Negative viscosity in the Gulf Stream?

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ABSTRACT

This paper discusses whether Webster's and Schmitz & Niiler's measurements verify the proposed existence of negative viscosity for two-dimensional flows put forward by Krause & Rüdiger. It leads to the conclusion, contrary to that of other workers, that only positive viscosity was observed for the Gulf Stream when cross stream directed mean motion, as found in the surveys mentioned above, is considered. This conclusion depends on the validity of our fundamental assumption that the turbulence is caused by an external force field.

1. Introduction

In order to investigate problems of hydrodynamics in turbulent regions it is necessary to know the influence of a mean motion, $\bar{\mathbf{u}}$, on the correlation tensor

$$Q_{ij} = \overline{u_i'(\mathbf{x}, t)\, u_j'(\mathbf{x}, t)} \tag{1}$$

since it appears in the Reynolds equation for mean motion,

$$\frac{\partial}{\partial t} \varrho \bar{u}_i + \frac{\partial}{\partial x_j} (\varrho \bar{u}_i \bar{u}_j + P\delta_{ij} - \varrho\nu(\bar{u}_{i,j} + \bar{u}_{j,i}) + \varrho Q_{ij}) = 0 \tag{2}$$

where ν denotes the molecular viscosity, ϱ the constant density and P represents pressure terms. Here and in what follows we wish to restrict ourselves to dealing with incompressible matter. Consequently the relation

$$\operatorname{div} \mathbf{u} = 0 \tag{3}$$

can be used for both the mean motion, $\bar{\mathbf{u}}$, and the fluctuating part, $\mathbf{u}'$. For example, the calculation of the rotation laws of the Earth's atmosphere as well as of Sun's convection zone seems to be a problem in which concepts about the structure of (1) are needed (Rüdiger, 1974). It is possible, however, to make direct measurements of the correlation tensor (1) in oceanic regions in which mean movements appear. In this connection Webster (1965) and Schmitz & Niiler (1969) have made several important surveys in the Gulf Stream and have obtained very interesting results.

The interaction of the fluctuating motions with the mean flow leads to the occurrence of the deformation tensor $\bar{u}_{i,j} + \bar{u}_{j,i}$ in the correlation tensor, which normally enhances the effective viscosity in the Reynolds equation. But Starr's (1968) conclusion from Webster's measurements is that turbulent viscosity seems, surprisingly, to be negative, and thus reduces the effective viscosity in (2). Indeed, in a recent paper Krause & Rüdiger (1974*b*) verify the possibility of the existence of negative viscosity in the case in which the turbulence is two-dimensional. Their fundamental assumption, i.e. that the (weak) turbulence is driven by an external force field which is thought to be independent of the mean velocity, was corroborated by Oort (1964). Therefore, a turbulent field is considered which may be present also for vanishing mean motion. It seems to be possible that the turbulence is forced by the oceanic circulation at greater depths or tropospheric circulations (Oort, 1964). But the used mathematical treatment of the influence of a mean motion on such "original" turbulence does not require exact statements concerning the outside source of turbulent energy.

In the above mentioned paper Krause & Rüdiger establish the shape of the correlation tensor for incompressible matter as

$$Q_{ij} = Q_{ij}^{(0)} - \nu_T(\bar{u}_{i,j} + \bar{u}_{j,i}) \tag{4}$$

where

$$\nu_T = \frac{1}{2}\iint \frac{\nu k^2(\nu^2 k^4 - \omega^2)}{(\nu^2 k^4 + \omega^2)^2} \hat{Q}_{l}^{(0)}(\mathbf{k}, \omega)\, d\mathbf{k}\, d\omega \tag{5}$$

All vector quantities are two-dimensional. ν is the molecular viscosity, $\hat{Q}_{ij}^{(0)}$ is the Fourier transform of the correlation tensor

$$Q_{ij}^{(0)}(\boldsymbol{\xi}, \tau) = \overline{u_i^{(0)}(\mathbf{x}, t)\, u_j^{(0)}(\mathbf{x} + \boldsymbol{\xi}, t + \tau)} \tag{6}$$

