The seasonal thermal structure of deep temperate lakes

By T. R. SUNDARAM[1] and R. G. REHM, *Aerodynamic Research Department, Cornell Aeronautical Laboratory, Buffalo, New York 14221, USA*

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ABSTRACT

A theoretical description is given of the seasonal variation of the thermal structure of deep, temperate lakes. The influence of the unsteady and nonlinear interaction between the wind-induced turbulence and the stable buoyancy gradients due to surface heating on the thermal eddy diffusivities is assumed to be given by a product of the eddy diffusivities under conditions of neutral stability and an appropriate function of a stratification parameter, which, in this case, is taken to be the gradient Richardson number. An implicit account is taken of the interaction between the current and thermal structures within the lake, and free convection is also included implicitly during the cooling portion of the annual cycle. The basic equation, which is integrated by finite difference means, is an unsteady, nonlinear diffusion equation describing the vertical temperature structure. The boundary condition on the heat exchange at the lake surface is formulated in terms of the concept of an equilibrium temperature. The seasonal stratification cycle of a lake is determined as the response of the lake to certain imposed "external parameters" which specify the exchange of mechanical and thermal energies between the lake and the environment. The results of calculations of the vertical thermal structure using this model agree with the observed qualitative features of the stratification cycle. For the one case in which quantitative comparisons are made, good agreement with observations is obtained. A brief discussion comparing the features of the present theory with those of other existing theories is also given.

I. Introduction

All deep bodies of water in temperate latitudes, including lakes, rivers and the ocean, exhibit, during part of their annual cycles, a characteristic thermal structure in which a well-mixed, warm upper layer is separated from a relatively colder bottom region. When a water body exhibits such a thermal structure, it is said to be stratified, and the layer of intense temperature gradient separating the almost homogeneous upper layer from the colder bottom waters is termed the thermocline. In a previous paper, the authors (1970; henceforth this paper will be referred to as paper I) have shown that a satisfactory theory for the mechanism of formation and maintenance of thermoclines in a temperate lake has to take into account two essential aspects. Firstly, the mechanism of formation and of maintenance of a thermocline is an unsteady process even when conditions above the body of water under consideration are steady. Therefore, a satisfactory theory for the thermocline must include this basic aspect. Secondly, the formation of a thermocline is by the nonlinear interaction between the wind-generated turbulence and the stable buoyancy gradients in the body of water under consideration. This nonlinearity, while making the equations difficult to analyze, is an essential feature of the interaction between the turbulence and prevailing temperature structure and as such has to be retained.

In paper I, the above concepts were used to formulate a theoretical model in which the interaction between wind-induced turbulence and buoyancy gradients due to surface heating were included explicitly by considering the eddy diffusivity for the vertical transport of heat to be dependent on a gradient Richardson number. This model was then used to calculate the processes of formation and maintenance of the thermocline in an idealized lake for the special cases when the surface temperature or the surface heat flux were held constant. It was shown that the behavior of the thermocline is

[1] Present address: Hydronautics, Laurel, Maryland 20810.

basically unsteady even when conditions at the surface are held constant, and that the characteristic time-scales of the unsteadiness can be calculated by using dimensional analysis in conjunction with the governing equation.

When considering the entire stratification cycle of a lake, in addition to the basic time-scale mentioned above, a time-scale characteristic of the variation of the surface conditions (or equivalently, the atmospheric conditions above the lake) arises. One of the major objectives of the present paper is to examine the effect of this additional time-scale on the thermal structure of the deeper layers of a temperate lake. In the following sections, a description of the theoretical model is first given and calculations are then presented. The physical significance of the calculated results are discussed and the calculations are also compared with actual field data. Finally, a brief comparison of the feature of the present theory with those of other existing ones, is given.

The concepts outlined above have also been utilized by the authors (1971) to study the effects of power-plant thermal discharges on the stratification cycle of lakes. Results from this paper, which will be referred to as paper II, will also be utilized in the present paper.

II. The model

In the model developed in paper I, the basic equations governing the vertical turbulent transfer of heat are,

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial Z}\left(K_H \frac{\partial T}{\partial Z}\right) \tag{1}$$

$$K_H = K_{H_0}(1 + \sigma_1 Ri)^{-1} \tag{2}$$

and

$$Ri = -\alpha_v g Z^2 \frac{\dfrac{\partial T}{\partial Z}}{w^{*2}} \tag{3}$$

where T is the temperature, t is the time, Z is the distance measured downward from the surface, K_H is the eddy diffusivity for the vertical transport of heat, K_{H_0} is the eddy diffusivity in the absence of stratification, α_v is the volumetric coefficient or thermal expansion of water, g is the acceleration due to gravity, $w^* = \sqrt{\tau_s/e}$ is the friction velocity due to the stress τ_s exerted by the wind at the surface and σ_1 is an empirical constant.

