The structure of non-linear processes

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ABSTRACT

The spectral energetics of a two-layer quasi-geostrophic model of the kind of Lorenz (1960*b*) is presented. The model has the channel geometry and is subject to friction and heating.

It is found that the barotropic transfers between three spectral components of the kinetic energy are governed by a conservation law similar to Fjörtoft's (1953): the kinetic energy flows from the intermediate scale towards the two extreme scales or conversely.

The baroclinic transfers between three spectral components of the available potential energy involve three individual conservative processes, relating two scales of motion only, so that a given scale may feed one of the two others and be fed by the third.

The structure of the baroclinic conversions between available potential energy and kinetic energy is formally similar to the structure of transfers between different spectral components of the available potential energy. For each set of non-linearly associated scales, the baroclinic production of kinetic energy is controlled by a function of the individual conversion rates, the weighting coefficients depending upon the static stability and the wave numbers of the associated scales.

Obviously, all the transfer and conversion rates are function of the geometry of the system.

Introduction

The fundamental studies of Charney (1947), Eady (1949), and Kuo (1949) have identified two essential mechanisms of evolution of the atmosphere. The baroclinic and barotropic instability criteria, however, are derived from linearized equations. Consequently, the following question arises: if two processes are simultaneously possible, each of the instability criteria being applied individually to both processes, what happens to the actual flow. Fluid motions being usually highly non-linear, the answer is far from self-evident. For instance, the possibility exists that the intervention of one of the possible mechanisms of instability at a certain time merely inhibits the other or, conversely, that the occurrence of a single one allows the appearance of the other. Moreover, the situation may change with time.

Obviously, a better understanding of the fluid motion must deal with this problem, which implies the description of the mutual interactions between different scales of motion. Several authors, Brown (1969), Fischer & Renner (1971), Baer (1968, 1971), Simons & Rao (1972) have recently contributed to this subject and reached very iteresting results on the interactions between a zonal flow and superimposed eddies. The lack of analytical instability criteria is advantageously replaced by a discussion of the energy exchange rates where, fortunately, barotropic and baroclinic processes remain discernible. In order to depict the system with satisfactory accuracy, a detailed breakdown of the energy processes is necessary so that the contribution of the different scales of motion can be identified. Such a procedure is followed here for a simple spectral model of the kind described and used by Lorenz (1960*b*, 1963), which is energetically consistent, and where we have been anxious to keep the interactions between several eddies and also the connection between the static stability and the disturbances, whereas this is generally overlooked.

We present here only the formal structure of the various energy processes associated with the interactions between different scales of motion and consider this paper as the design of a potential tool for the understanding of non-linear phenomena. Its usefulness, involving actual computations, is evidenced in another paper published in this issue of *Tellus*. (Quinet, 1973).

1. The model

In order to identify easily the different scales of motion, and to deal with a sufficiently simple and realistic case, we use the spectral form of the equations of a two-layer quasi geostrophic channel.

The behaviour of insulated non-dissipative models has been found to depend strongly upon the total energy of the model (Lorenz, 1960*a*). An easy way to avoid an *a priori* definition of the energy content is to consider an external energy source consisting of a thermal forcing whose intensity may be varied for different experiences. Moreover, it is then possible to introduce dissipative processes. Another advantage of such a procedure is that the scale interactions do not only redistribute the energy between the scales of motion but may also modify the interaction of the system with its environment. The quasi-geostropic model described by Lorenz (1963) is particularly well suited to this purpose and will be used in this paper. For details of formulation, the reader is referred to Lorenz' paper; the governing equations only will be briefly recalled.

The basic functions F_i of the spectral representation are

$$\begin{aligned} \varphi_{o,o} &= 1 \\ \varphi_{m,o} &= \sqrt{2}\cos(my/L) \\ \varphi_{m,n} &= 2\sin(my/L)\cos(nx/L) \\ \varphi'_{m,n} &= 2\sin(my/L)\sin(nx/L) \end{aligned} \tag{1.1}$$

where $o \leqslant x/L \leqslant 2\pi$, $o \leqslant y/L \leqslant \pi$, πL being the width of the channel and the x and y Cartesian coordinate axis being respectively oriented along and across the channel.