In (6) the symbol $^{(0)}$ is used to refer to the respective turbulent motion which is unaffected by the mean motion and also to the case of vanishing mean motion. This original turbulence field is assumed to be homogeneous and stationary. Following Bochner's theorem (Bochner, 1933), valid for homogeneous fluctuating processes, which states that the spectral tensor $\hat{Q}_{ij}^{(0)}$ forms a positive semidefinite form,

$$\hat{Q}_{ij}(\mathbf{k}, \omega) X_i X_j^* \geqslant 0 \tag{7}$$

(**X** arbitrary, see Krause & Roberts, 1973), we find the trace $\hat{Q}_{ii}^{(0)}$ is also positive semidefinite. The conditions under which (5) provides negative values are described in the paper referred to. Because of their complexity it is very important to analyze observations which seem to indicate the occurrence of negative values of ν_T. We use the term "viscosity" for this quantity because it appears in Reynolds equation in the form of molecular viscosity. In an earlier investigation we were able to prove the positive nature of turbulent viscosity in the case of three-dimensional turbulence (Krause & Rüdiger, 1974*a*). It is interesting to note that turbulent thermal conductivity is a positive quantity not only in the presence of three-dimensional turbulence but also in two-dimensional turbulence, if it exists. It becomes clear after a Fourier transformation of Krause's formula (1972), giving

$$\lambda_T = \frac{1}{2}\iint \frac{\lambda k^4 q(k, \omega)}{\omega^2 + \lambda^2 k^4}\, d\mathbf{k}\, d\omega \tag{8}$$

(λ molecular thermal conductivity). $q(k, \omega)$ is the spectral function of the two-dimensional turbulent field whose spectral tensor is represented by

$$\hat{Q}_{ij}^{(0)}(\mathbf{k}, \omega) = q(k, \omega)\,(k^2 \delta_{ij} - k_i k_j) \tag{9}$$

Bochner's theorem (7) is fulfilled if $q(k, \omega)$ is chosen as a positive semidefinite function,

$$q(k, \omega) \geqslant 0 \tag{10}$$

By inserting the two horizontal velocity components u and v in (4) we obtain

$$\overline{u'v'} = -\nu_T\left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right) \tag{11}$$

If we can neglect either the component $\bar{u}$ or its derivative with respect to y by rotating the y-axis in the direction of the mean movement, eq. (11) provides

$$\overline{u'v'} = -\nu_T \frac{\partial \bar{v}}{\partial x} \tag{12}$$

and after multiplication

$$\overline{u'v'}\frac{\partial \bar{v}}{\partial x} = -\nu_T\left(\frac{\partial \bar{v}}{\partial x}\right)^2 \tag{13}$$

This quantity possesses the opposite sign for ν_T; positive values of $\overline{u'v'} \cdot \partial\bar{v}/\partial x$ indicate the occurrence of negative viscosity.

At the maximum of the stream's velocity ($\bar{v}$-) profile the cross correlation $\overline{u'v'}$ vanishes. It must be exactly antisymmetrical if a symmetrical $\bar{v}$-profile exists. But Webster's result is very unsymmetrical: he has observed large positive values only on the shoreside of the Gulf Stream where $\partial\bar{v}/\partial x > 0$ and positive, although small, values at the maximum of the $\bar{v}$-profile ($\partial\bar{v}/\partial x = 0$). From the existence of positive cross correlations (12), in regions with positive $\partial\bar{v}/\partial x$ the conclusion was drawn that negative viscosity occurs. Though the measurements contain large errors we will assume them correct on principle. We shall discuss the influence of newly established higher terms in (11), which are based on the existence of the non-vanishing cross-stream directed mean velocity $\bar{u}$, which was also present during Webster's measurements. With regard to the sign of the eddy viscosity it is necessary to draw different conclusions than those by Starr. Of course, our conclusions are based on the validity of the fundamental assumption that there is an external force field driving the turbulence. From this point of view measurements of the correlation tensor can lead to a significant examination of this fundamental assumption.