The approximations on which Eqs. (1)–(3) are based have been discussed in detail in papers I and II and need not be repeated here. However, it should be emphasized that, while in Eq. (2) the form originally proposed by Rossby & Montgomery (1935) is used to relate the eddy diffusivity in the presence of stratification to that in the absence of stratification, the *qualitative* conclusions given in the present paper are *not* influenced by the specific form chosen. The functional form given in Eq. (2) was chosen primarily because this form seems to fit available experimental data (Kent & Pritchard, 1959) rather well.

Boundary conditions

The boundary conditions to be used with Eqs. (1)–(3) must describe the thermal and mechanical energy transfers at the air-water interface. The boundary condition describing the heat exchange between the lake and the atmosphere can be written in the form, see Edinger & Geyer (1967), Sundaram et al. (1969) and Edinger (1970).

$$q_s = -\varrho C_p \left(K_H \frac{\partial T}{\partial Z}\right)_{Z=0} = K(T_E - T_S) \tag{4}$$

where q_s is the surface heat flux (taken positive when downward), K is a heat-exchange coefficient, T_s is the surface temperature and T_E is a fictitious surface temperature, called the equilibrium temperature, at which there would be no net heat transfer to or from the lake surface. The equilibrium temperature and the heat-transfer coefficient are both functions of the environmental conditions above the lake and can be expressed as functions of the wind speed, air temperature and humidity, and net incoming (sky and solar) radiation. The annual variation of the equilibrium temperature over most temperate lakes can be represented by the simple sinusoidal relation,

$$T_E = \overline{T}_E + \delta T_E \sin(\omega t + \varphi) \tag{5}$$

where $\overline{T}_E$ is the average value of the equilibrium temperature over one annual cycle, δT_E is one half of the annual variation and $\omega = 2\pi/365$

days^{-1}. The value of φ will depend upon the conditions from which the analysis is begun. In most cases the annual variation of K is small and it can be taken, to sufficient accuracy, as a constant. Methods of evaluating T_E and K are described fully by Edinger & Geyer (1967) and by Sundaram et al. (1969).

The appropriate boundary conditions related to the transfer of mechanical energy across the interface due to wind stirring are the specifications of the eddy diffusivity, K_{H_0}, under neutral conditions, and the friction velocity, w^*. While these quantities are directly related, the wind speed above the lake and its annual variation can be arbitrary, in the present study, the friction velocity has been assumed to be a cyclic function of time of the form,

$$w^* = B_1 + B_2 \sin(\omega t + \psi) \quad (6)$$

where B_1, B_2 and ψ are constants to be determined from the known conditions above the lake, and ω is the angular frequency as before. A similar relation is assumed for the variation of K_{H_0}, namely

$$K_{H_0} = cw^* \quad (7)$$

where c is an appropriate constant.

Thus, the three major parameters necessary for the specification of the thermal and mechanical energy transfer at the surface of a lake are the equilibrium temperature, T_e, the heat-exchange coefficient, K, and the friction velocity, w^*. The effects of these conditions on the lake are closely coupled, of course, since the heat that can be received by the surface layers is dependent on the depth to which wind action can mix this heat, and conversely, the depth of wind mixing is itself a function of the degree of surface heating. The effect of wind mixing on the seasonal temperature cycle of a lake can be interpreted in terms of the changes in the values of either the Monin-Obukhov length (1954) or the over-all stability of the lake. Both of these interpretations have been discussed in detail by Sundaram et al. (1969, 1971).

Significance of the equilibrium temperature

The concept of the equilibrium temperature is an extremely useful one for studying the characteristic features of the stratification cycle of temperature lakes. The physical significance

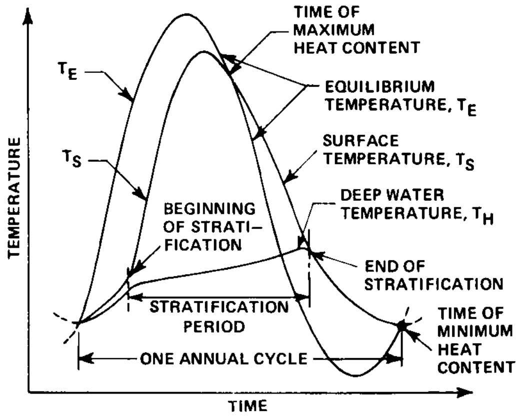

Fig. 1. Schematic representation of the annual temperature cycle.

of the equilibrium temperature is that the redistribution of heat within a lake always tends to be in such a manner as to drive the surface temperature towards the equilibrium temperature, as can be seen in the schematic representation of the stratification cycle shown in Fig. 1. Since the heat flux at the surface of the lake has to change sign during the time of minimum heat content, it is clear from Eq. (4) that the surface temperature, the deep-water temperature and the equilibrium temperature will all coincide at this time. During the spring months, the equilibrium temperature increases rapidly due to increasing solar radiation. While the surface temperature also increases during this period, it increases much more slowly than the equilibrium temperature, since the temperature of the surface layers depends not only on the rate of heating of these layers but also on the rate at which heat is removed, by turbulent mixing, from these layers to the deeper layers of the lake. During the early period, the lake remains nearly homothermal since, due to the low over-all stability, wind action is able to mix the surface heat into great depths. However, as the heating continues a thermocline forms, with the time of its formation usually coinciding with the time when the heat flux through the surface is a maximum. As the thermocline forms, the surface temperature begins to increase rapidly, since now the upper layers are heated preferentially in relation to the deeper layers (whose temperature increases only slightly).

