A note on the concepts of age distribution and transit time in natural reservoirs

By BERT BOLIN and HENNING RODHE, *Institute of Meteorology, University of Stockholm, Sweden*[1]

(Manuscript received October 4, 1972)

ABSTRACT

A brief review is given of the concepts *age distribution, transit time distribution, turn-over time, average age* and *average transit time* (residence time) and their relations. The characteristics of natural reservoirs are discussed in terms of these concepts, and a classification is proposed based on whether the average age is larger, equal to or smaller than the average transit time. Some examples illustrate the differences between these various cases.

1. Introduction

When concerned with transfer of matter in nature, particularly when attempting an overall treatment of how exchange takes place between the major natural reservoirs, it is common and useful to introduce concepts as age distribution, transit time, residence time, turn-over time etc. Obviously these different concepts are interrelated. To avoid misunderstandings and even erroneous conclusions it is important to introduce precise definitions and to use them with care. Eriksson (1961, 1971) has dealt with some of these problems in two important papers, to which reference is made. In this note we summarize certain results into a simple and rigorous form. We also wish to give some further examples which illustrate the way in which the concepts defined are dependant on reservoir characteristics. Since Eriksson's work particularly refers to hydrology it is of interest to give some examples from other fields of geophysics, such as meteorology. Obviously the following considerations are also of considerable interest in the development of models in ecology.

2. Basic concepts

Consider a reservoir (such as the atmosphere or a closed water body) at a given time t. Assume that this reservoir is in exchange with other reservoirs. We shall further limit the present study to steady state conditions, i.e. we assume that the total mass and the statistical distributions studied do not vary with time. Such a restriction may of course limit the applicability of the results obtained. A study of steady state conditions, however, necessarily must precede an analysis of transient developments.

Each element in such a reservoir can be characterized by the time τ that has elapsed since it entered the reservoir under consideration, i.e. τ is the "age" of the element. These elements can be arranged in a cumulative fashion whereby the cumulative function $M(\tau)$ is defined as the mass that has spent a time less or equal to τ in the reservoir. Strictly, such an arrangement is only possible by considering molecules or atoms. When mixing occurs, which is normally the case, we must be careful when talking about "a fluid element". We shall return to this problem later.

If the total mass of the reservoir is M_0 we obviously have

$$\lim_{\tau\to\infty} M(\tau) = M_0 \tag{1}$$

From $M(\tau)$ we define the frequency function $\psi(\tau)$ of mass with respect to age (i.e. the "*age distribution*" function of the particles in the fluid). Since we require that

$$\int_0^\infty \psi(\tau)\, d\tau = 1 \tag{2}$$

[1] Contribution No. 258.

it follows that

$$\psi(\tau) = \frac{1}{M_0}\frac{dM(\tau)}{d\tau} \tag{3}$$

or equivalenty

$$M(\tau) = M_0 \int_0^\tau \psi(\xi)\, d\xi \tag{4}$$

It should be noted that this definition is slightly different from that used by Eriksson (1961).

Next consider the flux to and from this reservoir. Since steady state prevails at any one time a constant amount equal to F_0 enters and leaves the reservoir per unit time. Each element of mass leaving the reservoir can be characterized by the time that has elapsed since it entered the reservoir. We arrange these elements in a cumulative fashion and define the function $F(\tau)$, as the mass leaving the reservoir per unit time which has spent a time less than or equal to τ in the reservoir. This function we call "*transit time*" function. Thus

$$\lim_{\tau\to\infty} F(\tau) = F_0 \tag{5}$$

The frequency function for the transit time $\varphi(\tau)$, (i.e. the age distribution for the particles leaving the fluid) is consequently

$$\varphi(\tau) = \frac{1}{F_0}\frac{dF(\tau)}{d\tau} \tag{6}$$

or equivalenty

$$F(\tau) = F_0 \int_0^\tau \varphi(\xi)\, d\xi \tag{7}$$

For the frequency function $\varphi(\xi)$ the normalization condition

$$\int_0^\infty \varphi(\xi)\, d\xi = 1 \tag{8}$$

holds.

In the case of *steady state* the two functions $M(\tau)$ and $F(\tau)$ are uniquely related by the equation

$$F_0 - F(\tau) = M_0\, \psi(\tau) = \frac{dM(\tau)}{d\tau} \tag{9}$$

This equation can be verified in the following way. $F_0 - F(\tau)$ is the flux out of the reservoir per unit time of fluid elements with an age larger than τ. In a steady state this must be balanced by the number of elements per unit time reaching the age τ, i.e. the number of elements $M_0 \cdot \psi(\tau)$ that at any one moment have an age lying within an interval of unit length around τ.