Closing this section we will note that the cross correlation (12), is not influenced by Coriolis force of rotating Earth. In our papers (Rüdiger, 1974) we have pointed out vanishing of correlation $Q_{v\varphi}$ in the case of independence of the angular velocity upon the latitude for two-dimensional as well as three-dimensional turbulence. Such a conclusion also holds for a non-linear treatment of global rotation if one of the coordinate axes is polward directed. However, a possible dependence of ν_T on the angular velocity is to consider at a later state of our theory.

2. The quadratic terms of the correlation tensor

There is no reason to suppose that only the deformation tensor appears in the correlation tensor (1). It is essential to show whether the quadratic terms,

$$\begin{aligned}&\bar{\mathbf{u}}^2\delta_{ij}, \quad \bar{u}_i\bar{u}_j,\\ &\bar{u}_{l,k}\bar{u}_{l,k}\delta_{ij}, \quad \bar{u}_{l,k}\bar{u}_{k,l}\delta_{ij}, \quad \bar{u}_{i,k}\bar{u}_{j,k},\\ &\bar{u}_{k,i}\bar{u}_{k,j}, \quad \bar{u}_{i,k}\bar{u}_{k,j}+\bar{u}_{j,k}\bar{u}_{k,i}\end{aligned} \tag{14}$$

also play an important role. Since they also appear in Reynolds equation, the coefficients of the first-line terms in (14) modify the values of the effective pressure and density. Now we shall examine the general statement for the correlation tensor

$$\begin{aligned}Q_{ij} = Q_{ij}^{(0)} &- \nu_T(\bar{u}_{i,j}+\bar{u}_{j,i}) - \chi_1\bar{u}^2\delta_{ij} - \chi_2\bar{u}_i\bar{u}_j\\ &- \varkappa_1\bar{u}_{i,k}\bar{u}_{j,k} - \varkappa_2(\bar{u}_{i,k}\bar{u}_{j,k}+\bar{u}_{k,i}\bar{u}_{k,j})\\ &- \varkappa_3(\bar{u}_{i,k}\bar{u}_{k,j}+\bar{u}_{j,k}\bar{u}_{k,i}) - \varkappa_4\bar{u}_{l,k}\bar{u}_{l,k}\delta_{ij}\\ &- \varkappa_5\bar{u}_{l,k}\bar{u}_{k,l}\delta_{ij}\end{aligned} \tag{15}$$

All vector quantities are horizontal; vector quantities different from those used shall not occur. Hence $Q_{ij}^{(0)}$ is proportional to a Kronecker tensor δ_{ij}. From (15) we find that the required cross correlation function $\overline{u'v'} \equiv Q_{12}$ must be represented by

$$Q_{12} = -\nu_T\frac{\partial\bar{v}}{\partial x} \tag{16}$$

in the case of the mean motion consisting exclusively of a movement in the y-direction, i.e. $\bar{u} \equiv 0$. The quadratic terms, (14), provide no contribution to the expression for Q_{12}. Measurements made under these circumstances give direct information about the sign of ν_T. Therefore, the simplest method of obtaining the value of ν_T seems to be to transform the measured correlation tensor, $\overline{u_l'u_k'}$, so that no cross-stream directed mean velocity, $\bar{u}$, appears. Hence,

$$Q_{ij} = \frac{\partial\tilde{x}_i}{\partial x_l}\frac{\partial\tilde{x}_j}{\partial x_k}\overline{u_l'u_k'} \tag{17}$$

must be calculated, where the new coordinates, $\tilde{x}$, arise after a rotation of the coordinate system such that

$$\begin{aligned}\tilde{x} &= \cos\varphi\, x - \sin\varphi\, y,\\ \tilde{y} &= \sin\varphi\, x + \cos\varphi\, y,\end{aligned} \qquad \tan\varphi = \frac{\bar{u}}{\bar{v}} \tag{18}$$

This gives

$$Q_{12} = \sin\varphi\cos\varphi(\overline{u'u'} - \overline{v'v'}) + (\cos^2\varphi - \sin^2\varphi)\,\overline{u'v'} \tag{19}$$

for the cross correlation function considered in the coordinate system, directed parallel to the mean motion, to which the simple eq. (16) refers.