If the non-dimensional spectral components of the stream function ψ of the vertically averaged wind, of the stream function τ of the vertical wind shear, of the diegence of the wind in the lower layer $\nabla^2\chi$ and of the vertically averaged potential temperature θ are respectively denoted by ψ_i, τ_i, ω_i $(i = 1, 2, \ldots, N)$ and θ_i $(i = 0, 1, \ldots, N)$ the equations of the model are (Lorenz, 1963)

$$\dot{\psi}_i = \sum_{j<k=1}^{N} f a_i^{-2}(a_j^2 - a_k^2)\, C_{ijk}(\psi_j\psi_k + \tau_j\tau_k) - K(\psi_i - \tau_i) \tag{1.2}$$

$$\dot{\tau}_i = \sum_{j<k=1}^{N} f a_i^2 (a_j^2 - a_k^2)\, C_{ijk}(\tau_j\psi_k + \tau_k\psi_j) - f a_i^{-2}\omega_i + K\psi_i - (K + 2K')\tau_i \tag{1.3}$$

$$\dot{\theta}_i = \sum_{j<k=1}^{N} f C_{ijk}(\theta_j\psi_k - \theta_k\psi_j) + f\sigma_o\omega_i - H(\theta_i - \sigma_i) + H\theta_i^* \tag{1.4}$$

$$\dot{\sigma}_o = -\sum_{i=1}^{N} f\theta_i\omega_i + H\theta_0 - (H + 2H')\sigma_o - H\theta_o^* \tag{1.5}$$

$$\theta_i = \tau_i \quad \text{if} \quad a_i^2 \neq 0 \tag{1.6}$$

In equations (1.2)–(1.6) the dot denotes a time derivative, Σ is a summation symbol, f is the usual Coriolis parameter (a constant in this model), $-a_i^2$ is the eigenvalue corresponding to the eigen function F_i of the two dimensional Cartesian Laplacian operator, θ_i^* are the spectral components of the thermal forcing and $2K$ and K' ($2H$ and H') are the coefficients of friction (heating) at the lower boundary and at the interface of the two layers. If J denotes a Jacobian and the bar a horizontal average, the interaction coefficients C_{ijk} are given by

$$C_{ijk} = L^2 \overline{F_i\, J(F_j, F_k)} \tag{1.7}$$

and satisfy

$$\begin{gathered} C_{ojk} = C_{ijj} = 0 \\ C_{ijk} = C_{kij} = -C_{jik} = -C_{ikj} \end{gathered} \tag{1.8}$$

Notice that, according to (1.5), the measure of the static stability σ has been restricted to a single component σ_0 so that σ depends only on time.

In Lorenz' study (1963) the spectra of the dependent variables contain at most two meridional waves, whose wave number m defines the mode, and one zonal wave with wave number n. We introduce here three zonal waves n, $2n$ and $3n$ for each of the first two modes ($m = 1, 2$). Any scalar quantity G is then represented by at most 15 spectral components following

$$G = \sum_{m=0}^{2} G_{m,o}\varphi_{m,o} + \sum_{m=1}^{2}\sum_{j=1}^{3} (G_{m,jn}\varphi_{m,jn} + G'_{m,jn}\varphi'_{m,jn}) \tag{1.9}$$

were jn takes the values n, $2n$ and $3n$. With the balance equation (1.6) which identifies the components of the vertical wind shear with those of the mean potential temperature field, the system is represented, after eliminating the ω_i, by thirty independent variables, σ_o, θ_0, ψ_i and τ_i with $i = 1, 2, ..., 14$ and by thirty evolution equations.

When the distinction between the functions $\varphi_{m,n}$ or $\varphi'_{m,n}$ in (1.7) is respectively taken into account by the subscript K_m^n or L_m^n and the subscript A_m refers to the functions $\varphi_{m,o}$, the only non-zero interaction coefficients C_{ijk} are

$$\frac{C_{A_1 K_1^r L_1^r}}{5} = \frac{C_{A_1 K_2^r L_2^r}}{4} = \frac{C_{A_2 K_1^r L_2^r}}{8} = \frac{C_{A_1 K_2^r L_1^r}}{8} = -r\frac{8\sqrt{2}}{15\pi} \quad (1.10)$$

where $r = n$, $2n$ or $3n$ and

$$C_{K_1^n K_1^{2n} L_1^{3n}} = C_{K_1^n K_2^{3n} L_1^{2n}} = C_{K_2^{3n} K_1^{2n} L_1^n} = C_{L_1^n L_1^{2n} L_2^{3n}} = \frac{n}{2} \quad (1.11)$$

$$C_{K_1^n K_2^n L_1^{2n}} = C_{K_1^n K_1^{2n} L_2^n} = C_{K_1^{2n} K_2^n L_1^n} = C_{L_1^n L_1^{2n} L_2^n} = \frac{3n}{2} \quad (1.12)$$

$$C_{K_1^n K_2^{2n} L_1^{3n}} = C_{K_1^n K_1^{3n} L_2^{2n}} = C_{K_1^{3n} K_2^{2n} L_1^n} = C_{L_1^n L_1^{3n} L_2^{2n}} = \frac{4n}{2} \quad (1.13)$$

$$C_{K_1^{2n} K_2^n L_1^{3n}} = C_{K_1^{2n} K_1^{3n} L_2^n} = C_{K_1^{3n} K_2^n L_1^{2n}} = C_{L_1^{3n} L_1^{2n} L_2^n} = \frac{5n}{2} \quad (1.14)$$

