On inertial oscillations in the oceans

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ABSTRACT

The interpretation of the observed spectral peaks of certain oceanic disturbances near the inertial latitudes in terms of inertial waves given by Munk & Phillips (1968) is reassessed in the light of the more complete theory given by Stewartson & Walton (1974). It is shown that these waves are not in fact confined to the neighbourhood of the inertial latitudes but decay only slowly on the equatorial side to about one-quarter of their maximum at the equator. Stewartson & Walton (1974) also discussed the character of waves of inertial frequency for varying values of a parameter Γ, a measure of the relative magnitude of the inertial frequency at the inertial latitude and the Brunt-Vaisala frequency. Munk & Phillips' waves constitute the limit $\Gamma \to 0$ and it is suggested here that the observed incoherence may be due to the non-vanishing of this parameter.

It seems to be generally agreed that there exist inertial oscillations in the oceans, even though they are intermittant in character and there are few signs of coherence between observations at different places. One type which has attracted considerable attention exhibits sharp peaks in the amplitude near the "inertial" latitudes at which the associated group velocity vanishes. For example, Fofonoff & Webster (reported in Munk & Phillips, 1968) find 10-db peaks in the energy per harmonic within a few percent of the local inertial frequency at a latitude of 30°.

Munk & Phillips themselves (hereafter referred to as MP) investigate whether there might be theoretical reasons for the occurrence of such peaks because of the existence of a class of planetary-gravity (pg) waves in a stratified ocean. It is well-known (Phillips, 1963) that these waves are trapped between the inertial latitudes, but MP further suggest that they are essentially confined to the neighbourhood of the inertial latitudes. Other kinds of waves exhibit trapping (e.g., inertial waves near the equator (Stern, 1963; Israeli, 1972; Stewartson, 1971) and Rossby waves (Longuet-Higgins, 1964)) but do not have peaks at the inertial latitudes.

On the assumption that $\gamma \gg 1$ where $\gamma = 2\Omega a r \pi / \tilde{N} h$ (Ω is the angular velocity of the earth, a its radius, r an integer, $\tilde{N}$ the mean Brunt-Vaisala frequency, and h the ocean depth), MP demonstrate that such *pg* waves can occur with frequency $\sigma\Omega$ where $\sigma = 2 \sin \phi_s$ and ϕ_s is the inertial latitude. At neighbouring latitudes the amplitude of the oscillation is

$$QAi[\gamma^{2/3} (\sin 2\phi_s)^{1/3} (\phi - \phi_s)] \tag{1}$$

where Ai is the Airy function and Q a constant. Since (1) decays exponentially for $\phi > \phi_s$ and algebraically for $\phi < \phi_s$, this result seems to indicate that the amplitude is peaked near ϕ_s and MP conclude that these solutions are essentially confined to a strip of order $\gamma^{-2/3}$ in $\phi - \phi_s$ and are therefore insensitive to conditions farther away. Specific latitudinal boundary conditions may then be ignored and the solution interpreted as a continuous spectrum. They go on to give a physical argument supporting this conclusion as follows. If there is a local generation of waves the disturbance will leak off except for the components associated with very small group velocity. The group velocity of a wave vanishes at $\phi = \phi_s$ if $\sigma = 2 \sin \phi_s$.

There is a mathematical sense in which these conclusions are correct, namely, when the fluid is inviscid and in the limits $h/a \to 0$, $\gamma \to \infty$, $\gamma (h/\pi r a)^{3/4} = \Gamma \to 0$. Further, when $\phi > \phi_s$ where

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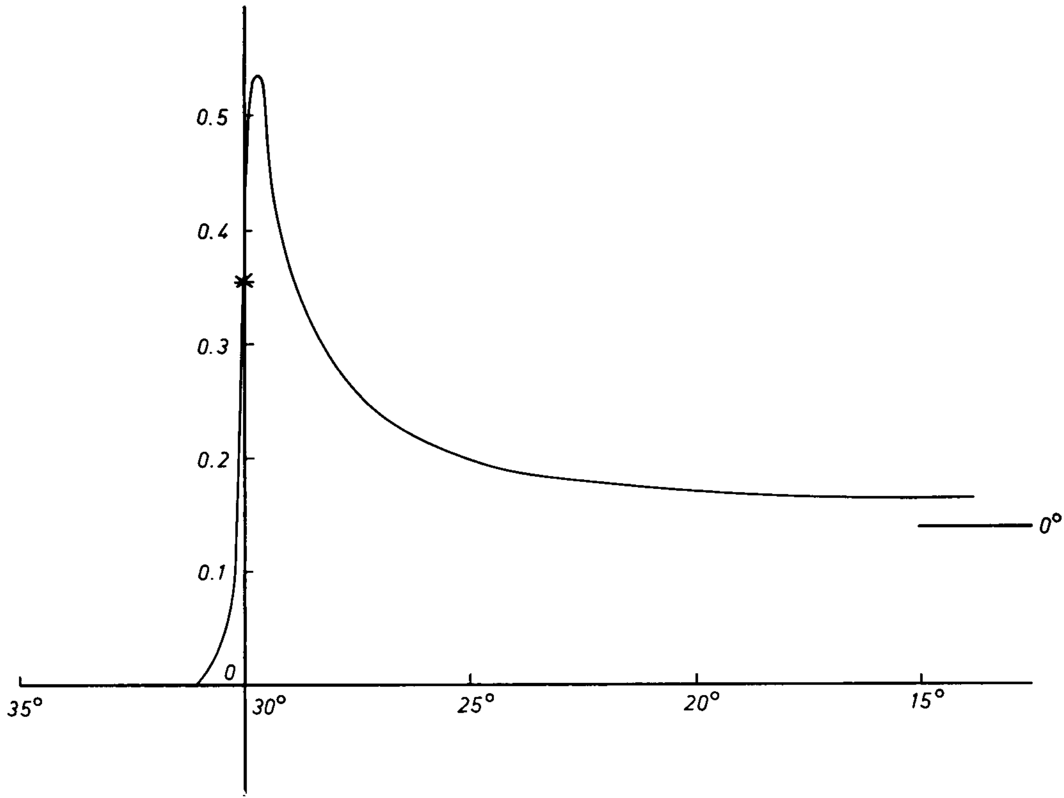

