The net magnetic moment of an assemblage of randomly oriented dipoles

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(Manuscript received November 20, 1972)

ABSTRACT

The expected distribution of net magnetic moments, as measured for a randomized assemblage of magnetic dipoles whose magnitudes have a lognormal distribution, is derived. The results are compared with experiments on rock chips.

1. Introduction

The results described below were obtained during research into the magnetism of small rock chips recovered in deep drilling for geothermal power in Iceland (Kristjansson, 1972). They include a description of some of the properties of the lognormal distribution function, rarely given prominence in statistical texts. These mathematical results are directly applicable in various other physical situations, such as: deposition of sediments; magnetic anomalies over detrital beds; domain structure; and turbulent motion. They are independent of the system of units used in measurements, but the experimental results of Figs. 2 and 3 are given in electromagnetic units, as is common in rock magnetic literature.

2. Statistical considerations

Let us take a lump of solid rock such as basalt, having density p and mass m. Its remanent magnetic moment is $M = J\, m/p$, where the remanence, J, is assumed to be uniform in direction throughout the rock. Its susceptibility is not relevant here. We crush this lump into a arge but finite number N of small pieces, say 1 000–10 000 pieces per gramme. They will be called "chips" here, to avoid confusion with the "grains" of magnetic mineral in the rock, whose number is assumed to be much greater than N. The chips should have varying sizes, but no preferred shape orientation with respect to their remanence directions. We further assume that the chips do not interact magnetically, and that the crushing does not alter their magnetic properties.

Out of the N chips, only a number n will contain magnetic minerals, while the others will be non-magnetic portions of phenocrysts, zeolites etc. We put all N chips into a vial and shake it thoroughly to randomize the directions of the individual chip moments. We then measure three orthogonal components of the resultant total moment of the chips, taking care to avoid any viscous buildup of secondary magnetization. The chips are held tightly together during the measurement, but are then taken out of the magnetometer, re-randomized, measured again etc.

What distribution of resultant total moments of this sample is going to be obtained in this series of measurements? This question has been considered in different contexts by e.g. Irving et al. (1961) and by Nagata (1961) who treated respectively the statistics of equal chip moment magnitudes and of a Maxwellian distribution of these; see also Noltimier (1972).

Let us consider a more general and realistic case, where the distribution of chip moment magnitudes is assumed to be a lognormal one. In a lognormal distribution, written here as

$$dn = f(v)\, dv = \frac{n}{v\sigma(2\pi)^{\frac{1}{2}}} \exp\left\{-\left[\frac{\log_e (v/v_m)}{\sqrt{2}\,\sigma}\right]^2\right\} dv \tag{1}$$

the variable f is a Gaussian function of the

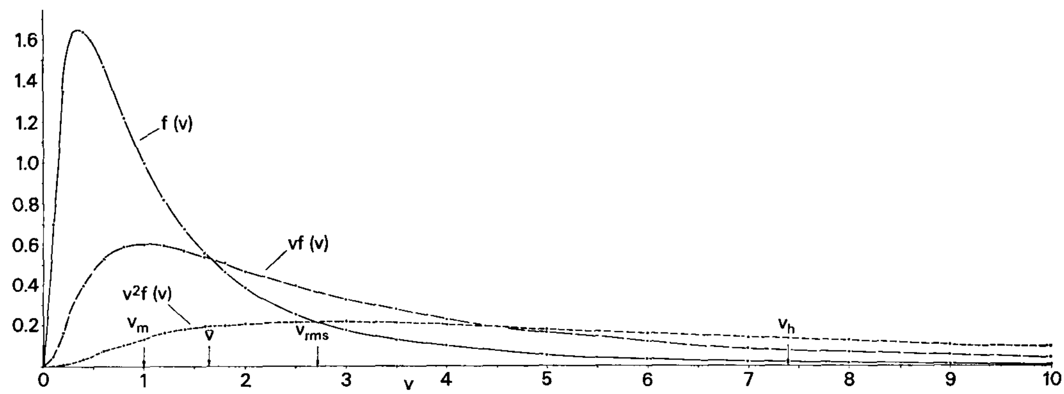

Fig. 1. A plot of the lognormal distribution $f(v)$ as defined in text, with a standard deviation of $\sigma = 1$. The abscissa is normalized to the geometric mean v_m, and the area under all the curves from zero to infinity is the same. See text for an explanation of $vf(v)$, $v^2f(v)$, $\bar{v}$, v_{rms} and v_h.