It would be tedious to give explicitly the equations of the model. However, in order to make things as clear as possible, Fig. 1 (where waves have been represented by two numbers, the first for the mode m, the second for the zonal wave number n) describes all the scale interaction types included in the model. The self-interaction of each wave with a zonal component of first mode ($m = 1$, $n = 0$) is not shown.

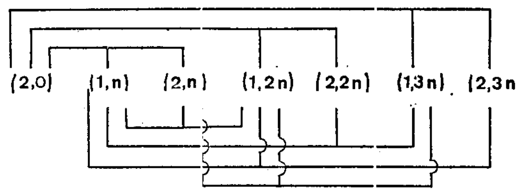

Fig. 1. Schematic representation of the triple interactions in the model. In each parenthesis, the first integer refers to the mode, the second one to the zonal wave number. The self-interaction of each wave with a zonal component of first mode (1, 0) is not shown.

2. The energetics of the model

One of the most powerful tools for the study of hydrodynamic systems is the consideration of the energy budget equations (Van Mieghem, 1973). These will be established for the model under consideration.

The spectral form of the kinetic energy of the horizontal motion (K.E.) can be expressed as

$$K = \frac{1}{2} C \left\{ \sum_{i=1}^{M} a_i^2 (\psi_i^2 + \tau_i^2) + \sum_{i=M+1}^{N} a_i^2 (\psi_i^2 + \tau_i^2) \right\} = K_z + K_E \quad (2.1)$$

where $C = 2\pi^2 L^4 p_0 g^{-1} f^2$ is a scale factor, p_0 being the pressure at the lower level and g the acceleration of gravity. As usual, K_Z is the zonal kinetic energy and K_E the eddy kinetic energy when the basic functions F_i have been ordered in such a way that $\partial F_i / \partial x = 0$ everywhere for $0 \leq i \leq M$. Moreover, each term of (2.1) may be considered as the i spectral component of the K.E.

For the available potential energy (A.P.E.), we have

$$A = \frac{C}{\sigma_o + \sigma_{om}} \left\{ \sum_{i=1}^{M} \theta_i^2 + \sum_{i=M+1}^{N} \theta_i^2 \right\} = A_Z + A_E \quad (2.2)$$

where A_Z and A_E are the zonal and the eddy A.P.E. respectively and where (Lorenz, 1960*b*)

$$\sigma_{om}^2 = \sigma_o^2 + \sum_{i=1}^{N} \theta_i^2 \quad (2.3)$$

From the definition (2.3) of σ_{om} it can be verified that for adiabatic non-dissipative flows, σ_{om} is a constant and, hence, is the maximum value of σ_o when $\theta_i = 0$ for $i = 1, 2, \dots N$. So, σ_{om} may be considered as characterizing the stratification of the reference state in the usual definition of the A.P.E. (Lorenz, 1955). As for the K.E., each term in (2.2) may be considered as the i spectral component of the A.P.E.

With (1.2)–(1.6) the energy budget equations assume the form

$$\frac{\partial A_Z}{\partial t} = -C_Z - C_A - C_{AR} + G_Z + G_{ZR} \quad (2.4)$$

$$\frac{\partial A_E}{\partial t} = -C_E + C_A + C_{AR} + G_E + G_{ER} \quad (2.5)$$

$$\frac{\partial K_Z}{\partial t} = C_Z - C_K - D_Z \quad (2.6)$$

$$\frac{\partial K_E}{\partial t} = C_E + C_K - D_E \quad (2.7)$$

where the different terms may be given the following interpretation:

$$C_Z = -Cf \sum_{i=1}^{M} \theta_i \omega_i \quad (2.8)$$

is the conversion rate of A_Z to K_Z,

$$C_E = -Cf \sum_{i=M+1}^{N} \theta_i \omega_i \quad (2.9)$$

is the conversion rate of A_E to K_E,

$$C_A = -\frac{2Cf}{\sigma_o + \sigma_{om}} \sum_{i+1}^{M} \theta_i \sum_{j<k<1}^{N} C_{ijk}(\theta_j \psi_k - \theta_k \psi_j) \quad (2.10)$$