After the formation of stratification, the surface temperature usually increases at a more rapid rate than the equilibrium temperature so that the heat flux into the lake begins to decrease. As the surface temperature continues to increase and the volume of the epilimnion increases, the lake begins to approach its maximum heat content. However in many lakes, the surface temperature beings to decrease *before* the maximum heat content is reached because of a decrease in over-all stability of the lake and the attendant rapid descent of the thermocline. As pointed out by Hutchinson (1957), the above behavior is exhibited by the Lunzer Untersee. It has also been found by Sundaram et al. (1969) to be characteristic of Cayuga Lake.

The lake attains its maximum heat content as the equilibrium and surface temperatures once again coincide. Beyond this point the equilibrium temperature falls below the surface temperature, and the lake begins to lose heat. During this period, the stratification in the upper layers of the lake is statically unstable and the thermocline continues to descend rapidly into the deeper layers of the lake. When, in late fall, the thermocline descends to the bottom of of the lake, the lake again attains homothermy and cools uniformly while losing heat. The minimum heat content (and the end of the cycle) is reached when the equilibrium temperature once more equals the surface temperature. The cycle is then repeated.

Required modifications of the form of the eddy diffusivity

In paper I, it was pointed out that the form of the eddy diffusivity given in Eq. (2) has to be modified after the formation of stratification to account for the observed changes in the current structure. Thus before the time of formation of the thermocline, when the temperature decreases smoothly with increasing depth, Eq. (2) can be used to describe the eddy diffusivity over the entire depth of the lake. After the thermocline forms, Eq. (2) can be an adequate representation for the eddy diffusivity only in the epilimnion since the deeper layers are "protected" from direct wind effects by the stratification. The procedure for the representation of the eddy diffusivity after the formation of the thermocline (that is, after the maximum value of the temperature gradient occurs at some point below the surface) that is used in the present study is to assume that Eq. (2) is valid only in the region above the level at which the eddy diffusivity attains a minimum value. The value of the eddy diffusivity in the hypolimnion is taken to be equal to the minimum value of the eddy diffusivity predicted by Eq. (2). This procedure, yields a value of the hypolimnetic diffusivity which decreases continuously with the progress of the stratification, and is in accordance with observations (see Hutchinson (1957) and Sundaram et al. (1969)).

During late summer, as the lake begins to lose heat to the environment, the stratification in the upper layers of the lake becomes statically unstable. In othei words, in these layers the turbulence is augmented, rather than suppressed, by the buoyancy gradients. When the rate of cooling is relatively small, as in the early parts of the cooling season, the degree of augmentation of the wind-induced mechanical turbulence by the convective turbulence is still governed by the Richardson number. However when the rate of cooling becomes large, as in the later parts of the cooling period, the Richardson number no longer constitutes a meaningful parameter (and Eq. (2) will not be valid), since now the dominant transport mechanism in the upper layers is the convective turbulence and wind-induced turbulence has very little effect on the mixing processes. This does *not* mean that the quantity of mechanical energy transferred to the water by wind shear is small compared with the quanitity of convective energy. Indeed, free convection is initiated when the above two energies are approximately equal. As Lumley & Panofsky (1964) point out in connection with atmospheric turbulence, the structures of mechanical and convective turbulences are quite different, the latter being a far more efficient transporting agent. The mechanical eddies are usually quite small while the eddies produced by convection are relatively large, their size being of the order of the thickness of the unstable layer. Thus the latter provide larger correlations between the fluctuating quantities, and hence larger transport, than the former.

Formulae appropriate to free convection have been developed by various authors, (see Lumley & Panofsky (1964) and Phillips (1966)), and these can be used to describe the transport processes during the cooling period. However, it should be pointed out that during the cooling parts of the stratification cycle both stable and unstable conditions will exist simultaneously, though at different levels, in the lake. Thus while the upper layers will be statically unstable, the lower layers, including the thermocline region, will exhibit a stable stratification. Therefore the appropriate procedure for representing the transport processes during the cooling period would be to use a suitable free-convection formula to represent the upper layers of the lake and to match this to Eq. (2) at the level separating the unstable and stable regions. Clearly, this procedure will be quite complex, and it has not been adopted in the present study. Instead a relatively simple procedure for representing free convection has been chosen.

It can be seen from Eq. (2) that, during the cooling season, the maximum value of the eddy diffusivity will occur at some depth below the surface. This depth represents the depth at which the rate of generation of turbulent energy by buoyancy forces is a maximum. In other words, convective eddies of the size of this critical depth provide the largest correlations between the fluctuating quantities. In the present study, the eddy diffusivity is assumed to be well represented by Eq. (2) below the critical depth, while above this depth the eddy diffusivity is assumed to be constant everywhere and equal to the value at the critical depth. That is, the region above the critical depth is assumed to be kept well mixed by the convective turbulence.