If, like Webster, one has not carried out such a transformation one must discuss the formula (15) with regard to the occurrence of both vector components, $\bar{u}$ and $\bar{v}$. We then get

$$Q_{12} = -\nu_T\left(\frac{\partial\bar{v}}{\partial x}+\frac{\partial\bar{u}}{\partial y}\right) - \chi_2\bar{u}\bar{v} - \varkappa_1\frac{\partial\bar{u}}{\partial x}\left(\frac{\partial\bar{v}}{\partial x}-\frac{\partial\bar{u}}{\partial y}\right) \tag{20}$$

by making use of the incompressibility condition $\partial\bar{u}/\partial x = -\partial\bar{v}/\partial y$. Let the y-axis be nearly directed downstream,

$$\bar{u} < \bar{v} \tag{21}$$

We can cancel the y-derivatives, obtaining

$$Q_{12} = -\nu_T\frac{\partial\bar{v}}{\partial x} - \chi_2\bar{u}\bar{v} - \varkappa_1\frac{\partial\bar{u}}{\partial x}\frac{\partial\bar{v}}{\partial x} \tag{22}$$

Since it is equal to $-\partial\bar{v}/\partial y$ the value of $\partial\bar{u}/\partial x$ can also be expected to be small. According to (21) $\bar{u}$ is a small quantity, too. But we have to allow for the possibility that the values of χ_2 or $\varkappa_1$ may be large in a crucial sense.

3. Evaluation of χ_2

We start with the equation for a fluctuating velocity

$$\frac{\partial u_i'}{\partial t} - \nu \Delta u_i' = -\frac{1}{\varrho}\frac{\partial p'}{\partial x_i} - \frac{\partial}{\partial x_k}(\bar{u}_i u_k' + \bar{u}_i' \bar{u}_k) + F_i' \quad (23)$$

where $\mathbf{F}'$ represents an external force field maintaining the turbulence. Eq. (23) holds for weak turbulence fulfilling the condition

$$\min\left(\frac{u'L}{\nu}, \frac{u'T}{L}\right) < 1 \quad (24)$$

Supposing the mean motion to be constant we can introduce the functionals $\hat{\mathbf{u}}$, $\hat{p}$, $\hat{\mathbf{F}}$ defined by

$$\mathbf{u}'(\mathbf{k}, t) = \iint \hat{\mathbf{u}}(\mathbf{k}, \omega) \exp(i(\mathbf{kx} - \omega t))\, d\mathbf{k}\, d\omega,$$

$$p'(\mathbf{x}, t) = \iint \hat{p}(\mathbf{k}, \omega) \exp(i(\mathbf{kx} - \omega t))\, d\mathbf{k}\, d\omega,$$

$$\mathbf{F}'(\mathbf{x}, t) = \iint \hat{\mathbf{F}}'(\mathbf{k}, \omega) \exp(i(\mathbf{kx} - \omega t))\, d\mathbf{k}\, d\omega \quad (25)$$

and obtain

$$(-i\omega + \nu k^2)\, \hat{\mathbf{u}} = -i\frac{\hat{p}}{\varrho}\mathbf{k} - i(\mathbf{k}\bar{\mathbf{u}})\, \hat{\mathbf{u}} + \hat{\mathbf{F}} \quad (26)$$

Because of the incompressibility the scalar product $\mathbf{k}\cdot\hat{\mathbf{u}}$ vanishes, giving rise to

$$-i\frac{\hat{p}}{\varrho} = -\frac{\mathbf{k}\hat{\mathbf{F}}}{k^2} \quad (27)$$