$$= \frac{2Cf}{\sigma_o + \sigma_{om}} \sum_{i=M+1}^{N} \theta_i \sum_{j<k=1}^{N} C_{ijk}(\theta_j \psi_k - \theta_k \psi_j) \quad (2.10')$$

is the transfer rate of A_Z to A_E resulting from sensible heat advection,

$$C_{AR} = -\frac{Cf}{(\sigma_o + \sigma_{om})^2} \sum_{i=1}^{M} \theta_i \sum_{j=1}^{N} \theta_j(\theta_i \omega_j - \theta_j \omega_i) \quad (2.11)$$

$$= \frac{Cf}{(\sigma_o + \sigma_{om})^2} \sum_{i=M+1}^{N} \theta_i \sum_{j=1}^{N} \theta_j(\theta_i \omega_j - \theta_j \omega_i) \quad (2.11')$$

is the transfer rate of A_Z to A_E due to the evolution of σ_o.

$$G_Z = \frac{2C}{\sigma_o + \sigma_{om}} \sum_{i=1}^{M} H\theta_i(\theta_i^* - \theta_i) \quad (2.12)$$

is the generation rate of A_Z by external heating,

$$G_E = \frac{2C}{\sigma_o + \sigma_{om}} \sum_{i=M+1}^{N} H\theta_i(\theta_i^* - \theta_i) \quad (2.13)$$

is the generation rate of A_E by external heating

$$G_{ZR} = \frac{C}{\sigma_{om}(\sigma_o + \sigma_{om})} \left\{ H(\theta_0^* - \theta_0) + (2H' + H)\sigma_o + \frac{H}{\sigma_{om} + \sigma_o} \sum_{j=1}^{N} \theta_j(\theta_j - \theta^*_j) \right\} \sum_{i=1}^{M} \theta_i^2 \quad (2.14)$$

is the generation rate of A_Z due to the influence of non-adiabatic effects on σ_o and the reference state,

$$G_{ER} = \frac{C}{\sigma_{om}(\sigma_o + \sigma_{om})} \left\{ H(\theta_0^* - \theta_0) + (2H' + H)\sigma_o + \frac{H}{\sigma_{om} + \sigma_o} \sum_{j=1}^{N} \theta_j(\theta_j - \theta_j^*) \right\} \sum_{i=M+1}^{N} \theta_i^2 \quad (2.15)$$

is the generation rate of A_E due to the influence of non-adiabatic effects on σ_o and the reference state,

$$C_K = -Cf \sum_{i=1}^{M} \sum_{j<k=1}^{N} (a_j^2 a_k^2) C_{ijk}\{\psi_i(\psi_j \psi_k + \tau_j \tau_k) + \tau_i(\tau_j \psi_k + \tau_k \psi_j)\} \quad (2.16)$$

$$= Cf \sum_{i=M+1}^{N} \sum_{j<k=1}^{N} (a_j^2 - a_k^2) C_{ijk}\{\psi_i(\psi_j \psi_k + \tau_j \tau_k) + \tau_i(\tau_j \psi_k + \tau_k \psi_j)\} \quad (2.16')$$

is the transfer rate of K_Z to K_E,

$$D_Z = CK \sum_{i=1}^{M} a_i^2(\psi_i - \tau_i)^2 + 2CK' \sum_{i=1}^{M} a_i^2 - \tau_i^2 \quad (2.17)$$

is the dissipation rate of K_Z by friction,

$$D_E = CK \sum_{i=M+1}^{N} a_i^2(\psi_i - \tau_i)^2 + 2CK' \sum_{i=M+1}^{N} a_i^2 \tau_i^2 \tag{2.18}$$

is the dissipation rate of K_E by friction.

Let us mention that the condition of conservation of the total energy of the insulated non-dissipative system allows the formulation of the rules that the truncation of the serial representation of the various fields must satisfy. Here, energetic consistency is achieved by simply using the same type of development for ψ, τ, θ and $\nabla^2\chi$. This may not be the case for other models, especially when the Coriolis parameter is no longer constant.

Let us now discuss the influence of scale interactions on the energy budgets. These interactions generate two kinds of terms; those including the zonal flow $(1 \leqslant i \leqslant M)$ and those involving pure scale interactions between three different eddy scales $(M+1 \leqslant i \leqslant N)$.