While the above procedure for representing free convection is admittedly crude, it appears to give an accurate representation in practice. Other more sophisticated methods are, of course, possible. For example, the convection velocity due to surface cooling can be included explicitly into Eq. (1). However since, in the present study, the coupling between the velocity and thermal structures is included only implicitly, the effect of the free-convection velocity can also be included only implicitly.

III. Numerical results

By using the appropriate modifications for the eddy diffusivity given in the last section, Eqs. (1)–(3) can be integrated, when the equilibrium temperature, the heat-exchange coefficient and the friction velocity are specified. These three conditions are the only primary input conditions required for the present analytical model, and it should be noted that they are all concerned with the environmental conditions above the lake. Specifically, no knowledge of the temperature profiles within the lake, or eddy diffusivities derived from them are needed. Some *a priori* knowledge of the thermal structure of the waterbody is a prerequisite in most of the existing models for the stratification cycle.

The starting point for the calculations is from the time of the minimum surface temperature of the lake, and at this time the lake is assumed to be homothermal. However, it should be pointed out that the minimum temperature of the lake and the time at which the minimum temperature occurs are, a priori, not known. Nevertheless, because of the cyclic nature of the imposed boundary conditions, the solution will ultimately tend to a cyclic one (regardless of the initial conditions) provided that the computation is carried out over several cycles. On the other hand, if the initial conditions are chosen appropriately, the solution can be expected to return to the same conditions after just one cycle. Thus when information on the temperature of the lake is available for some period during spring homothermy, the calculations can be greatly simplified.

In the present study a number of calculations, corresponding to different input conditions were carried out. The details of the numerical computational procedure are described in the report of Sundaram et al. (1970).

Figs. 2–5 show plots of temperature, thermal diffusivity and thermocline depth for a particular calculation. For this calculation the following input conditions were used:

$$T_E = 11 + 16 \sin\left(\frac{2\pi}{365}t + 0.531\right), \text{ °C}$$

$$K = 488\ \text{CAL/m}^2\ \text{day °C}$$

$$w^* = 0.0305 + 0.00763 \sin\left(\frac{2\pi}{365}t + 2.61\right)\text{m/sec}$$

$$K_{H_0} = 2.82 \times 10^{-2}\, w^*,\ \text{m}^2/\text{sec}$$

Based on available experimental data the value of the semi-empirical constant σ_1 in Eq. (2) has been assumed to be equal to 0.1 (Sundaram et al., 1970). In addition to the above input conditions, the depth of the lake was assumed to be about 60 m and the minimum temperature during spring homothermy was assumed to be equal to 2.9°C. All the conditions given above

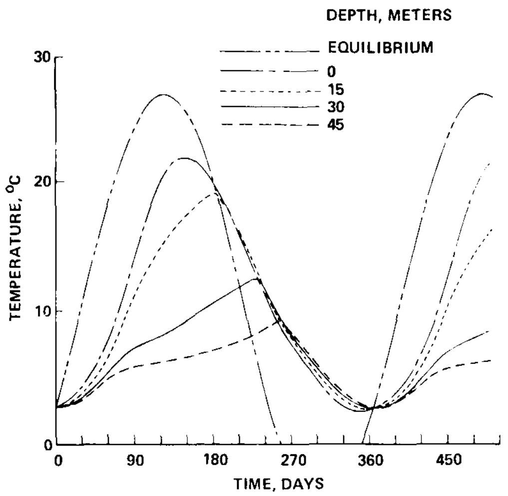

Fig. 2. The stratification cycle.

correspond approximately to conditions in Cayuga Lake, New York and were chosen since extensive data on the thermal structure of this lake are available from the work of Sundaram et al. (1969).

The calculated seasonal variation in the thermal structure is shown in Fig. 2 which shows five temperatures, the equilibrium temperature (which is an input condition for the calculations), the surface temperature and the temperatures at depths of 15, 30 and 45 m, as functions of time for approximately 1 1/4 cycles. A solution for, and a discussion of, the response when the problem is linearized (i.e., when the thermal diffusivity K_H is taken as constant) has been given by Sundaram et al. (1969). According to this solution, for fixed values of the thermal diffusivity and of the heat transfer rate at the surface, the temperature variation at any depth would have a smaller amplitude and a phase shift with respect to the imposed equilibrium temperature variation; the amplitude would decrease and the phase shift increase with increasing depth. Fig. 2 clearly displays the nonlinear behavior resulting from the interaction of the turbulence with the buoyancy field. Stratification in the lake can be seen to occur at around 60 days where the temperature plots at the greatest depths display severe deviation from the sinusoidal behavior predicted from a linearized model. As the thermocline descends, between 180 and 270 days, the temperature plots at depths below the surface again follow a sinusoidal curve after the thermocline has swept past that depth.

