the two-fluid system and also the exponential density profile as specific examples of their general theory.

Another density distribution to consider is the linear density profile. Stratification of this type is perhaps of most interest to experimentalists because of the ease with which it can be created in the laboratory using the method of Oster (1965); in fact it was the need to interpret some internal wave motions in our own laboratory[1] which motivated the present investigation. Previous experiments using this stratification have been reported by Thorpe (1963), for example, in his investigation of progressive internal waves and by Browand and Winant (1972) in their study of upstream blocking. Theoreticians have also sometimes used the linear profile to approximate density changes in oceans and atmospheres. One can cite the recent work of Maxworthy and Redekopp (1976) who have adopted the linear density distribution as a model for the Jovian atmosphere in their study of the Great Red Spot.

It is perhaps worth noting that the present analysis for linear stratification affords an opportunity to study the behavior of internal solitary waves in a fluid which does not possess the special feature of having a constant Brunt-Väisälä frequency of oscillation. Although one would not anticipate any major differences, the aforementioned studies for exponential stratification by Peters and Stoker (1960), Long (1965), Benney (1966), and Benjamin (1966) are all characterized by a constant Brunt-Väisälä frequency since the fluid is necessarily non-Boussinesq.

[1] Private communication with Mr R. Amen and Dr T. Maxworthy concerning the collapse of a mixed region in a linearly stratified fluid.

2. Problem statement

Dubreil-Jacotin (1937) and later Long (1953) have shown that the Navier-Stokes equation for the steady, two-dimensional motion of a heterogeneous, incompressible fluid can be reduced to a single equation, now commonly referred to as Long's equation. In terms of the displacement $\delta'(x', y')$ of a fluid particle above its equilibrium height, we have (Long, 1953)

$$\nabla^2\delta' - \frac{1}{\rho}\frac{d\rho}{dy'_0}\left[\frac{(\nabla\delta')^2}{2} - \frac{\partial\delta'}{\partial y'} + \frac{g}{c^2}\delta'\right] = 0. \tag{2.1}$$

Here c is the constant horizontal velocity in the undisturbed region both upstream ($x' \to +\infty$) and downstream ($x' \to -\infty$) of the stationary wave, g the gravitational constant, $\rho(y'_0)$ the fluid density distribution, and y'_0 is the elevation of the fixed streamline through the point (x', y') in the undisturbed region. It should be realized at the outset that in the derivation of (2.1) it has been assumed that the streamfunction is a monotonic function of y'_0 and, hence, there can be no reversal of streamlines. The flow is bounded below by the rigid wall $y' = 0$, and the upper surface at $y' = h$ may be either fixed or free. At fixed surfaces $\delta' = 0$. The generalized Bernoulli equation for this problem is obtained by setting the expression in brackets in (2.1) equal to zero and, consequently, the dynamic boundary condition at the free surface is

$$\frac{(\nabla\delta')^2}{2} - \frac{\partial\delta'}{\partial y'} + \frac{g}{c^2}\delta' = 0 \quad \text{on } y' = h + \delta' \tag{2.2}$$

Since the free surface is a streamline, the kinematic condition there is satisfied identically.

We choose to follow the solution procedure used by Long (1965) to show how it can be applied to free surface problems. Following his notation, we introduce the non-dimensional variables

$$x = \frac{x'}{\lambda}, \quad y = \frac{y'}{h}, \quad \delta = \frac{\delta'}{a} \tag{2.3}$$

where λ and a are the wavelength and the maximum wave amplitude, respectively. When the fluid density decreases linearly with height we have

$$\rho = \rho_0(1 - by'_0) \tag{2.4}$$

and the non-dimensional form of (2.1) for this particular stratification becomes

$$(1 - \beta(\alpha\delta + y))\left(\frac{\partial^2\delta}{\partial y^2} + \gamma\frac{\partial^2\delta}{\partial x^2}\right) + \left[\alpha\beta\left\{\left(\frac{\partial\delta}{\partial y}\right)^2 + \gamma\left(\frac{\partial\delta}{\partial x}\right)^2\right\} - \beta\frac{\partial\delta}{\partial y} + \sigma^2\delta\right] = 0. \tag{2.5}$$