From (2.10′) and (2.16′) it can be checked that pure scale interactions give a zero net contribution to C_A and C_K. In addition, interactions of each scale with the zonal fields modify both the eddy and zonal portions of A.P.E. and K.E. by altering the transfer rates C_A and C_K. However, the total A.P.E. and the total K.E. are individually conserved.

Moreover, each scale contributes to C_Z and C_E representing the (baroclinic) conversions between potential and kinetic energy. This process is such that the sum of A.P.E. and K.E. is conserved. When considering the role of eventual supplementary scales of motion, it must be stressed that they do not only add extra terms to (2.8) and (2.9) but imply a modification of the previously existing terms also. This is obvious because the dependent variables ω_i should be expressed in terms of all the independent variables ψ_i, τ_i and σ_0.

3. Spectral energetics

We discuss now some specific terms of the various energy conversion and transfer rates defined in the previous paragraph. To specify the particular interactions which may be considered, reference is made to Fig. 1.

(a) *The transfers between spectral components of K.E. (2.16) or (2.16′)*

As the differences $a_j^2 - a_k^2$ to be associated with the non-vanishing C_{ijk} (1.10) in (2.16) and (2.16′) are all zero in the case of transfers between the zonal flow of first mode ($m=1$, $n=0$) and any single wave n, $2n$ or $3n$ of first or second mode ($m=1$ or 2), this process does not take place in the present model. This, as mentioned by Lorenz, already appears in (1.2).

For transfers between three different scales (i, j, k) of motion, two possibilities do exist. In a first case (upper connections in Fig. 1), one of the three scales (i, j, k) is the second mode of the zonal flow ($m=2$, $n=0$) and the associated energy transfer consists in a transfer between K_Z and K_E. A second type of transfer process (lower connections in Fig. 1) involves three scales of the eddy motion and consists of a transfer between components of K_E.

Using (2.16′) for C_K, this latter type of process is represented by

$$\left(\frac{\partial K_E}{\partial t}\right)_i^* = Cf(a_j^2 - a_k^2)\,T \tag{3.1}$$

$$\left(\frac{\partial K_E}{\partial t}\right)_j^* = Cf(a_k^2 - a_i^2)\,T \tag{3.2}$$

$$\left(\frac{\partial K_E}{\partial t}\right)_k^* = Cf(a_i^2 - a_j^2)\,T \tag{3.3}$$

where

$$T = {\sum_{j<k}}' C_{ijk}(\psi_i\psi_j\psi_k + \psi_i\tau_j\tau_k + \psi_j\tau_k\tau_i + \psi_k\tau_i\tau_j), \tag{3.4}$$

$(\ )_i$ denotes the contribution of scale i to the K_E budget and * recalls that interactions between scale i and only scales j and k are considered, the symbol Σ' meaning a sum over all terms characterizing these interactions. Obviously, the process is conserving K_E. When one of the three scales is the zonal flow of second mode ($m=2$, $n=0$), (2.16′) gives rise to two budget relations only, the third one appearing in (2.16). The global process in this case is conserving the total K.E.

Ordering the a_i's in such a way that

$$a_i^2 < a_j^2 < a_k^2 \tag{3.5}$$

(3.1)–(3.3) show that the barotropic K.E. transfers are such that the energy flows from the intermediate scale towards the extreme scales or reversely as in the case of a purely barotropic flow (Fjörtoft, 1953). Hereafter, scales i, j and k will always be considered as ordered following (3.5).

(b) *The transfers between spectral components of A.P.E. (2.10) or (2.10′)*

Direct transfer of zonal A.P.E. of first mode ($m=1$, $n=0$) into eddy A.P.E. is now possible. For instance, the transfer rate of A.P.E. between scales (1,0) and (1, r) deduced from (2.10) is

$$\left(\frac{\partial A_Z}{\partial t}\right)^*_{1,0} = \frac{2Cf}{\sigma_0+\sigma_{om}} C_{A_1 K_1^r L_1^r} \theta_{1,0}(\theta_{1,r}\psi'_{1,r} - \theta'_{1,r}\psi^*_{1,r})$$

By virtue of (1.10), $C_{A_1 K_1^r L_1^r} < 0$, so that *if* $\theta_{1,0} > 0$, zonal A.P.E. of the first mode is transferred to eddy A.P.E. of scale $(1, r)$ if the temperature field $(1, r)$ lags behind (upstream from) the stream field $(1, r)$.