Some of the above features can be seen more clearly in Fig. 3 which displays temperature as a function of depth for various times during the annual cycle. Note that in all of the figures, temperature is plotted in degrees centigrade, depth in meters and time in days. The lake was assumed to be homothermal at 2.9°C initially, corresponding approximately to conditions in March. The first plot, at 30 days, displays a nearly homothermal profile at about 4°C resulting from the simultaneous heating at the surface and the complete wind mixing of the lake. The next plot, at 90 days and corresponding approximately to conditions which would be expected in June, clearly displays the thermocline. The third plot (at 180 days) in Fig. 3 displays the thermal structure of the lake as it starts into the cooling portion of the cycle. In this plot the thermocline is somewhat deeper than at 90 days and is much sharper due to the convective mixing which now supplements the wind-induced mixing in the surface (nearly homothermal) layer. The next three plots show the effects of progressive cooling and the enhanced mixing due to cooling upon the temperature structure. At 240 days (corresponding approximately to November) the thermocline has descended beyond 30 m showing a completely mixed upper layer and a characteristic temperature decay in the hypolimnion. At 300 days the mixed layer extends to the bottom of the lake (60 m) and a slight temperature inversion exists. At 360 days the lake is once again homothermal at about 3°C.

The vertical variations of the eddy diffusivity

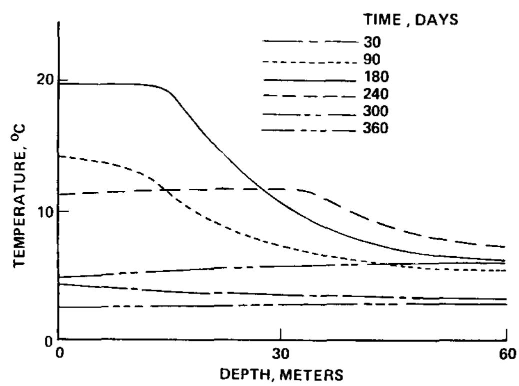

Fig. 3. Vertical temperature distributions.

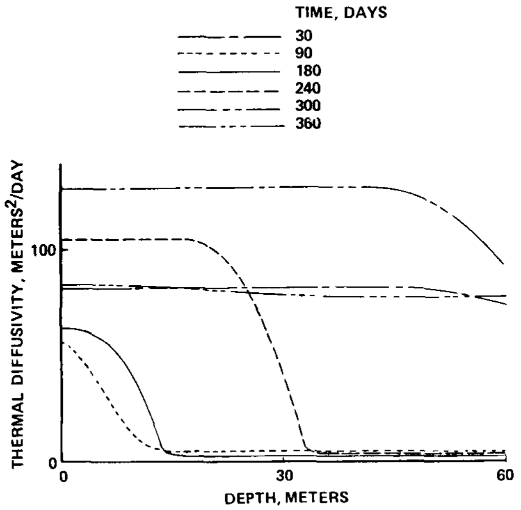

Fig. 4. Vertical distributions of the thermal diffusivities.

for various times during the stratification cycle are shown in Fig. 4. At 30 days, before the formation of the thermocline, the eddy diffusivity remains nearly invariant with depth. However, as the thermocline forms, there is a drastic reduction in the values of the diffusivity in the lower layers as can be seen from the plots for 90, 180 and 240 days. In the plot for 240 days, the initiation of free convection (with the attendant increase in the value of the eddy diffusivity) can be seen in the upper layers. By 300 days, a large part of the lake is dominated by free convection. At 360 days, the lake is again homothermal and the eddy diffusivity is nearly invariant with depth.

Fig. 5 is a plot of the depth of the thermocline versus time. The thermocline is seen to form at about 60 days. The depth of the thermocline is then seen to decrease to a minimum value of about 11 m between 120 and 150 days. As noted above the depth of the thermocline increases rapidly beyond 180 days when the cooling part of the cycle beings. The thermocline reaches the bottom, and the lake attains homothermy, at about 270 days, so that the length of the stratification period is about 210 days.

The above results describe all the well-known characteristic features of the stratification cycle (see, for example, Hutchinson, 1957). The present model also predicts the specific features of stratification discussed earlier. For example, it can be seen from Fig. 2 that the maximum surface temperature of the lake is reached at about 150 days after the start of the calculations, while the maximum heat content of the lake does not occur till after thirty days later. Again, the present model predicts the formation of the thermocline some time *after* the period of maximum spring homothermy (with an intermediate period of smoothly varying temperature distributions), but *before* the lake attains its maximum heat content and begins to lose heat to the atmosphere.

Comparison of the results with observations

As mentioned earlier, the boundary conditions for the calculations presented were roughly those appropriate to Cayuga Lake, New York. Extensive information on the thermal structure of this lake is available from the works of Henson et al. (1961) and of Sundaram et al. (1969). Comparisons of the calculated and observed values for the temperature cycle, the temperature profiles and the thermocline depth are shown in Figs. 6–8. In all the figures, the measured values shown are the averages of the values for the years 1950, 1951, 1952 and 1968.