The new parameters are defined as

$$\alpha = \frac{a}{h}, \quad \beta = bh, \quad \gamma = \frac{h^2}{\lambda^2}, \quad \sigma^2 = \frac{\beta g h}{c^2} \tag{2.6}$$

and α, β and γ measure the effects of non-linearity, stratification and dispersion, respectively. Although the analyses of Benney (1966) and Benjamin (1966) show that permanent internal waves are not limited to weakly stratified fluids (apparently their theories are limited only by the requirement $\alpha\beta \ll 1$), here it is assumed that the three scales α, β and γ are small parameters in order to avoid undue complications. Under these restrictions $c^2 = 0(\beta g h)$ and, hence, the eigenvalue σ^2 is of order one. In non-dimensional form the lower boundary condition is written

$$\delta = 0 \quad \text{on } y = 0 \tag{2.7}$$

and at the upper boundary we have either

$$\delta = 0 \quad \text{on } y = 1 \tag{2.8}$$

or

$$\sigma^2\delta = \beta\frac{\partial\delta}{\partial y} - \frac{\alpha\beta}{2}\left[\left(\frac{\partial\delta}{\partial y}\right)^2 + \gamma\left(\frac{\partial\delta}{\partial x}\right)^2\right] \quad \text{on } y = 1 + \alpha\delta \tag{2.9}$$

depending on whether the surface is fixed or free.

It should be noted that any assumed strength of stratification fixes the divergence or, alternately, the stiffness of the fluid and, therefore, the relative importance of the free surface. Benjamin (1966) has shown that for weakly stratified fluids the free surface effect can be crucial. In his example for exponential stratification all the modes change sign in going from a rigid upper boundary to a free surface; e.g., the mode 1 internal wave of elevation for a fixed boundary becomes a wave of depression for a free surface.

Assuming the asymptotic eigenfunction and eigenvalue expansions

$$\delta(x,y) = \delta_{00} + \alpha\delta_{10} + \beta\delta_{01} + \alpha\beta\delta_{11} + \dots \quad (2.10)$$

$$\sigma^2 = \sigma_{00}^2 + \alpha\sigma_{10}^2 + \beta\sigma_{01}^2 + \alpha\beta\sigma_{11}^2 + \dots, \quad (2.11)$$

Long (1965) has shown that solutions for the rigid lid which die out as $|x| \to \infty$ are possible for $\gamma = \alpha^m\beta^n$ only if $m = n = 1$ (odd modes) or when $m = 1$, $n = 2$ (even modes); all other integer combinations of $m,n \geqslant 0$ fail to yield acceptable steady-state solutions. Although Long did not consider the free surface problem, it is apparent from the work of Benjamin (1966) that in this case all the modes are determined when $m = n = 1$.

3. Rigid lid solutions

3.1. *Odd modes*

We set $\gamma = \alpha\beta$ and insert expansions (2.10) and (2.11) into the equation of motion (2.5) to obtain an ordered set of equations for the eigenfunctions δ_{ij}, assuming $\alpha \ll 1$ and $\beta \ll 1$. The first four equations are

$$\delta_{00}'' + \sigma_{00}^2\delta_{00} = 0 \quad (3.1)$$

$$\delta_{10}'' + \sigma_{00}^2\delta_{10} = -\sigma_{10}^2\delta_{00} \quad (3.2)$$

$$\delta_{01}'' + \sigma_{00}^2\delta_{01} = -\sigma_{01}^2\delta_{00} + \delta_{01}' + y\,\delta_{00}'' \quad (3.3)$$

$$\delta_{11}'' + \sigma_{00}^2\delta_{11} = -\sigma_{11}^2\delta_{00} - \sigma_{01}^2\delta_{10} - \sigma_{10}^2\delta_{01} + \delta_{00}\delta_{00}'' + y\,\delta_{10}'' + \delta_{10}' - \tfrac{1}{2}(\delta_{00}')^2 - (\delta_{00})_{xx} \quad (3.4)$$

with the primes now denoting differentiation with respect to y. The eigenvalues σ_{ij} are then determined from the boundary conditions

$$\delta_{ij} = 0 \text{ on } y = 0, 1 \quad (3.5)$$

appropriate for the rigid plate configuration.