This turns (26) into the form

$$\left(1 + \frac{i(\mathbf{k}\bar{\mathbf{u}})}{-i\omega + \nu k^2}\right)\hat{\mathbf{u}} = \frac{\hat{\mathbf{F}} - (\mathbf{k}\hat{\mathbf{F}})\,\mathbf{k}/k^2}{-i\omega + \nu k^2} \quad (28)$$

The right-hand side of this equation must be equal to $\hat{\mathbf{u}}^{(0)}$ if we further postulate the occurrence of a turbulent field, $\mathbf{u}^{(0)}$, also in the case of vanishing mean motion. Thus, (28) can be written as

$$\hat{\mathbf{u}} = \frac{\hat{\mathbf{u}}^{(0)}}{1 + \dfrac{i(\mathbf{k}\bar{\mathbf{u}})}{-i\omega + \nu k^2}} \quad (29)$$

from which we are led to the equation

$$Q_{ij} = \iint \frac{\hat{Q}_{ij}^{(0)}(\mathbf{k}, \omega)}{\left(1 + \dfrac{i(\mathbf{k}\bar{\mathbf{u}})}{-i\omega + \nu k^2}\right)\left(1 - \dfrac{i(\mathbf{k}\bar{\mathbf{u}})}{i\omega + \nu k^2}\right)}\, d\mathbf{k}\, d\omega \quad (30)$$

Using (9) this results in the cross-correlation function

$$Q_{12} = -\iiint \frac{(\omega^2 + \nu^2 k^4)\, k_1 k_2\, q(k, \omega)\, dk_1\, dk_2\, d\omega}{\omega^2 - 2\omega(k_1\bar{u} + k_2\bar{v}) + (k_1\bar{u} + k_2\bar{v})^2 + \nu^2 k^4} \quad (31)$$

By series expansion valid for $|T\bar{u}/L| \lesssim 1$ (T, L correlation time and length) we obtain a more convenient formula, namely

$$Q_{ij} = Q_{ij}^{(0)} + \frac{1}{4}\oint\int \frac{3\omega^2 - \nu^2 k^4}{(\omega^2 + \nu^2 k^4)^2}\, \hat{Q}_l^{(0)} k^2\, d\mathbf{k}\, d\omega \times \left(\tfrac{3}{2}\bar{\mathbf{u}}^2 \delta_{ij} - \bar{u}_i \bar{u}_j\right) \quad (32)$$

Combining this with (5) we can write

$$Q_{ij} = Q_{ij}^{(0)} + \frac{2\mu_T - \nu_T}{2\nu}\left(\tfrac{3}{2}\bar{\mathbf{u}}^2 \delta_{ij} - \bar{u}_i \bar{u}_j\right) \quad (33)$$

where we have introduced the positive expression

$$\mu_T = \frac{1}{2}\iint \frac{\nu k^2 \omega^2 \hat{Q}_{11}^{(0)}}{(\omega^2 + \nu^2 k^4)^2}\, d\mathbf{k}\, d\omega > 0 \quad (34)$$

From the definitions of χ's we obtain

$$\chi_1 = \frac{3}{4}\frac{\nu_T - 2\mu_T}{\nu}, \quad \chi_2 = \frac{1}{2}\frac{2\mu_T - \nu_T}{\nu} \quad (35)$$

Only for negative values of ν_T does the sign of the χ's prove to be definite. In that case χ_2 is positive. Correspondingly, negative values of χ_2 can be only expected for positive viscosity ν_T, especially if the condition

$$3\omega^2 < \nu^2 k^4 \quad (36)$$

is valid ($\nu_{H_2O} \approx 10^{-2}$ cm²/s).