It can be seen from the figures that the agreement between the measured and computed values is surprisingly good, in spite of the fact that the "external parameters" used in the calculations represent the conditions over Cayuga Lake only in a rough manner.

It should also be pointed out that in a lake in which the area of the cross section changes with

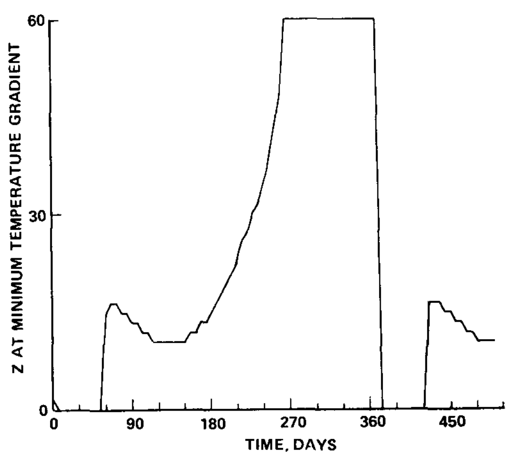

Fig. 5. Seasonal variation of the thermocline depth.

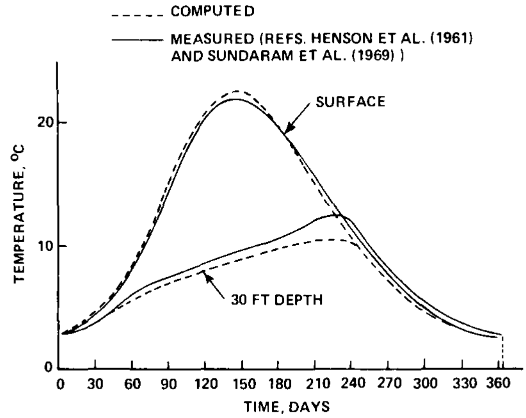

Fig. 6. Comparison of the computed and observed stratification cycles of Cayuga Lake, New York.

depth, the assumption of horizontal homogeneity necessarily implies a distortion of the vertical scale. This fact was pointed out by Birge (see discussions of Hutchinson (1957) and Sundaram et al. (1969)) a number of years ago in connection with the calculation of the heat budget of a stratified lake. Thus a direct comparison (that is, without taking vertical distortion into effect) of the computed and measured values is not compatible with the assumption of horizontal homogeneity.

The points mentioned above have to be resolved by carrying out systematic calculations using the present model before definitive conclusions on the quantitative accuracy of the present scheme can be reached. Since such calculations have not yet been performed, it can only be stated here that the present scheme appears to give quite adequate quantitative accuracy. However it should be emphasized that the present scheme gives excellent *qualitative* agreement with *all* of the observed features of stratification in temperate lakes. Indeed, many of these features have never before been predicted analytically. Moreover, the qualitative conclusions given in the present paper are *not* influenced by the specific form of the eddy diffusivity in Eq. (2). It was shown in paper I that the qualitative feature of the formation and maintenance of the thermocline under steady external conditions were identical for two considerably different forms of the eddy diffusivity. A similar conclusion is valid in the present case as well.

Comparison with other theories

Finally, it is relevant to consider the similarities as well as differences between the present theory and other existing theories. In the present paper, as in paper I, it was emphasized that the nonlinear interaction between wind-induced turbulence and buoyancy gradients due to surface heating should be included in a proper theory for describing the features of stratification of a temperate waterbody. Theories which do not include this interaction do not predict properly even the *qualitative* features of stratification. Fcr example, Dake & Harleman (1969) have recently proposed a theory in which the major phenomenon responsible for the observed feature of stratification of a lake is postulated to be the differential rates of absorption of the incident solar radiation at different levels of the lake. This theory predicts the formation of a thermocline, and a well-mixed upper layer, only after the onset of surface cooling in late summer. On the other hand, the observed evidence indicates that a thermocline forms in most deep temperate lakes sometime in spring before the surface heat flux into the lake reaches its maximum value.

Therefore, in the present section a comparative discussion of the present theory with only those theories which include the interaction between the turbulence and the thermal structure will be given. Specifically, a brief discussion will be given of the similarities and differences

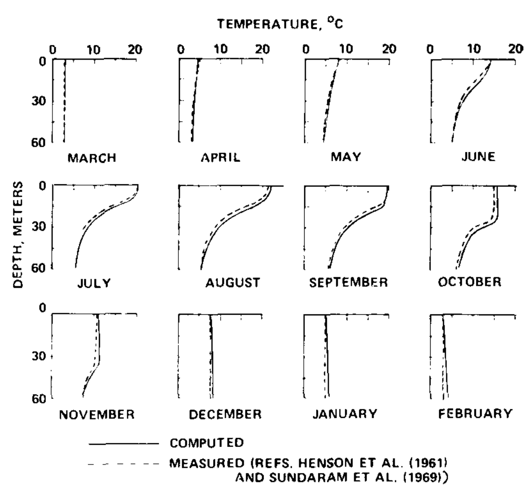

Fig. 7. Comparison of the computed and observed temperature profiles for Cayuga Lake, New York.