The solutions of the first three equations are readily shown to be

$$\delta_{00} = f(x)\sin n\pi y, \quad \sigma_{00}^2 = n^2\pi^2 \quad (3.6)$$

$$\delta_{10} = g_1(x)\sin n\pi y, \quad \sigma_{10}^2 = 0 \quad (3.7)$$

$$\left.\begin{aligned}\delta_{01} &= f_1(x)\sin n\pi y + \frac{f}{4}\Big[y\sin n\pi y + n\pi(y^2 - y)\cos n\pi y\Big]\\ \sigma_{01}^2 &= -\tfrac{1}{2}n^2\pi^2\end{aligned}\right\} \quad (3.8)$$

where $f(x)$, $f_1(x)$ and $g_1(x)$ are at this point undetermined. The solution of (3.4) vanishing at $y = 0$ is given by

$$\delta_{11} = f_2(x)\sin n\pi y + \tfrac{5}{6}f^2\cos n\pi y - \tfrac{3}{4}f^2 - \tfrac{1}{12}\cos 2n\pi y + \frac{g_1}{4}y\sin n\pi y + \frac{1}{2n\pi}\left(f_{xx} + \sigma_{11}^2 f - \frac{n^2\pi^2}{2}g_1\right)y\cos n\pi y + \frac{n\pi}{4}g_1 y^2\cos n\pi y \quad (3.9)$$

and the upper boundary condition is satisfied if

$$\frac{(-1)^n}{2n\pi}\left(f_{xx} + \sigma_{11}^2 f\right) + \tfrac{5}{6}f^2[(-1)^n - 1] = 0. \quad (3.10)$$

Thus we have two equations for $f(x)$ depending on whether n is even or odd. For n even there is no solution which vanishes for $|x| \to \infty$. For n odd, however, we have

$$f_{xx} + \sigma_{11}^2 f + \frac{10n\pi}{3}f^2 = 0 \quad (3.11)$$

with solution vanishing at both upstream and downstream infinity given by

$$f(x) = \frac{-9\sigma_{11}^2}{20n\pi}\operatorname{sech}^2\left(\frac{i\sigma_{11}x}{2}\right) \quad (3.12)$$

provided $\sigma_{11}^2 < 0$. Regarding α in (2.6) as the maximum non-dimensional disturbance amplitude, we obtain

$$\sigma_{11}^2 = -\tfrac{20}{9}n\pi \quad (3.13)$$

and the solution to this order written in terms of the dimensional (primed) variables is given by

$$\left.\begin{aligned}\frac{\delta'}{h} &= \alpha \sin n\pi \frac{y'}{h}\,\mathrm{sech}^2\left[\frac{(5n\pi\alpha\beta)^{1/2}}{3}\frac{x'}{h}\right]\\ \sigma^2 &= n^2\pi^2 - \frac{n^2\pi^2}{2}\beta - \tfrac{20}{9}n\pi\alpha\beta\end{aligned}\right\} \quad (3.14)$$

3.2. *Even modes*

The even modes are recovered by the balance $\gamma = \alpha\beta^2$ and this leads to a new ordered set of equations for δ_{ij}. The first three equations are exactly those given by (3.1), (3.2) and (3.3) and, hence, have the solutions reported in the previous section. Also, the differential equation for δ_{11} is just that given by (3.4) with the $(\delta_{00})_{xx}$ term missing so that condition (3.10) becomes

$$\frac{(-1)^n}{2n\pi}\sigma_{11}^2 f + \tfrac{2}{6}f^2[(-1)^n - 1] = 0 \quad (3.15)$$

which gives $\sigma_{11}^2 = 0$ for n even. For n odd we obtain $f = 0$ and, consequently, we are henceforth restricted to even values of n.