Fig. 1. The meridional velocity u as a function of latitude ϕ taking $N = 2.10^{-3}$ sec, $h = 4.10^3$ m, $r = 10$, $\phi_s = 30°$. (For $\phi < \phi_s$ the envelope of the maxima of u is shown.)

(1) decays exponentially there seems little doubt that they are correct for the finite values of h/a, γ observed in the oceans. On the other side, however, this decay is algebraic as indeed was recognised by MP and others. The implications are however more serious than they envisaged, for (1) is proportional to $\gamma^{-1/6}(\phi_s - \phi)^{-1/4}$ when $\gamma^{2/3}(\phi - \phi_s)$ is large and negative. The structure of (1) is then oscillatory and derivatives with respect to ϕ in fact become unbounded as $(\phi - \phi_s)\gamma^{2/3} \to \infty$. It is therefore important to examine the structure of this solution at latitudes substantially less than ϕ_s. This study has recently been undertaken by the present authors (Stewartson & Walton, 1974, hereafter referred to as SW) who find that the continuation of (1) into the region $|\phi| < \phi_s$ takes the form

$$Q\gamma^{-1/6}\pi^{-1/2}(1-\mu_s^2)^{1/3}\mu_s^{-1/6}[(\mu_s - \mu)(1-\mu^2)]^{-1/4} \times \cos\left[\gamma(g(\mu) - g(\mu_s)) - \frac{\pi}{4}\right]$$

where $\mu = \sin\phi$, $\mu_s = \sin\phi_s$, $g(\mu) = -2[E(\mu/\mu_s, \mu_s^2) - (1-\mu_s^2)K(\mu/\mu_s, \mu_s^2)]$ and K, E are elliptic integrals of the first and second kinds, confirming the rather slow decay for large γ. Indeed, for the example quoted by MP ($N = 2.10^{-3}$ sec^{-1}, $h = 4.10^3$ m, $r = 10$, $\phi_s = 30°$) the maximum amplitude near ϕ_s is only about four times that at the equator. The maximum amplitude is plotted in Fig. 1 and should be compared with the observed amplitude presented in Fig. 3 of MP.

SW show that waves of frequency $\omega\Omega$ cover the whole sphere if $N/\Omega < \omega < 2$ but for larger values of N a cut-off latitude exists which is higher than the inertial latitude and is coincident with it in the limit $\Gamma \to 0$. Taking $r = 10$ and using the numbers quoted by MP, $\Gamma \sim 0.3$ which is not very small. Phillips (1963) notes that "in the deep ocean and near the surface after a storm, $N(z)$ may be as small as 10^{-4} sec^{-1}" which gives $\Gamma \approx 2$. MP consider the limit $\Gamma \to 0$ and the only other mathematical solutions available (SW) are for $\Gamma \to \infty$. These exhibit singularities on the rays which touch the inner boundary at the inertial latitudes and their subsequent reflections. Of course, such singularities cannot be expected to occur in practice but it is quite possible that the velocity distribution due to such an inertial wave will exhibit a non-sinusoidal variation with depth and in particular have peaks at these rays. It follows that a spectral analysis of the observed velocity at different depths on the basis that it is composed of inertial waves varying sinusoidally with depth is unlikely to show much coherence. Similar remarks might be relevant to observations at different latitudes also. Finally, the variation of Γ with time may well be a significant factor in the observed intermittancy of these waves.

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ОБ ИНЕРЦИОННЫХ КОЛЕБАНИЯХ В ОКЕАНЕ

Интерпретация наблюдаемых спектральных пиков некоторых океанических возмущений вблизи инерционных широт в терминах инерционных волн, предложенная Манком и Филлипсом (1968), пересматривается в свете более полной теории, предложенной Стюартсоном и Уолтоном (1974). Показано, что эти волны фактически не ограничены районами инерционных широт, а слабо убывают в сторону экватора, где достигают 1/4 своего максимума на экваторе. Стюартсон и Уолтон (там же) разобрали характер волн с инерционной частотой для различных значений параметра Γ — меры величины инерционной частоты на критической широте по отношению к частоте Брэнта–Вяйсяля. Волны Манка и Филлипса получаются в пределе при $\Gamma \to 0$. Высказывается предположение, что наблюдаемая некогерентность может объясняться отличием этого параметра от нуля.