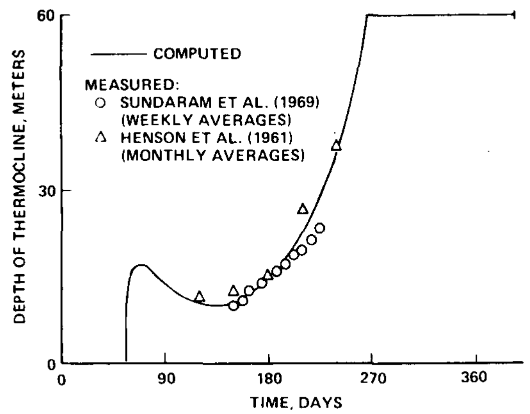

Fig. 8. Comparison of the measured and observed thermocline depths for Cayuga Lake, New York.

between the present theory and the theories of Munk & Anderson (1948), Kraus & Turner (1967) and Moore & Jaluria (1972).

In the theory proposed by Munk & Anderson (1948), the interaction between turbulence and stratification is included explicitly by considering the functional forms of the eddy diffusivities for the vertical transport of heat and momentum to be similar to that given in Eq. (2). These are then used in the classical *steady-state* equations for the Ekman Spiral with the additional assumption that the *heat flux is independent of depth* to solve for the depth of the thermocline. However the steady-state approximation leads to certain inconsistencies and, as the authors point out, "it appears that the distributions of current and temperature cannot both be stationary at the same time" (Munk & Anderson, 1948).

In the present theory, the assumption of the constancy of heat flux with depth (which under conditions of horizontal homogeneity necessarily restricts the analysis to steady-state conditions) has been eliminated. Instead it has been assumed that the generation of wind-induced turbulence in the deeper layers can be expressed in terms of surface conditions (specifically the friction velocity, w^*) alone, and accordingly a modified form of the Richardson number is defined. This assumption effectively uncouples the thermal and current structures even though the essential features of the interaction between wind-induced turbulence and the thermal structure are retained.

The fact that the heat flux

$$q = -\varrho C_p K_H \frac{\partial T}{\partial Z} \tag{8}$$

changes with depth implies, through Eq. (1), that the thermal structure of the subsurface layers changes with time even when surface conditions are steady. In paper I it was pointed out that a time scale characterizing the unsteadiness is given by

$$\tau = w^{*2} / \alpha_v g \left(\frac{q_s}{\varrho C_p} \right) \sigma \tag{9}$$

where q_s is a *steady*, imposed surface heat flux. The quantity τ is also a measure of the time lag between a change in the surface heat flux and the first nonlinear manifestation of this changed surface condition on the conditions at the level of the thermocline.

Thus the major improvement of the present theory over that of Munk & Anderson is that the assumption of constancy of heat flux with depth is relaxed and the essential unsteady nature of the problem (even when surface conditions are steady) is recognized. An alternate assumption which has been made to make the problem tractable is that the effects on the deeper layers of the lake of wind-induced turbulence can be described in terms of surface conditions alone. With this theory a rather simple, self-consistent set of equations has been obtained to model the thermal structure of a lake, and this model, as seen above, agrees quite well with observations.

In addition to the basic unsteadiness inherent in the phenomenon responsible for the maintenance of the thermocline another unsteady effect will exist if the surface conditions are themselves unsteady. This latter effect has been investigated by Kraus & Turner (1967) who accounted for the interaction between turbulence and thermal stratification by using the zeroth and first moments of Eq. (1) in conjunction with the global equation for turbulent-energy conservation. These authors assume that a completely-mixed upper layer is formed by wind action (and, during the later parts of stratification, by convective cooling) and that the temperature structure of the region below this layer remains unchanged. Essentially, the method is equivalent to assuming that the thermal structure can always be represented by a two-parameter family of profiles, with the two characteristic parameters being the temperature and the depth of the well-mixed layer. The two parameters are solved for by using the integral forms of the zeroth and first moments of Eq. (1).

Implicit in the specific form of the temperature profile used by Kraus & Turner is the assumption that the eddy diffusivity is effectively very large within the mixed layer and that it is zero below the mixed layer. The depth of the bottom of the mixed layer, which is the location at which eddy diffusivity changes discontinuously from a very large value to a zero value, is determined using the turbulent energy conservation equation. This procedure is compatible with, and is a simplified form of, the features expressed in Eqs. (2) and (3).

Another assumption which is inherent in the assumption of the existence of a homothermal layer is that the characteristic time scale for the variation of surface conditions is large compared to the characteristic time scale for the adjustment of the conditions in the deeper layers to the changed surface conditions. In other words a nearly homothermal upper layer can form only when surface conditions change so slowly that the conditions at the deeper layers are able to equilibrate to each changed condition. At any given time, a measure of the former of the two time scales mentioned above can be taken to be the quantity $[q_s/(dq_s/dt)]$ while a measure of the latter time scale is the quantity τ defined in Eq. (9). If the ratio of these two time scales is denoted by Ω, then, in general, Ω is quite small during the early parts of the heating cycle (the period following spring homothermy) and increases as heating continues.