Since $\lim_{\tau\to 0} F(\tau) = 0$ it follows from eq. (9) that

$$\lim_{\tau\to 0} \psi(\tau) = \psi(0) = \frac{F_0}{M_0} \tag{10}$$

With the aid of eq. (6) we may transform eq. (9) into

$$\varphi(\tau) = -\frac{M_0}{F_0}\frac{d\psi(\tau)}{d\tau} \tag{11}$$

$M(\tau)$ and $F(\tau)$ are, by definition, non-decreasing functions, and it follows from eq. (9) that $\psi(\tau)$ is non-increasing. $\varphi(\tau)$, on the other hand, may well have a less regular shape.

The concepts discussed above can be well illustrated by an analogy with a human population. Let $M(\tau)$ be the number of people of an age equal to or less than τ (i.e. measured in years). $M_0\psi(\tau)$ is then the age distribution measured as the number of people in each year class. $F(\tau)$ is the yearly number of deaths among that part of the population that is younger than or equal to τ and eq. (9) expresses the fact that in a steady state the number of deaths among people older than τ years must be balanced by the number of persons each year reaching the age of τ.

3. Characteristic types of frequency functions

In many applications it is desirable with a simple description of the frequency functions $\psi(\tau)$ or $\varphi(\tau)$. We shall next discuss some of the time constants that are used commonly to characterize these functions.

A. *The turn-over time*, τ_0, is usually expressed as the ratio of the total mass in the reservoir to the total flux

$$\tau_0 = \frac{M_0}{F_0} \tag{12}$$

B. *The average age*, τ_a, of particles in the reservoir at any one time is given by

$$\tau_a = \int_0^\infty \tau\psi(\tau)\,d\tau = \frac{1}{M_0}\int_0^\infty \tau\,dM(\tau) \qquad (13)$$

C. *The average transit time*, τ_t, of particles leaving the reservoir (= the expected life time for newly incorporated particles) is given by

$$\tau_t = \int_0^\infty \tau\varphi(\tau)\,dt \qquad (14)$$

Making use of eq. (11) this expression for τ_t may be transformed

$$\tau_t = -\frac{M_0}{F_0}\int_0^\infty \tau d\psi(\tau) = -\frac{M_0}{F_0}\tau\psi(\tau)\Big/_0^\infty$$

$$+\frac{M_0}{F_0}\int_0^\infty \psi(\tau)d\tau = \frac{M_0}{F_0}\int_0^\infty \psi(\tau)\,d\tau = \frac{M_0}{F_0} = \tau_0 \qquad (15)$$

Thus the average age of particles leaving the reservoir is identical with the turn-over time.

As an alternative name for "average transit time" one may use the word *residence time*. Especially for a reservoir where no flux in the physical sense occurs within the reservoir, residence time should be preferred. It is important to note at this point, that Eriksson (1961, 1971) uses the word residence time for what we propose more appropriately should be called the average age.

The relation between $\tau_t(=\tau_0)$ and τ_a is determined by the form of the frequency function $\psi(\tau)$ (or $\varphi(\tau)$). We may distinguish three cases

a) $\tau_a < \tau_t = \tau_0$

b) $\tau_a = \tau_t = \tau_0$

c) $\tau_a > \tau_t = \tau_0$

Case a: $\tau_a < \tau_t$

$\psi(\tau)$ and $\varphi(\tau)$ have typical shapes as shown in Fig. 1*a* and are characterized by the fact that few elements leave the reservoir soon after having entered it, i.e. $\varphi(\tau)$ is small for small τ.

A reservoir with modest transport velocities and source and sink regions far apart belongs to this case. One example of such a reservoir is particulate matter being introduced high up in the stratosphere and removed at the tropopause by transfer into the troposphere (i.e. nuclear bomb testing). Another example is given by the age of water in a lake with inlet and outlet at opposite sides.

Referring to the analogy with a human population, a country with low infant mortality represents a case of this kind. In Sweden for example, the expected length of life of a newly born child (τ_t) is about 70 years, while the average age (of living people) (τ_a) is about 35 years.