We now seek a determination of $f(x)$ at higher order in our expansion scheme. The next three equations in the set are given by

$$\delta_{20}'' + \sigma_{00}^2\delta_{20} = -\sigma_{20}^2\delta_{00} - \sigma_{10}^2\delta_{10} \quad (3.16)$$

$$\delta_{02}'' + \sigma_{00}^2\delta_{02} = -\sigma_{02}^2\delta_{00} - \sigma_{01}^2\delta_{01} + \delta_{01}' + y\,\delta_{01}'' \quad (3.17)$$

$$\begin{aligned}\delta_{12}'' + \sigma_{00}^2\delta_{02} = &-\sigma_{12}^2\delta_{00} - \sigma_{10}^2\delta_{02} - \sigma_{01}^2\delta_{11} - \sigma_{11}^2\delta_{01}\\ &- \sigma_{02}^2\delta_{00} + \delta_{00}\delta_{01}'' + \delta_{01}\delta_{00}'' - \delta_{00}'\delta_{01}' + \delta_{11}'\\ &+ y\,\delta_{11}'' - (\delta_{00})_{xx}\end{aligned} \quad (3.18)$$

and the solutions for δ_{20} and δ_{02} are found to be

$$\delta_{20} = g_2(x)\sin n\pi y, \quad \sigma_{20}^2 = 0 \quad (3.19)$$

$$\begin{aligned}\delta_{02} = &f_3(x)\sin n\pi y + \frac{f_1}{4}y\sin n\pi y + \frac{f}{32}\left(5 - n^2\pi^2\right)\\ &\times y^2\sin n\pi y + \frac{n^2\pi^2}{16}fy^3\sin n\pi y\\ &- \frac{n^2\pi^2}{32}fy^4\sin n\pi y\\ &- \frac{n\pi}{16}\left(f + 4f_1\right)y\cos n\pi y\\ &+ \frac{n\pi}{8}\left(2f_1 - f\right)y^2\cos n\pi y + \tfrac{3}{16}n\pi f y^3\cos n\pi y\end{aligned}$$

$$\sigma_{02}^2 = -\frac{(1 + n^2\pi^2)}{16} \quad (3.20)$$

Lengthy but straightforward computation shows that the solution of (3.18) satisfying the lower boundary condition is

$$\begin{aligned}\delta_{12} = &f_4(x)\sin n\pi y - (l_0 + q_0)\cos n\pi y\\ &+ (h_1 y + h_2 y^2 + h_3 y^3 + h_4 y^4)\sin n\pi y\\ &+ (k_1 y + k_2 y^2 + k_3 y^3)\cos n\pi y\\ &+ (l_0 + l_1 y)\cos 2n\pi y + (p_0 + p_1 y\\ &+ p_2 y^2)\sin 2n\pi y + q_0 + q_1 y\end{aligned} \quad (3.21)$$

where

$$\left.\begin{aligned}h_1(x) &= \frac{f_2}{4} + \frac{5n\pi}{24}f^2; \quad h_3(x) = \frac{n^2\pi^2}{16}g_1\\ h_2(x) &= \frac{(5 - n^2\pi^2)}{32}g_1 - \frac{5n\pi}{24}f^2\\ h_4(x) &= -\frac{n^2\pi^2}{32}g_1\end{aligned}\right\} (3.21\text{a})$$

$$\left.\begin{aligned}k_1(x) &= \frac{f_{xx}}{2n\pi} + \tfrac{5}{24}f^2 + \frac{\sigma_{12}^2}{2n\pi}f - \frac{n\pi}{16}g_1\\ &\quad - \frac{n\pi}{4}f_2\\ k_2(x) &= -\frac{n\pi}{8}g_1 + \frac{n\pi}{4}f_2\\ k_3(x) &= \frac{3n\pi}{16}g_1\end{aligned}\right\} (3.21\text{b})$$

$$l_0(x) = -\frac{ff_1}{6}; \quad l_1(x) = -\tfrac{1}{8}f^2 \quad (3.21\text{c})$$

$$\left.\begin{aligned}p_0(x) &= \frac{f^2}{8n\pi}; \quad p_1(x) = -\frac{n\pi}{24}f^2\\ p_2(x) &= \frac{n\pi}{24}f^2\end{aligned}\right\} (3.21\text{d})$$

$$q_0(x) = -\frac{3n^2\pi^2}{2}ff_1; \quad q_1(x) = -\tfrac{9}{8}n^2\pi^2 f^2. \quad (3.21\text{e})$$