logarithm of the variable v. The standard deviation of the log plot, σ, is related to the range of orders of magnitude in v spanned by the population of chips in the sample; if 68 % of the number n have values of v within a range of b factors of ten (centered, in the log plot, on the geometric mean v_m), then $\sigma = 1.15b$. Fig. 1 shows a lognormal distribution with $\sigma = 1$; a Maxwellian distribution resembles in shape a lognormal distribution with $\sigma = 0.4$–0.5.

Other useful properties of the lognormal distribution include:

(*a*) The arithmetic mean value of v is

$$\bar{v} = \int_0^\infty vf(v)\,dv/n \qquad (2)$$

where $vf(v)\,dv$ itself describes a lognormal distribution with a geometric mean at $v_{\mathrm{rms}} = v_m e^{\sigma^2} = \bar{v}e^{\sigma^2/2}$ and a logarithmic standard deviation of σ. The arithmetic s.d. is $\bar{v}(e^{\sigma^2} - 1)^{\frac{1}{2}}$.

(*b*) The root-mean-square value of v is

$$v_{\mathrm{rms}} = \left[\int_0^\infty v^2 f(v)\,dv\right]^{\frac{1}{2}} \Big/ n^{\frac{1}{2}} \qquad (3)$$

where $v^2 \cdot f(v)\,dv$ again describes a lognormal distribution, centered on $v_h = v_m\, e^{2\sigma^2}$ and having a logarithmic standard deviation of σ.

(c) The distribution of components of chip magnetic moments along an arbitrary direction, say the coordinate axis x of a magnetometer sample holder, is the even function $q(v_x)$, where

$$q(v_x)dv_x = (ne^{\sigma^2}/2\bar{v})\{1 - F[\log_e (v_x/\bar{v})^{1/\sigma} + 3\sigma/2]\}\, dv_x \qquad (4)$$

with

$$F(a) = (1/2\pi)^{\frac{1}{2}} \cdot \int_{-\infty}^{a} e^{-t^2/2}\, dt \qquad (5)$$

From equation (4) or from general principles it may be shown that the r.m.s. value of v_x is

$$v_{x\mathrm{rms}} = v_{\mathrm{rms}}/\sqrt{3} = v_m\, e^{\sigma^2}/\sqrt{3} \qquad (6)$$

According to the central limit theorem (see Jenkins & Watts, 1968) the measured values of any component, say μ_x, of the net random moment μ of the chips, should in a large number of measurements have a Gaussian distribution about zero with a standard deviation of

$$\mu_{x\mathrm{rms}} = n^{\frac{1}{2}}\, v_{x\mathrm{rms}} = (n/3)^{\frac{1}{2}}\,(\bar{v}e^{\sigma^2/2}) = (3n)^{-\frac{1}{2}}\,(Me^{\sigma^2/2}) \qquad (7)$$

since the pre-crushing moment $M = \bar{v}n$. One sees from this formula that a lump crushed into finer and finer chips should yield finite random moments whose average magnitude is inversely proportional to the square root of the number n of magnetic chips. On the other hand, the average random moment expected from specimens of increasing size taken from a large sample, is directly proportional to the square root of n.

If the remanence moments of individual chips in a specimen of crushed rock are too small to be measured, an estimate of their remanence J can still be obtained by the following simple procedure:

(i) Obtain an estimate of μ_{rms} by measuring

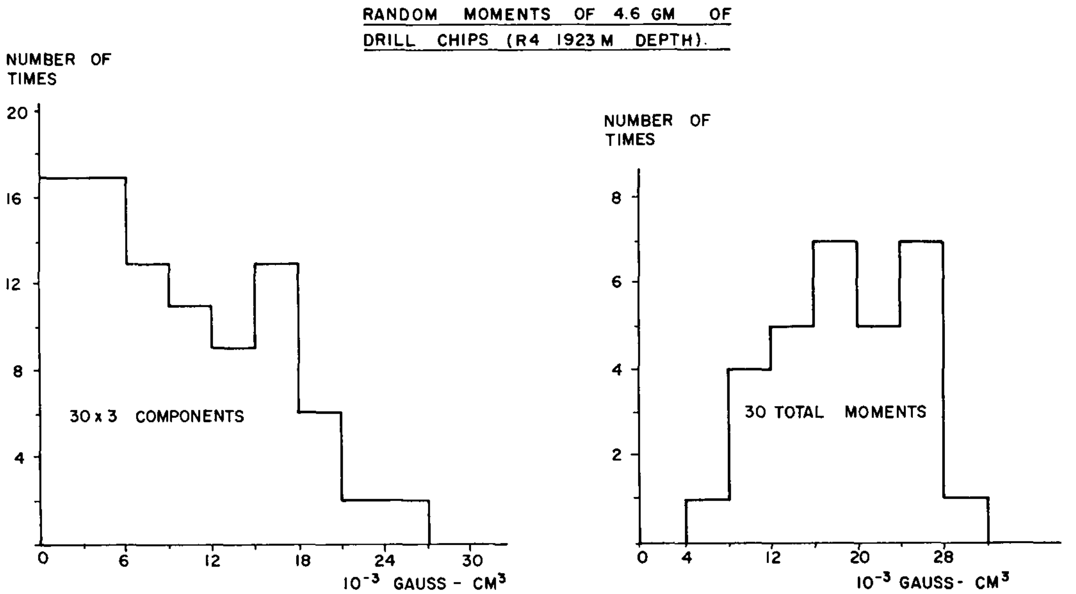

Fig. 2. Frequency histogram of random magnetic moment magnitudes μ (right) and orthogonal component magnitudes for a specimen of Reykjavík drill chips, containing approximately 8 000 magnetic chips. Measured with a spinner in nulled fields, the specimen being shaken or stirred thoroughly between measurements.

the three components of μ several times, randomizing the chips in the specimen between the sets of measurements.

(ii) Thoroughly demagnetize the specimen in alternating fields or by heating in a zero field.

(iii) Re-magnetize the chips by bringing them near a strong magnetic field.

(iv) Measure their new, field-aligned moment M', and then again randomize their directions, as described in (i).

(v) Measure several times the random moment of the specimen, μ', with this new magnetization.

Then it is easily shown that

$$J/J' = M/M' = \mu_{rms}/\mu'_{rms} \qquad (8)$$

provided the distribution of chip moment magnitudes after step (iii) has a similar shape, i.e. the same σ, as $f(v)$.

3. Measurements

It is simple to test the conclusions derived from the central limit theorem, namely that the distribution of component magnitudes of the specimen net random moment should be Gaussian, and that the r.m.s. value of these magnitudes should increase proportionally to the root of n. Fig. 2 shows the observed frequency histograms of net moment and component magnitudes (x, y and z) in a specimen from a sample of drill chips from a deep drill hole in Reykjavik, SW Iceland. These results were obtained by randomizing and remeasuring the specimen 30 times, using a spinner magnetometer. The distribution was found, by a χ^2-test, to be not significantly different from a Gaussian one in the case of the components; the distribution of moment magnitudes is similarly a Maxwellian one to a good approximation.

Fig. 3 is a plot of component variance (assuming a mean of zero) as a function of specimen mass for another sample of Reykjavik drill chips. Each data point represents 36 measurements of component magnitudes, with 80% confidence bars as obtained from Fig. 3.10 of Jenkins & Watts (1968). The results show a linear relation, as expected for specimens taken from a large homogeneous sample.