Case b: $\tau_a = \tau_t$

As seen from eqs. (13) and (14) the condition for τ_a to be equal to τ_t is

$$\int_0^\infty \tau[\psi(\tau)-\varphi(\tau)]\,d\tau = 0 \qquad (16)$$

and thus a *sufficient* condition is that

$$\psi(\tau) = \varphi(\tau) \qquad (17)$$

It then follows from eqs. (6) and (9) that

$$\psi(\tau) = -\frac{M_0}{F_0}\frac{d\psi(\tau)}{d\tau} \qquad (18)$$

and thus

$$\psi(\tau) = \varphi(\tau) = \frac{1}{\tau_a}e^{-\tau/\tau_a} \qquad (19)$$

If, on the other hand, one of the frequency functions can be shown to have an exponential form, it follows from eq. (11) that the other frequency function must be identical. This case is illustrated in Fig. 1*b*.

Exponential frequency functions characterize reservoirs, in which all elements have a certain constant probability of being removed per unit time. Well-known examples of such sink processes are radioactive decay and first order chemical transformation.

A reservoir, in which the sink region is isolated and the removal probability of an element is constant once it has reached this region, is characterized by an exponential age distribution only if the reservoir is "well-mixed". In such a case all elements in the reservoir will "touch" the sink region an equal number of times per unit time, independently of their initial position in the reservoir. It should be remarked, however, that even though the fre-

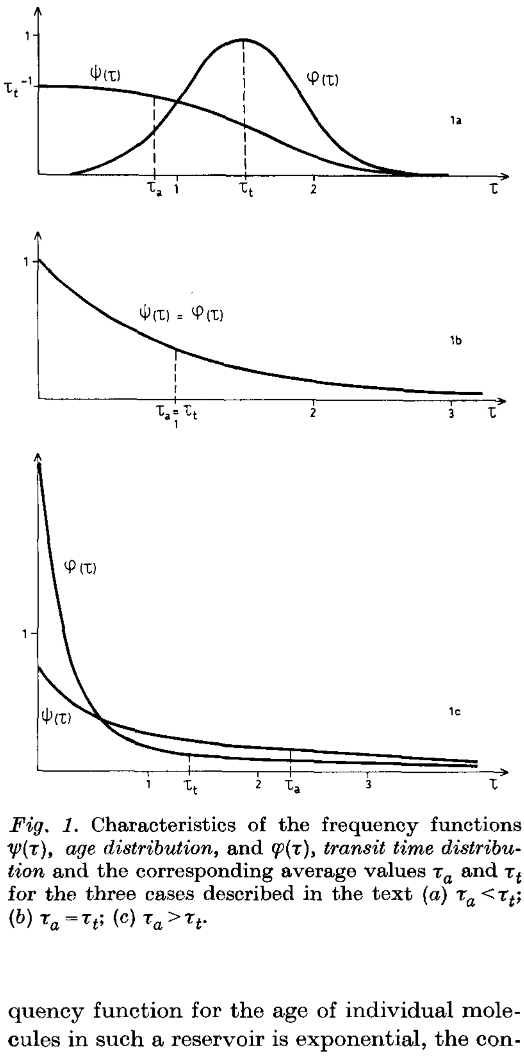

Fig. 1. Characteristics of the frequency functions $\psi(\tau)$, *age distribution*, and $\varphi(\tau)$, *transit time distribution* and the corresponding average values τ_a and τ_t for the three cases described in the text (*a*) $\tau_a < \tau_t$; (*b*) $\tau_a = \tau_t$; (*c*) $\tau_a > \tau_t$.

quency function for the age of individual molecules in such a reservoir is exponential, the concept "well-mixed" of course implies that any *finite* element of the reservoir contains molecules of all ages. ***It is therefore impossible to establish the frequency functions by direct observations in such a case.***

Case c: $\tau_a > \tau_t$

This case is characterized by a situation in which most of the elements that enter the reservoir stay there for a short period of time, making τ_t small. On the other hand those elements that remain for a longer period of time stay sufficiently long to make τ_a comparatively large.

The following two examples are of geophysical interest.