We now apply the condition $\delta_{12} = 0$ at $y = 1$ which requires

$$k_1 + k_2 + k_3 + l_1 + q_1 = 0 \tag{3.22}$$

for even values of n. This yields the equation

$$f_{xx} + \sigma_{12}^2 f - \frac{n\pi}{12}\left(27n^2\pi^2 - 2\right) f^2 = 0 \tag{3.23}$$

with the solution vanishing as $|x| \to \infty$ given by

$$f(x) = \frac{18\sigma_{12}^2}{n\pi(27n^2\pi^2 - 2)} \operatorname{sech}^2\left(\frac{i\sigma_{12}x}{2}\right) \tag{3.24}$$

as long as σ_{12}^2 is negative. Thus we have

$$\sigma_{12}^2 = -\frac{n\pi}{18}\left(27n^2\pi^2 - 2\right) \tag{3.25}$$

and to the present order of approximation

$$\left.\begin{aligned} \frac{\delta'}{h} &= -\alpha \sin n\pi \frac{y'}{h} \operatorname{sech}^2 \\ &\quad \times \left[\left(\frac{27n^2\pi^2 - 2}{72}\right)^{1/2} (n\pi\alpha\beta^2)^{1/2} \frac{x'}{h}\right] \\ \sigma^2 &= n^2\pi^2 - \frac{n^2\pi^2}{2}\beta - \frac{(1 + n^2\pi^2)}{16}\beta^2 \\ &\quad - \frac{n\pi}{18}\left(27n^2\pi^2 - 2\right)\alpha\beta^2 \end{aligned}\right\} \tag{3.26}$$

4. Free surface solutions

For $\gamma = \alpha\beta$ we are faced with solving the set of eqs. (3.1)–(3.4) subject to the boundary conditions (2.7) and (2.9). Since the Bernoulli equation must be evaluated at $y = 1 + \alpha\delta$, the variable δ and its derivatives in (2.9) are first expanded about the linearized position of the free surface, $y = 1$. The ordered set of free surface conditions are then obtained by inserting the asymptotic developments (2.10) and (2.11) in the expanded form of (2.9). This gives

$$\sigma_{00}^2 \delta_{00} = 0 \quad \text{at } y = 1 \tag{4.1}$$

$$\sigma_{00}^2(\delta_{10} + \delta_{00}\delta_{00}') + \sigma_{10}^2\delta_{00} = 0 \quad \text{at } y = 1 \tag{4.2}$$

$$\sigma_{01}^2\delta_{00} + \sigma_{00}^2\delta_{01} = \delta_{00}'' \quad \text{at } y = 1 \tag{4.3}$$

$$\begin{aligned} &\sigma_{11}^2\delta_{00} + \sigma_{01}^2\delta_{10} + \sigma_{10}^2\delta_{01} + \sigma_{00}^2 \\ &\quad \times (\delta_{11} + \delta_{00}\delta_{01}' + \delta_{01}\delta_{00}') = \delta_{10}' \\ &\quad + \delta_{00}\delta_{00}'(1 - \sigma_{01}^2) - \frac{(\delta_{00}')^2}{2} \quad \text{at } y = 1 \end{aligned} \tag{4.4}$$

for the respective differential eqs. (3.1)–(3.4). If one has solved the fixed boundary problem, it is a simple matter to make the necessary adjustments for the free surface. As an example, we take Long's results for exponential stratification and solve the free surface problem. Denoting his mode 1 solutions (Long (1965), eqs. (30), (33), (36) and (38)) with an overbar ($\bar{\delta}_{00}$, $\bar{\delta}_{10}$ etc.) one readily finds that application of (4.1)–(4.3) gives the free surface solutions