4. Discussion of the measurements

Let us return to eq. (22). At the maximum of the $\bar{v}$-profile Webster has got small positive Q_{12}

for positive $\bar{u}$ and $\bar{v}$ (see table 2 of his paper). Our expression (22) provides

$$Q_{12} = -\chi_2 \bar{u}\bar{v} \quad \text{if} \quad \frac{\partial \bar{v}}{\partial x} = 0 \tag{37}$$

It follows that only negative values of χ_2 can explain the experimental results. Negative χ_2, however, applies only to positive values of the eddy viscosity ν_T—as shown in the previous section. To be more precise, the relation

$$\nu_T > 2\mu_T \tag{38}$$

must hold, where μ_T represents a positive definite quantity. Also the results of Schmitz & Niiler fulfil a relation (37) with $\chi_2 < 0$. What is more it seems that all of their measurements can be represented by such a dependence.

One would think eq. (36) represents a sufficient condition to satisfy (38). But the scale of the horizontal fluctuations under consideration ($L \approx 10^7$ cm, $T \approx 10^6$ s) does not allow such a condition. Hence there is little prospect of understanding the observed positive cross-correlation using (36). The possibility of fulfilling a condition like (36) seems to arise from replacing the molecular viscosity, ν, by an enlarged expression. This could be explained by the action of quasi-isotropic three-dimensional small-scale turbulence.

On no account, however, can one interprete the presented positive cross-correlations at the maximum of $\bar{v}$, the mean velocity profile, by negative viscosity, ν_T. Only if the terms in the relationship

$$\nu_T \gtrless 2\mu_T \tag{39}$$

are very nearly equal can we explain the smallness of the observed correlation at the maximum of the $\bar{v}$-profile.

Furthermore, eq. (22) implies

$$Q_{12} = \nu_T \left(-\frac{\partial \bar{v}}{\partial x} + \frac{\bar{u}\bar{v}}{2\nu} \right) - \frac{\mu_T}{\nu} \bar{u}\bar{v} - \varkappa_1 \frac{\partial \bar{u}}{\partial x} \frac{\partial \bar{v}}{\partial x} \tag{40}$$

In the Appendix we shall show that $\varkappa_1$ is a positive definite quantity. The value of $\partial \bar{v}/\partial x$ is negligible in comparison with $\bar{u}\bar{v}/(2\nu)$ if

$$\bar{u} \gg \frac{2\nu}{\bar{v}} \left| \frac{\partial \bar{v}}{\partial x} \right| \tag{41}$$

is valid. With Webster's values it is easy to fulfil this condition. Therefore, eq. (40) can be turned into the form

$$Q_{12} = \frac{\nu_T}{2\nu} \bar{u}\bar{v} - \frac{\mu_T}{\nu} \bar{u}\bar{v} - \varkappa_1 \frac{\partial \bar{u}}{\partial x} \frac{\partial \bar{v}}{\partial x} \tag{42}$$

Since in the experiment under consideration $\partial \bar{u}/\partial x \, \partial \bar{v}/\partial x$ was positive a supposed negative value of the viscosity would always lead to negative values of Q_{12}. This is contrary to observation. Webster's positive cross-correlations measured in the range conceived,

$$\left| \frac{2\nu}{\bar{v}} \frac{\partial \bar{v}}{\partial x} \right| \ll |\bar{u}| \ll |\bar{v}| \tag{43}$$

must, however, be interpreted as evidence for the existence of positive turbulent viscosity.

5. Conclusions

Using the fundamental assumption that an external force field exists which drives the turbulence, we cannot find that negative values of viscosity are indicated by Webster's measurements in the Gulf Stream. Drawing such a conclusion we have ignored any anisotropy in the horizontal plane which might be produced by the nearby shore. Satisfactory results can be obtained after a transformation of the coordinate system so that the mean motion contains one component only. The large uncertainties in the measurements under consideration and the detailed presentation of new expressions for the correlation tensor in our paper, lead to considerable interest in new experimental results, not only from fluids but also from the Earth's atmosphere.