Calculations to obtain the average original remanence of these chips, and the parameter σ,

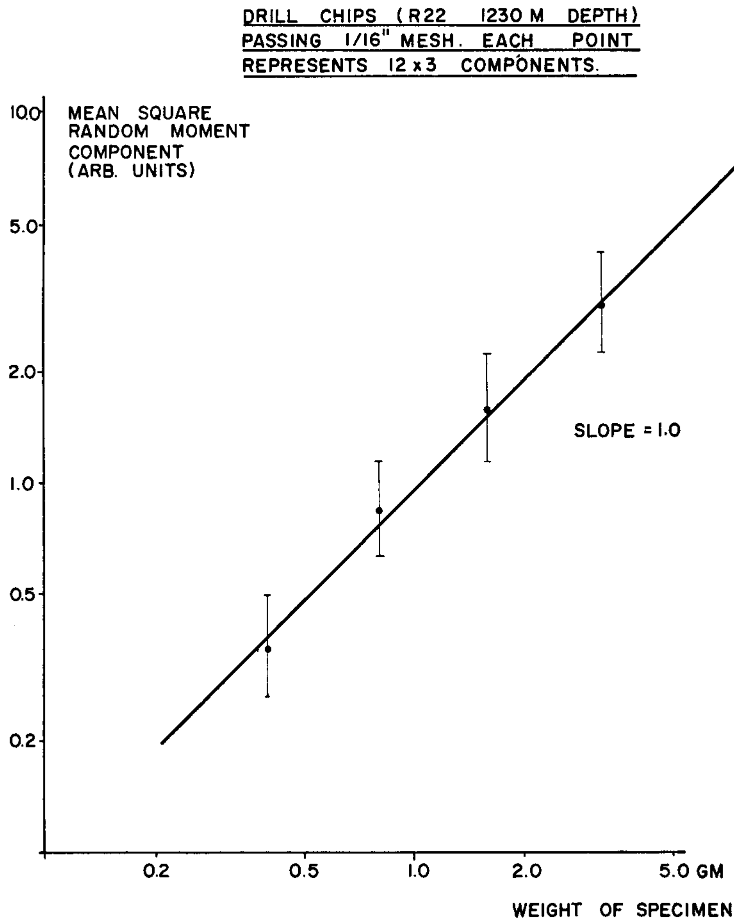

Fig. 3. Mean square component magnitude of random moment for specimens from a sample of drill chips from Reykjavík, plotted against specimen mass. Vertical bars indicate 80 % confidence limits (see text). The most coarse fraction (one third by weight) of the sample was discarded, to reduce the possible dominating effect of a few large chips on the random moment.

from measurements as described above, did not always yield results in agreement with measurements on single large chips from the same samples. This may be caused by the difficulty of estimating n in these samples, and also because some of the chips were plate-shaped with the remanence direction tending to be at right angles with the plate plane. The latter effect, which is probably due to piezomagnetic remanence acquired during the drilling process (Kristjansson, 1972) would violate the assumption of no preferred shape orientation stated in section 2.

Finally, one may ask what is the minimum number n of magnetic chips needed in order that lognormal statistics may be applied to experimental results on real specimens of rock chips, to obtain significant values of J and σ. Since the size of the specimen may often be limited by experimental conditions, it is better to change the question and ask: what is the maximum value of σ for which a lognormal model may be applied, in a specimen with given n, say $n = 10^4$ chips. An estimate of this value of σ may be obtained by requiring that not more than half of the value of the integral $\int_0^\infty v f(v)\, dv$ derives from those 2–5 % of the chips which have the highest values of v. This is found to be equivalent to σ lying between 3 and 4; i.e. the moments v of the central 68 % out of the n chips should not range over more than 2.5–3.5 orders of magnitude. Hence, if the magnetic moment of a rock chip is roughly proportional to its volume, the dimensions of those in the central 68%-range should in this case not span more than one order of magnitude.

4. Acknowledgements

The assistance and encouragement of Dr E. R. Deutsch and Mr A. P. Annan of Memorial University of Newfoundland is gratefully acknowledged. This work was supported by the Science Fund of Iceland.

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ПОЛНЫЙ МАГНИТНЫЙ МОМЕНТ АНСАМБЛЯ СЛУЧАЙНО ОРИЕНТИРОВАННЫХ ДИПОЛЕЙ

Выводится ожидаемое распределение полного магнитного момента, измеряемого для случайного ансамбля магнитных диполей, величины которых имеют логарифмически-нормальное распределение. Результаты сравниваются с данными экспериментов по скальным осколкам.