$$\delta_{00} = \bar{\delta}_{00}, \quad \sigma_{00}^2 = n^2\pi^2 \tag{4.5}$$

$$\delta_{10} = \bar{\delta}_{10}, \quad \sigma_{10}^2 = 0 \tag{4.6}$$

$$\delta_{01} = \bar{\delta}_{01} + \frac{f}{n\pi} y \cos n\pi y, \quad \sigma_{01}^2 = 2 \tag{4.7}$$

and the solution for δ_{11} satisfying the lower boundary condition is

$$\delta_{11} = \bar{\delta}_{11} + \frac{g_1}{n\pi} y \cos n\pi y. \tag{4.8}$$

The free surface condition (4.4) is satisfied if

$$f_{xx} + \sigma_{11}^2 f - \frac{7n\pi}{3} f^2 = 0 \quad (n \text{ odd}) \tag{4.9}$$

$$f_{xx} + \sigma_{11}^2 f + 3n\pi f^2 = 0 \quad (n \text{ even}) \tag{4.10}$$

showing that the even as well as the odd modes are determined when $\gamma = \alpha\beta$. Solutions of (4.9) and (4.10) vanishing at infinity agree with the results obtained by Benjamin (1966).

Now we return to the linear denisty profile problem. Again denoting the rigid plate solutions (eqs. (3.6)–(3.9)) with an overbar, we find results identical with eqs. (4.5)–(4.8) except that now

$$\sigma_{01}^2 = 2\left(1 - \frac{n^2\pi^2}{4}\right). \tag{4.11}$$

The free surface condition (4.4) then yields

$$f_{xx} + \sigma_{11}^2 f + \frac{n\pi}{3} f^2 = 0 \quad (n \text{ odd}) \tag{4.12}$$

$$f_{xx} + \sigma_{11}^2 f + 3n\pi f^2 = 0 \quad (n \text{ even}) \tag{4.13}$$

and the solutions to this order are given by

$$\left.\begin{aligned}\frac{\delta'}{h} &= \alpha \sin n\pi \frac{y'}{h}\, \mathrm{sech}^2\left[\left(\frac{\alpha\beta n\pi}{18}\right)^{1/2} \frac{x'}{h}\right] \\ \sigma^2 &= n^2\pi^2 - 2\left(\frac{n^2\pi^2}{4} - 1\right)\beta - \frac{2n\pi}{9}\,\alpha\beta\end{aligned}\right\} (n \text{ odd}) \tag{4.14}$$

$$\left.\begin{aligned}\frac{\delta'}{h} &= \alpha \sin n\pi \frac{y'}{h}\, \mathrm{sech}^2\left[\left(\frac{\alpha\beta n\pi}{2}\right)^{1/2} \frac{x'}{h}\right] \\ \sigma^2 &= n^2\pi^2 - 2\left(\frac{n^2\pi^2}{4} - 1\right)\beta - 2n\pi\alpha\beta\end{aligned}\right\} (n \text{ even}) \tag{4.15}$$

A comparison of these results with eqs. (3.14) and (3.26) shows that while the free surface produces no qualitative change in the streamline patterns for the odd modes, the even mode streamlines with a free surface are all essentially inverted from what they were with a rigid upper boundary.

The shape $\tilde{\delta}(x)$ of the free surface can be calculated from the equation

$$\tilde{\delta}(x) = \delta(x, 1) = \delta_{00}(x, 1) + \alpha\delta_{10}(x, 1) + \beta\delta_{01}(x, 1) + \dots \tag{4.16}$$

and the first contribution comes in at order β. The maximum free surface distortion is readily found to be

$$(\tilde{\delta})_{\max} = (-1)^n \frac{\alpha\beta}{n\pi} \tag{4.17}$$

and, therefore, the odd modes are accompanied by a small depression in the free surface and the even modes have a small elevation. By comparing (4.17) with (4.14) and (4.15), one finds that both the even and odd mode streamlines experience a 180-degree phase shift in a thin layer (of order $\beta h/n^2\pi^2$) just below the free surface.