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APPENDIX

We designate that part of the correlation tensor which contains the quadratic forms of $\bar{u}_{l,k}$ as $Q_{ij}^{(2)}$. If we specify our coordinate system as in (18) by which the mean motion possesses only an x-component, that is $\bar{u}$, we obtain from (15)

$$Q_{22}^{(2)} - Q_{11}^{(2)} = \varkappa_1 \left(\frac{d\bar{u}}{dy}\right)^2 \quad \text{(A 1)}$$

To calculate the left-hand side of this equation we have to start from the Navier–Stokes equation (23), which describes the behaviour of fluctuating velocities. The equations for the linear and quadratical parts of the turbulent velocity are

$$\frac{\partial u_i^{(1)}}{\partial t} - \nu \Delta u_i^{(1)} = -\frac{1}{\varrho}\frac{\partial p^{(1)}}{\partial x_i} - \frac{\partial}{\partial x_k}(u_i^{(0)}\bar{u}_k + \bar{u}_i u_k^{(0)}) \quad \text{(A 2)}$$

$$\frac{\partial u_i^{(2)}}{\partial t} - \nu \Delta u_i^{(2)} = -\frac{1}{\varrho}\frac{\partial p^{(2)}}{\partial x_i} - \frac{\partial}{\partial x_k}(u_i^{(1)}\bar{u}_k + \bar{u}_j u_k^{(1)}) \quad \text{(A 3)}$$

We solve these equations with the help of a Fourier transformation after introducing the series expansion

$$\bar{u}_k(\mathbf{x}) = \bar{u}_k + x_l \bar{u}_{k,l} \quad \text{(A 4)}$$

whereby the Fourier transforms must result as

$$\hat{u}_i^{(1)}(\mathbf{x}, \mathbf{k}, t, \omega) = (\hat{u}_i^{(1)}(\mathbf{k}, \omega) + x_l \hat{u}_{il}^{(1)}(\mathbf{k}, \omega))\, e^{i(\mathbf{kx} - \omega t)},$$

$$\hat{u}_i^{(2)}(\mathbf{x}, \mathbf{k}, t, \omega) = (\hat{u}_i^{(2)}(\mathbf{k}, \omega) + x_l \hat{u}_{ij}^{(2)}(\mathbf{k}, \omega) + x_l x_m \hat{u}_{ilm}^{(2)}(\mathbf{k}, \omega))\, e^{i(\mathbf{kx} - \omega t)} \quad \text{(A 5)}$$

The required correlation tensor is calculated by considering the expression

$$Q_{ij}^{(2)} = \int (\overline{\hat{u}_i^{(1)}(\mathbf{k}', \omega')\, \hat{u}_j^{(1)}(\mathbf{k}, \omega)} + \overline{\hat{u}_i^{(0)}(\mathbf{k}', \omega')\, \hat{u}_j^{(2)}(\mathbf{k}, \omega)} + \overline{\hat{u}_i^{(2)}(\mathbf{k}', \omega')\, \hat{u}_j^{(0)}(\mathbf{k}, \omega)}) \times \exp\,(i((\mathbf{k} + \mathbf{k}')\,\mathbf{x} - (\omega + \omega')\,t))\, d\mathbf{k}\, d\mathbf{k}' d\omega\, d\omega' \quad \text{(A 6)}$$

using the well-known relation

$$\overline{\hat{u}_i^{(0)}(\mathbf{k}', \omega')\, \hat{u}_j^{(0)}(\mathbf{k}, \omega)} = \hat{Q}_{ij}^{(0)}(\mathbf{k}', \omega')\, \delta(\mathbf{k} - \mathbf{k}', \omega - \omega') \quad \text{(A 7)}$$