For the general case of A.P.E. transfers between the three scales (i, j, k), we get from (2.10′)

$$\left(\frac{\partial A_E}{\partial t}\right)^*_i = \frac{2Cf}{\sigma_0+\sigma_{om}} (\;T_1 \qquad - T_3) \quad (3.6)$$

$$\left(\frac{\partial A_E}{\partial t}\right)^*_j = \frac{2Cf}{\sigma_0+\sigma_{om}} (-T_1 + T_2 \qquad) \quad (3.7)$$

$$\left(\frac{\partial A_E}{\partial t}\right)^*_k = \frac{2Cf}{\sigma_0+\sigma_{om}} (\qquad -T_2 + T_3) \quad (3.8)$$

where

$$T_1 = \sum_{j<k}{}' C_{ijk}\theta_i\theta_j\psi_k \quad (3.9)$$

$$T_2 = \sum_{j<k}{}' C_{ijk}\theta_j\theta_k\psi_i \quad (3.10)$$

$$T_3 = \sum_{j<k}{}' C_{ijk}\theta_k\theta_i\psi_j \quad (3.11)$$

where Σ' and $(\;)^*_i$ have the same meaning as previously. Obviously the transfer process conserves the sum of A.P.E. contained in scales i, j, and k. The transfer rates are now inter-connected by pairs so that there is not a definite direction for energy transfers from (towards) the intermediate scale towards (from) the two others as for the barotropic transfers of K.E. Nevertheless equations (3.6)–(3.8) make possible the identification of individual transfer rates between two scales of motion only. Moreover, the explicit form of (3.9)–(3.11) allows also an interpretation of the non-linear evolution mechanisms in term of the flow configuration. For instance, in the case of scales

$$i \equiv (2,0),\; j \equiv (1,3) \text{ and } k \equiv (2,3),$$

we have, with

$$\alpha \equiv C_{A_2 K_1^3 L_2^3} < 0,$$

$$T_1 = \alpha\theta_{2,0}(\theta_{1,3}\psi'_{2,3} - \theta'_{1,3}\psi_{2,3}) \quad (3.12)$$

$$T_2 = \alpha\psi_{2,0}(\theta_{1,3}\theta'_{2,3} - \theta'_{1,3}\theta_{2,3}) \quad (3.13)$$

$$T_3 = \alpha\theta_{2,0}(\theta_{2,3}\psi'_{1,3} - \theta'_{2,3}\psi_{1,3}) \quad (3.14)$$

Accordingly, assuming $\theta_{2,0} > 0$, A.P.E. of scale (1,3) is transferred towards A.P.E. of scale (2,0) ($T_1 > 0$, see eq. (3.6)) if the wave (2,3) of the stream field lags behind the wave (1,3) of the temperature field. The T_2 and T_3 terms allow analogous interpretations of the transfers between scales (2,0) and (2,3) and between scales (1,3) and (2,3).

(c) *Conversions between A.P.E. and K.E. (2.8), (2.9)*

In (2.8) and (2.9), the partition among the different scales is very clear. However, the influence of scale interactions is hidden. Therefore, and though we are aware that such a procedure is always more or less artifical, we modify (2.8) and (2.9) in such a way that scale interactions appear.

Using (1.4) to express ω_i, substituting then in this expression τ_i for $\dot\theta_i$ in accordance with (1.6) and finally taking into account (1.3) for $\dot\tau_i$ we get

$$Cf\theta_i\omega_i = \frac{a_i^2}{1+\sigma_0 a_i^2} Cf\theta_i \sum_{j<k=1}^{N} [C_{ijk}a_i^{-2}(a_j^2 - a_k^2) \times(\tau_j\psi_k + \tau_k\psi_j) - C_{ijk}(\theta_j\psi_k - \theta_k\psi_j)] + \frac{a_i^2}{1+\sigma_0 a_i^2}\frac{C}{\sigma_0}\theta_i[K\psi_i - (K+2K')\tau_i + H(\theta_i - \theta_i^*)] \quad (3.15)$$

The contributions of *all* scales of the horizontal motion to the i-component of the vertical motion ω_i appear in the first bracket at the r.h.s. of (3.15). The second square bracket represents non-adiabatic effects and will be left out of consideration here. The second term of the first r.h.s. square bracket gives a contribution which is proportional to the transfer (2.10) between different components of the A.P.E. Consequently, conversions between A.P.E. and K.E. imply also transfers between different components of A.P.E. and reciprocally.