5. Discussion and conclusion

We now compare the results for a linearly stratified fluid with those obtained by previous investigators for an exponentially stratified fluid. For $\beta \ll 1$, the non-dimensional streamline deviations for both cases can be written

$$\delta = \mathrm{sgn}(\alpha)\,|\alpha|\, \sin n\pi y\, \mathrm{sech}^2\, \omega x \tag{5.1}$$

and the associated non-linear phase speeds are then

$$c^2 = c_n^2(1 + 4\omega^2), \quad c_n = \text{linear wave speed} \tag{5.2}$$

The values of c_n^2, ω^2 and sgn (α) are listed in Table 1 for both the rigid upper boundary and free surface problems. Not included in (5.1) is the $O(\beta)$ correction to the vertical modal structure; with the rigid upper boundary removed, and for the weak denisty stratification under consideration, this correction shows up as a phase reversal very near the free surface for each internal wave mode for both stratifications.

We first observe from Table 1 that with a rigid upper boundary, both density profiles produce qualitatively similar wave shapes for both even and odd modes. Also, with the exception of the odd modes for linear stratification, all modes shapes for a free surface are essentially inverted from what they were with a rigid top plate. It is remarkable that such profound changes can be brought about by simply removing the rigid boundary. Although there is a distinct character change for the odd mode internal waves with a free surface in going from an exponential to a linear density profile, this is apparently not a manifestation of a basic difference between constant and variable Brunt frequency fluids. Rather, all the above sensitive changes in mode shape seem to be due to the strong influence of the free surface when the fluid is weakly stratified. Supporting evidence is given in the following paragraph.

It is possible to look at the solution for exponential stratification in the limit $\beta \gg 1$, i.e., for a heavily stratified fluid where one might expect the free surface to behave like a rigid boundary. Prefixing references to Benjamin's (1966) equations with a B, we note that his solution for a rigid plate (B4.20, B4.21, B4.22) is valid in the limit

$$\alpha \ll \frac{1}{\beta} \ll 1 \tag{5.3}$$

which, although it is perhaps of little physical interest, does satisfy the limitation of his theory, $\alpha\beta \ll 1$. (Note that our β corresponds to his βh_0.)

Table 1

	Exponential stratification		Linear stratification		
	Top plate	Free surface	Top plate	Free surface	
$\frac{c_n^2}{\beta g h / n^2\pi^2}$	$1-\frac{\beta^2}{4n^2\pi^2}$	$1-\frac{2\beta}{n^2\pi^2}$	$1+\frac{\beta}{2}$	$1+\left(\frac{n^2\pi^2-4}{2n^2\pi^2}\right)\beta$	All modes
ω^2	$\frac{n\pi}{9}\alpha\beta$	$\frac{7n\pi}{18}\alpha\beta$	$\frac{5n\pi}{9}\alpha\beta$	$\frac{n\pi}{18}\alpha\beta$	Odd modes
sgn (α)	(+)	(−)	(+)	(+)	Odd modes
mode 1 wave	Elevation	Depression	Elevation	Elevation	Odd modes
ω^2	$\frac{n\pi}{36}\alpha\beta^2$	$\frac{n\pi}{2}\alpha\beta$	$\frac{n\pi(27n^2\pi^2-2)}{72}\alpha\beta^2$	$\frac{n\pi}{2}\alpha\beta$	Even modes
sgn (α)	(−)	(+)	(−)	(+)	Even modes
mode 2 wave	Bulge	Pinch	Bulge	Pinch	Even modes

For the corresponding free surface problem, the eigenvalue satisfying B4.27 takes the form

$$\lambda_n h_0 = n\pi\left(1-\frac{4}{\beta h_0}+\frac{16}{(\beta h_0)^2}\right)+0\left(\frac{1}{(\beta h_0)^3}\right) \quad (5.4)$$

and an asymptotic evaluation of Benjamin's I, εJ, and K integrals, taken in the limit $\beta h_0 \gg 1$, show that the streamlines for both even and odd modes do not experience any qualitative change when the rigid top plate is removed: the mode 1 wave of elevation remains an elevation wave; the mode 2 bulge remains a bulge. Moreover, the free surface propagates as a small wave of elevation for both the even and odd modes and there are no phase reversals just below the free surface.