For the specific numerical example considered earlier, the value of Ω is about 0.4 at thirty days (from the time of minimum surface temperature) while it is nearly 50 at sixty days. As expected, the numerical computations show that at thirty days the temperature decreases smoothly from top to bottom and that a mixed upper layer forms only after about sixty days.

Thus the assumption of the existence of a well-mixed upper layer right from the beginning of the heating season is unrealistic and is not in accordance with observations. At the early part of the stratification cycle, the time evolution of the thermal structure cannot be represented by a simple, two-parameter family of profiles. Instead, the thermal structure has to be solved for by a detailed consideration of the relevant transport processes, as has been done in the present paper.

Recently Moore & Jaluria (1972) have also proposed a model for the calculation of thermal structure in temperate lakes. They are concerned primarily with the thermal effects of power-plant discharges on the lakes. In their model, as in that of Kraus & Turner, the form of the temperature profiles during stratification is assumed, and the time variations of parameters which specify the profile are determined from overall heat balance. Once the lake has become stratified and during the time when the lake receives heat, the temperature distribution as a function of depth is assumed to vary linearly from its maximum value T_s at the surface to its minimum value, T_h, the hypolimnetic temperature, at the base of the epilimnion which is at depth d. During the late summer when cooling occurs, the model requires the development of an isothermal upper layer of depth a. The equations for thermal energy conservation are then used, together with a stability criterion based upon the Richardson number and an assumed minimum critical heat flux, to determine the parameters T_s, T_h and d as functions of time during stratification.

The calculation of the thermal structure from this model is both simple and very approximate. As in the model of Kraus & Turner the form of the temperature profiles is assumed, and, therefore, no detail of the variation with depth can be expected. The very important nonlinear interaction between the current structure and the stratification, which determines both the time of thermocline formation and the depth of the thermocline during stratification, is included in this theory only through the assumed stability criterion. Unfortunately it is not clear that the application of this stability criterion in the manner used in the model is adequate to represent the nonlinear interaction and to complete the mathematical description of the thermal structure. In paper I the nonlinear, unsteady process of formation and maintenance of a thermocline was treated in some detail; in particular, the evolution of the thermal structure as a function of both depth and time was calculated for imposed steady surface conditions on temperature or heat flux. In the present paper the model of paper I has been extended to calculate the complete thermal cycle of a temperate lake when the conditions imposed at the surface vary cyclically in an appropriate fashion. No *a priori* assumption is made on the depth dependence of the temperature profiles during the cycle.

In summary, while the present theory has included the salient feature of the theories of Munk & Anderson (1948), Kraus & Turner (1967), and Moore & Jaluria (1972), several important new features have been introduced. It is shown that the mechanisms responsible for the maintenance of a thermocline are basically unsteady even when the surface conditions are steady and this unsteady aspect has to be accounted for in a self-consistent theory. It is also shown that due to the fairly rapid changes in surface conditions during the early part of the stratification cycle an integral method, in which the temperature profiles are postulated to be of a given shape, leads to unrealistic results. The temperature structure during this early part of the stratification cycle has to be solved for by a detailed consideration of the basic differential equations.

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СЕЗОННАЯ ТЕРМИЧЕСКАЯ СТРУКТУРА ГЛУБОКИХ ОЗЕР СРЕДНЕГО РАЗМЕРА

Дается теоретическое описание сезонных вариаций термической структуры глубоких озер среднего размера. Влияние нестационарного и нелинейного взаимодействия между возбуждаемой ветром турбулентностью и устойчивыми градиентами плавучести, возникающими благодаря нагреванию поверхности воды, на коэффициенты турбулентного теплообмена предполагается заданным произведением коэффициента теплообмена при нейтральной статификации и соответствующей функции параметра стратификации, который в этом случае берется в виде динамического числа Ричардсона. Делается неявный учет взаимодействия между течением и термической структурой внутри озера, а также свободной конвекции в течение периода остывания для годового цикла. Основное уравнение, которое интегрируется методом конечных разностей — это нестационарное нелинейное уравнение диффузии, описывающее вертикальную структуру температуры. Граничное условие для теплообмена на поверхности озера формулируется с использованием концепции температуры равновесия. Сезонный цикл стратификации озера определяется как реакция озера на определенные заданные «внешние параметры» которые описывают обмен механической и тепловой энергиями между озером и окружающей средой. Результаты расчетов вертикальной термической структуры по этой модели качественно согласуются с особенностями цикла стратификации. Для одного случая, в котором выполнено количественное сравнение, получено хорошее согласие с результатами наблюдений. Дается также краткое обсуждение особенностей данной теории в сравнении с результатами других существующих теорий.