From (3.15), the adiabatic contribution $(\theta_l \omega_l)^*$ to the conversions $\theta_l \omega_l$ associated with the i, j, and k scale interactions may be written

$$-(\theta_i \omega_i)^* = (1+\sigma_o a_i^2)^{-1}\{\ \beta_1 T_1 \qquad -\beta_2 T_3\} \tag{3.16}$$

$$-(\theta_j \omega_j)^* = (1+\sigma_o a_j^2)^{-1}\{-\beta_3 T_1 + \beta_3 T_2 \qquad \} \tag{3.17}$$

$$-(\theta_k \omega_k)^* = (1+\sigma_o a_k^2)^{-1}\{ \qquad -\beta_1 T_2 + \beta_3 T_3\} \tag{3.18}$$

where, according to (3.5),

$$\beta_1 = a_i^2 + a_k^2 - a_j^2 > 0, \quad \beta_2 = a_i^2 + a_j^2 - a_k^2 \gtrless 0,$$
$$\beta_3 = a_j^2 + a_k^2 - a_i^2 > 0$$

and where T_1, T_2 and T_3 have been defined in section b.

According to (3.16)–(3.18) a conservation law expressing a constraint on the possible conversions of the A.P.E. involved in the three non-linearly associated scales i, j and k into the K.E. of these scales may be written

$$\begin{aligned}(1+\sigma_o a_j^2)^{-1}&(1+\sigma_o a_k^2)^{-1}\beta_3(\theta_i \omega_i)^* \\ &+ (1+\sigma_o a_k^2)^{-1}(1+\sigma_o a_i^2)^{-1}\beta_1(\theta_j \omega_j)^* \\ &+ (1+\sigma_o a_i^2)^{-1}(1+\sigma_o a_j^2)^{-1}\beta_2(\theta_k \omega_k)^* = 0\end{aligned} \tag{3.19}$$

Considering the particular case of the interaction between scales (1,0) and (1,r), it is easy to see that when $\theta_{1,0} > 0$, zonal A.P.E. is converted into zonal K.E. if the wave (1,r) of the ψ field lags behind the wave (1,r) of the θ field. The reverse holds for conversion of eddy A.P.E. into eddy K.E., a classical result of the theory of baroclinic disturbances. Recalling the results obtained in section (b), it appears that baroclinic processes involving fields (1,0) and (1,r) lead to one of the following energy chains

$$K_z \leftarrow A_z \leftarrow A_E \leftarrow K_E \tag{3.20}$$

or

$$K_z \rightarrow A_z \rightarrow A_E \rightarrow K_E \tag{3.21}$$

Note that neither of these chains corresponds to the atmospheric case where it is known that A_Z feeds both K_Z and A_E. It should be recalled that the atmospheric energy cycle depends also on the terms in the second square bracket on the r.h.s. of (3.15) and namely on the thermal forcing terms, proportional to the generation of A.P.E., allowing conversion of zonal A.P.E. into zonal K.E. In this respect, it can be concluded that balanced baroclinic models which are free of thermal forcing give rise to unrealistic atmospheric energy cycles at least as far as interactions with the zonal flow of the first mode are concerned.

In the case of scales $i \equiv (2,0)$, $j \equiv (1,3)$ and $k \equiv (2,3)$, the energy chains are represented on Fig. 2. The direction of the arrows corresponds to positive values for T, T_1, T_2 and T_3. According to Fig. 2, the baroclinic processes between two arbitrary eddy scales l and m are linked following

$$K_l \rightarrow A_l \rightarrow A_m \rightarrow K_m \tag{3.22}$$

or conversely. Likewise, Fig. 3 represents the energetic structure of the interactions between scales $i \equiv (1,3)$, $j \equiv (2,3)$ and $k \equiv (1,6)$. As a consequence of the scale depending coefficients in (3.16)–(3.18), Fig. 3 differs from Fig. 2 and the energy chains between scales (i) and (k) or (j) and (k) are now of the kind

$$K_l \leftarrow A_l \rightarrow A_k \rightarrow K_k \tag{3.23}$$

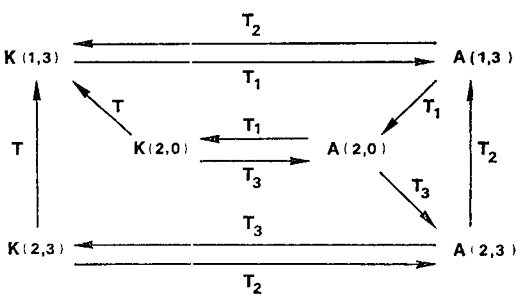

Fig. 2. Schematic representation of the simultaneous exchanges between the (2, 0), (1, 3) and (2, 3) spectral components of the kinetic energy K and the available potential energy A associated with the interactions between the (2, 0), (1, 3) and (2, 3) scales. T refers to the barotropic kinetic energy transfers, T_1, T_2 and T_3 to the baroclinic energy transfers and conversions. (See equations (3.1)–(3.3), (3.6)–(3.8) and (3.16)–(3.18).) The direction of the arrows corresponds to positive values of T, T_1, T_2 and T_3.