The terms containing $\bar{\mathbf{u}}$ are already accounted for eq. (32). The terms containing $\mathbf{x}$ extend them so that they are also valid when $\bar{\mathbf{u}}$ is not a constant vector quantity, but $\bar{\mathbf{u}} = \bar{\mathbf{u}}(\mathbf{x})$. Consequently we need solve only a shortened system of equations. We give here only the following solution by reference to $\bar{u}_{i,k} = \delta_{1i}\delta_{2k} d\bar{u}/dy$:

$$\begin{aligned} Q_{ts} = \left(\frac{d\bar{u}}{dy}\right)^2 \oint \int \Bigg\{ & \frac{1}{\omega^2 + \nu^2 k^4} \Bigg[\left(\frac{2k_1 k_t}{k^2} - \delta_{1t}\right) \\ & \times \left(\frac{2k_1 k_s}{k^2} - \delta_{1s}\right) \hat{Q}_{22}^{(0)} + \frac{2\nu k_1 k_2}{i\omega + \nu k^2} \\ & \times \left(\frac{2k_1 k_s}{k^2} - \delta_{1s}\right) \hat{Q}_{t2}^{(0)} + \frac{2\nu k_1 k_2}{-i\omega + \nu k^2} \\ & \times \left(\frac{2k_1 k_t}{k^2} - \delta_{1t}\right) \hat{Q}_{s2}^{(0)} + \frac{4\nu^2 k_1^2 k_2^2}{\omega^2 + \nu^2 k^4} \hat{Q}_{ts}^{(0)} \Bigg] \\ & + \frac{1}{(-i\omega + \nu k^2)^2} \\ & \times \Bigg[\left(\frac{8k_1^2 k_2 k_t}{k^4} - \frac{2k_1^2 \delta_{2t}}{k^2} - \frac{2k_1 k_2 \delta_{1t}}{k^2}\right) \hat{Q}_{2s}^{(0)} \\ & + \left(\frac{8k_1^2 k_2 k_s}{k^4} - \frac{2k_1^2 \delta_{2s}}{k^2} - \frac{2k_1 k_2 \delta_{1s}}{k^2}\right) \hat{Q}_{2t}^{(0)} \Bigg] \\ & + \frac{2\nu k^2}{(-i\omega + \nu k^2)^3} \Bigg[\left(\frac{6\nu k_1^2 k_2^2}{k^2(-i\omega + \nu k^2)} - \frac{k_1^2}{k^2}\right) \\ & \times (\hat{Q}_{st}^{(0)} + \hat{Q}_{ts}^{(0)}) + \frac{3k_1 k_2}{k^2}\left(\frac{2k_1 k_t}{k^2} - \delta_{1t}\right) \hat{Q}_{2s}^{(0)} \\ & + \frac{3k_1 k_2}{k^2}\left(\frac{2k_1 k_s}{k^2} - \delta_{1s}\right) \hat{Q}_{2t}^{(0)} \Bigg] \Bigg\} d\mathbf{k}\, d\omega \end{aligned} \quad \text{(A 8)}$$

from which, with (9), follows the expression

$$Q_{22}^{(2)} - Q_{11}^{(2)} = \frac{1}{2} \oint \int \frac{\nu^4 k^{10} q(k, \omega)}{(\omega^2 + \nu^2 k^4)^3}\, d\mathbf{k}\, d\omega \left(\frac{d\bar{u}}{dy}\right)^2 \quad \text{(A 9)}$$

Therefore

$$\varkappa_1 = \frac{1}{2} \oint \int \frac{\nu^4 k^{10} q(k, \omega)}{(\omega^2 + \nu^2 k^4)^3}\, d\mathbf{k}\, d\omega \quad \text{(A 10)}$$

Because $q \geqslant 0$ $\varkappa_1$ proves to be a positive definite quantity.

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ОТРИЦАТЕЛЬНАЯ ВЯЗКОСТЬ В ГОЛЬФСТРИМЕ?

Обсуждается вопрос о том, являются ли измерения Вебстера (1965) доказательством существования отрицательной вязкости для двумерного турбулентного течения, указанной в работе Краузе и Рюдигера (1974). Во всех измерениях Вебстера появляются небольшие поперечные компоненты среднего движения, учет которых приводит к выводу, что для Гольфстрима может существовать только положительная вязкость. Этот вывод зависит от нашего фундаментального предположения, что турбулентность обусловлена внешними силами.