It is the author's present feeling that results similar to those above for $\beta \gg 1$ will be found for other stable density profiles, and in this limit the free surface will not bring about significant differences in the internal waveforms. It appears, therefore, that the free surface effect is entirely responsible for the radical changes in the mode shapes calculated for $\beta \ll 1$. Were these qualitative changes due to effects derived from the particular shape of the density profile, they should be evident with the rigid lid in place and we see from Table 1 that no such differences exist. The transition from profound to slight changes wrought by the free surface probably occurs when $\beta = 0(1)$.

The reader may have noticed, however, that there is a small, qualitative difference in the *linear* wave speeds for the two density profiles with the upper surface fixed: there is no $O(\beta)$ correction for the exponential stratification. That this idiosyncracy is in fact special to fluids having a constant Brunt-Väisälä frequency is easily demonstrated. Assuming a stable, slowly varying density distribution $\rho(\beta y)$ we write the Taylor series expansion about $y = 0$,

$$\rho = \rho(0) + \rho'(0)\beta y + \rho''(0)\frac{(\beta y)^2}{2!} + \cdots \quad (5.5)$$

and substitution into the non-dimensional form of Long's eqn. (2.1) shows that the eigenvalue for a weakly stratified fluid (bounded above by a rigid plate) is given by

$$\sigma^2 = n^2\pi^2\left\{1-\tfrac{1}{2}\left[\frac{\rho''(0)}{\rho'(0)}-\frac{\rho'(0)}{\rho(0)}\right]\beta\right\} + 0(\beta^2) \quad (5.6)$$

The two profiles which make the term in brackets disappear are the exponential profile

$$\rho = \rho_0 \exp(-\beta y) \quad (5.7)$$

and the quadratic function

$$\rho = \rho_0\left(1-\beta y+\frac{(\beta y)^2}{2}\right) \quad (5.8)$$

which is nothing more than the expansion of the exponential profile (5.7) to order $(\beta y)^2$. Hence, the $O(\beta)$ difference in the first correction to the linear phase speed of the waves is indeed derived from the unique behavior of the exponential density profile.

Finally, it is worthwhile to note that the nonlinear behavior of internal waves in a linearly stratified fluid cannot be obtained from the exponential solution in the limit $\beta \to 0$, as one might be led to believe.

In conclusion, the results given here and also by Benjamin (1966) indicate the dramatic changes which take place due to free surface effects. It is particularly curious that the first internal wave mode can *ever* exist as a wave of depression (cf. Table 1). An experimental demonstration of this possibility would certainly be worthwhile, but perhaps difficult. Effects which are neglected in the present analysis, namely surface tension and viscous effects due to surface contamination, may substantially alter the free surface boundary condition and, therefore, would have to be carefully minimized. This should provide ample challenge for any experimentalist.

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ВНУТРЕННИЕ УЕДИНЕННЫЕ ВОЛНЫ В ЛИНЕЙНО СТРАТИФИЦИРОВАННОЙ ЖИДКОСТИ

В приближении слабой стратификации определяется структура мод внутренних уединенных волн, распространяющихся в линейно стратифицированной жидкости конечной глубины. Решения, полученные как при условии твердой крышки наверху, так и при условии свободной поверхности, сравниваются с известными ранее результатами для экспоненциально стуратифицированной жидкости. Существенное различие результатов, полученных при условиях твердой крышки и свободной поверхности объясняется сильным влиянием свободной поверхности в случае слабой стратификации. Приводятся доводы, подтверждающие предположение о том, что в случае сильно стратифицированной жидкости свободная поверхность не вызывает качественных изменений в структуре мод.