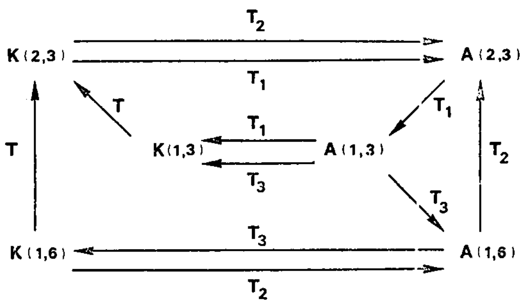

Fig. 3. Same as Fig. 2 but for scales (1, 3), (2, 3) and (1, 6).

or conversely where l stands for i or j while an energy chain of type (3.22) is maintained between scales (i) and (j). Figs. 2 and 3 are the two sole possibilities of a distinctive behaviour depending on the size of the interacting scales: Fig 2 corresponds to $a_i^2+a_j^2>a_k^2$ while Fig. 3 corresponds to $a_i^2+a_j^2<a_k^2$.

Summary

The spectral form of the energy budgets involves two classes of interaction terms: 1) interactions between a zonal scale and one or two eddies; 2) interactions between three eddies.

Each type of energy transformation is governed by a conservation law for each set of interacting scales.

Barotropic transfers of K.E. ((3.1) (3.2) and (3.3)) between three interacting scales are expressed by a single common triple product T of the ψ- and τ-field components and by specific coefficients depending upon the wave numbers of the interacting scales. The exchanges are from the intermediate scale towards the extreme scales or reversely. This generalizes to the model considered, Fjörtoft's result for barotropic flow.

In transfer processes between three different spectral components of A.P.E. ((3.6), (3.7) and (3.8)), the transfer rate of each scale is made of two parts and in such a way that the interacting scales are connected by pairs. Consequently and contrary to barotropic processes, one scale may simultaneously feed another scale and be fed by the third one. The conversion rates do not explicitly depend upon the wave numbers but upon the static stability.

The baroclinic conversions between the spectral components of A.P.E. and K.E. ((3.16), (3.17) and (3.18)) have a structure similar to the A.P.E. transfers except that wave numbers play now an explicit role. Obviously, these conversions conserve the sum of A.P.E. *and* K.E. But there also exists a conservation law concerning only the A.P.E. *or* the K.E. involved in three non-linearly associated scales. This constraint upon the baroclinic production of K.E. appears as a weighted function (3.19) of the individual conversion rates, the weighting coefficients depending both upon the wave numbers and the static stability.

Finally, it must be stressed that an increase of the number of spectral components in the representation of the fields of motion not only introduces supplementary individual conversion and transfer rates but also implies a modification of the previously existing ones. The crucial point of non-linear mechanisms is that there is no guarantee that these modifications are weak when the supplementary scales are weak. This is more particularly evidenced in an accompanying paper published in this issue of *Tellus*.

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СТРУКТУРА НЕЛИНЕЙНЫХ ПРОЦЕССОВ

Представлена спектральная энергетика двухслойной квазигеострофической модели типа модели Лоренца (1960в). Модель имеет геометрию канала и подвержена трению и нагреванию. Найдено, что баротропный обмен между тремя спектральными компонентами кинетической энергии управляется законом сохранения, аналогичным закону Фьортофта (1953): кинетическая энергия течет от промежуточного масштаба к двум предельным масштабом, или наоборот. Бароклинный обмен между тремя спектральными компонентами доступной потенциальной энергии сопровождается тремя индивидуальными процессами сохранения, связывающими только два масштаба, так что данный масштаб может отдавать энергию одному из двух других и получать ее от третьего. Структура бароклинных превращений между доступной потенциальной энергией и кинетической энергией формально подобна структуре обмена между различными спектральными компонентами доступной потенциальной энергии. Для каждого набора нелинейно связанных масштабов бароклинное производство кинетической энергии контролируется некоторой функцией индивидуальных скоростей превращений, причем весовые коэффициенты зависят от статической устойчивости и волновых чисел соответствующих масштабов. Все функции переноса и скорости превращений, очевидно, зависят от геометрии системы.