1. The source and sink regions are located close to each other or even coincide (the "short circuit case"). An important example is a gaseous constituent *in* the atmosphere *for which* the sea is the main source *and* sink and where the exchange is rapid in comparison with the *transfer* away from the sea surface into the bulk of the atmosphere. Water vapour in the atmosphere may be taken as a specific example. The source of water vapour is evaporation from the earth's surface, mainly the oceans. This evaporation is, however, only the net result of an exchange process that involves flux of water molecules both to *and* away from the water surface. Thus the process of evaporation represents both a source and sink for atmospheric water vapour. Since the net evaporation usually is quite a lot smaller than the gross fluxes, also precipitation as a sink for atmospheric water vapour is smaller than the direct condensation on the ocean surface. The average transit time (or residence time) τ_t, for water vapour in the atmosphere is often quoted to be about 10 days. Then only precipitation is considered as the sink mechanism. If we, however, wish to compute the average *time* spent *in* the *atmosphere* by each water vapour molecule leaving the ocean surface i.e. the true value of τ_t as defined here, we would (probably) find a considerably smaller value. The average age of water molecules in the atmosphere, τ_a, on the other hand, may well be of the order of 10 days, since at anyone time there are only a relatively small number of molecules close to the ocean surface that have very recently left the surface and soon will return to it by direct impaction. It is important to recognize these facts, whenever one compares the residence time (or average age) of other atmospheric trace substances with that of water vapour.

2. The removal is caused by two different physical processes of which one, for some reason, is confined to "young" particles. This may for example be the case with sulfur dioxide emitted into the atmosphere within a city. Near the source, where the concentration of other pollutants is high, an appreciable part of the sulfur dioxide is oxidized (catalytic oxidation). On the other hand, the part of the gas that escapes into comparatively clean air further away from the source is affected by less effective sink processes such as photochemical oxidation and direct uptake at the earth's surface.

A formal example of frequency functions that satisfy the condition $\tau_a > \tau_t$ is given by

$$\varphi(\tau) = Ae^{-\alpha\tau} + Ae^{-\beta\tau}$$

$$\psi(\tau) = \frac{F_0}{M_0}\left[\frac{A}{\alpha}e^{-\alpha\tau} + \frac{B}{\beta}e^{-\beta\tau}\right] \tag{20}$$

if the constants A, B, α and β are chosen suitably ($A \gg B$, $\alpha \gg \beta$; note further that $A/\alpha + B/\beta = 1$ in view of eq. (8)). The combination of two exponential functions with significantly different decays illustrates the kind of situation which was referred to above. The characteristic features of such distributions are shown in Fig. 1*c* (in which case it has been assumed that $A = 2$, $B = 0.2$, $\alpha = 4$, $\beta = 0.4$).

4. Concluding remarks

The brief comments given in the preceeding section only indicate some general properties of natural reservoirs. Clearly a closer analysis in *specific* cases may yield considerably more precise information.

Observations of the behaviour of natural reservoirs usually give insufficient information to determine accurately the frequency functions. Most commonly they permit some conclusions regarding the average age, τ_a, of particles in the reservoir and possibly some general features of the age distribution. For most applications of reservoir theory, however, the average transit time (residence time), τ_t, is the most relevant parameter and we have seen that τ_t may be quite different from τ_a. The general relations and results obtained in the previous sections may then be of help to deduce the average transit time. Quite generally they may be of value for interpreting observations of the behaviour of natural reservoirs (cf. also Rohde & Grandell, 1972).

Acknowledgements

Discussions with Dr Jan Grandell, Institute of Mathematical Statistics, University of Stockholm, were helpful in the course of preparing this note and is gratefully acknowledged.

This work has been supported by the Swedish Natural Science Council, Contract 138 G Dnr 0223-050 and by the National Swedish Environmental Protecting Board, Contract 7-110/70 Dnr 1182-7-70.

REFERENCES

Eriksson, E. 1961. Natural reservoirs and their characteristics. *Geofisica International 1* (2), 27–43.

Eriksson, E. 1971. Compartment models and reservoir theory. *Annual Review of Ecology and Systematics 2*, 67–84.

Rodhe, H. & Grandell, J. 1972. On the removal time of aerosol particles from the atmosphere by precipitation scavenging. *Tellus 24*, No. 5, 442–454.

ЗАМЕЧАНИЕ О КОНЦЕПЦИЯХ РАСПРЕДЕЛЕНИЯ ПО ВОЗРАСТУ И ПО ВРЕМЕНИ ПРОХОЖДЕНИЯ В ЕСТЕСТВЕННЫХ РЕЗЕРВУАРАХ

Дается короткий обзор концепций распределения по возрасту, распределения по времени прохождения, времени обращения, среднего возраста и среднего времени прохождения (времени пребывания) и соотношений между ними. С помощью этих терминов обсуждаются характеристики естественных резервуаров и предложена их классификация, основанная на том, больше, равен или меньше средний возраст в сравнении со средним временем прохождения. Некоторые примеры иллюстрируют разницу между различными случаями.