On the dispersion of a developing droplet spectrum in a turbulent cloud

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ABSTRACT

It is shown that the combined effects of condensation and shear-induced coalescence are able to produce significant dispersion in an initially narrow droplet spectral density centred on a radius of about 10 μm. Provided that the degree of supersaturation of the ambient air is sufficiently large, the time scale for this dispersion process is small compared with the expected life-time of a cloud.

1. Introduction

When a parcel of air becomes supersaturated, water commences to condense on aerosol particles forming the nuclei of water droplets which rapidly grow to a radius of about 10 μm. Accounting for the dependence of the local supersaturation upon the updraft and the volume of condensed water, Warner (1969) follows the development due to condensation of a distribution of droplets within a turbulent cloud. It is found that the variable rate of condensation caused by the turbulence does not produce significant broadening of the droplet spectrum; this result is confirmed by Bartlett & Jonas (1972). A further calculation by Warner (1973) shows that mixing between a cloudy parcel of air and its clear air environment also does not produce a realistic broadening of a spectrum controlled by condensation alone.

The growth of a droplet spectrum is determined essentially by two factors: the rate of condensation or evaporation at a droplet surface and the rate of coalescence between droplets. Gravitational coalescence—that is coalescence due to larger droplets having larger terminal velocities and so overtaking smaller droplets beneath them—is not effective in broadening a narrow spectrum because the relative terminal velocity of colliding droplets is negligible for droplets of comparable size. Moreover, for droplets less than about 20 μm in radius, the collision efficiency (or effective cross-section of a collision) is so small that very little dispersion is produced in a spectrum (Bartlett, 1970). On the other hand experiments by Jonas & Goldsmith (1972) show that a uniform shear flow gives rise to a significant rate of coalescence between small droplets of comparable size. These results can be explained if it is assumed that associated with the wake of a droplet is an interaction region such that other droplets swept into this region by the shear flow must collide with the former droplet (Manton, 1974*a*). Because the vertical extent of this interaction region is considerably larger than the horizontal dimension, it can be shown that only the vertical gradients of the horizontal velocity components of the ambient flow contribute significantly to the rate of coalescence of small droplets. Thus an estimate of the rate of coalescence between small droplets of comparable size in a turbulent field is found to be that produced by a uniform vertical shear of magnitude equal to the root mean square strain rate of the turbulence (Manton, 1974*b*).

It would seem therefore that shear-induced coalescence could perhaps broaden an initially narrow droplet spectrum. However, coalescence alone produces a rather low rate of dispersion because a collision between two droplets of radius r yields a single droplet of double the volume of each initial droplet, that is of radius $2^{1/3}r$. Thus if the initial spectrum is narrow and centred on a radius of 10 μm, say, then droplets of radius 20 μm are formed by shear-induced coalescence only after third order collisions have occured. Hence if T is the time scale for coalescence between droplets of comparable size, then a time of order $3T$ is required for the peak of a narrow spectrum of droplets to move from a radius of 10 to 20 μm.

In this work it is demonstrated that the combined effects of condensation and coalescence are able to produce dispersion in an initially narrow spectrum in a time small compared with the life-time of a cloud, provided that the supersaturation of the air is sufficiently large. For droplets less than about 10 μm in radius condensation dominates the growth process, but coalescence becomes of increasing importance with increasing radius so that condensation is insignificant for radii greater than about 20 μm. As condensation causes the radius r at the peak of a narrow droplet spectrum to increase with time from 10 μm, shear-induced coalescence gives rise to a secondary peak in the spectrum extending from $2^{1/3}r$ back towards the primary peak at r. Thus after a time typically of order 400 sec, the primary peak occurs at a radius of about 17 μm while the secondary peak extends from about 19 μm to 22 μm. Such dispersion ought to be sufficient for gravitational coalescence alone to produce subsequent growth of the spectrum.

2. Equation for the conservation of droplets

We consider the spectral density $n(r, t)$ of a homogeneous distribution of droplets in a cloud, such that n is the number of droplets per unit radius in a unit volume of air at time t and with radius r. The equation for the conservation of droplets may be written as

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial r}\left(\frac{dr}{dt} n\right) = p \tag{2.1}$$

where dr/dt is the rate of increase in radius of a droplet due to condensation and p is the rate of increase of droplets with radius r due to coalescence and to the activation of droplet nuclei. A solution of (2.1) is sought in the domain $t > 0$, $r > 0$ subject to an initial condition of the form

$$n = n_0(r) \quad \text{at } t = 0 \tag{2.2}$$

The radius of a droplet is related to the supersaturation S, in percent, by (Warner, 1969)

$$(r+\alpha)\frac{dr}{dt} = G(S - a/r + b/r^2)$$

The radius α accounts for the possibility that not all water molecules hitting the droplet surface actually condense; its value is uncertain but may be as large as 5 μm. The diffusivity of water vapour in air G is of order 10^{-8} cm^2 sec^{-1} at a temperature of 285 K. The quantity a is equal to 0.115 μm and it accounts for the effect of surface tension, while b accounts for the presence of the solute in the droplet and is less than 1 μm^3 for nuclei of mass less than 10^{-13} g. Thus, provided that the radius r is greater than a few microns, the effect of condensation on droplet growth is described well by

$$dr/dt = GS/r \tag{2.3}$$

Now the effect of condensation is clearly to reduce the local supersaturation of the air and so S depends upon the spectral density n. However, the calculations of Warner (1969) and Bartlett & Jonas (1972) show that S is essentially correlated to the large scale behaviour of the updraft and it is typically a few tenths of a percent in magnitude. We therefore assume that S is a fixed constant in (2.3); but the appropriate value of S increases with the mean updraft in the cloud.

Provided that the initial condition (2.2) corresponds to a spectrum for which the available nuclei have been activated, we may assume that the only significant contribution to the source term p in (2.1) is from droplet coalescence, and so we write

$$p(r,t) = \frac{1}{2}\int_0^r r^2(r^3 - r_1^3)^{-2/3} K((r^3 - r_1^3)^{1/3}, r_1) \times n((r^3 - r_1^3)^{1/3}, t)\, n(r_1, t)\, dr_1 - n(r,t)\int_0^\infty K(r, r_1)\, n(r_1, t)\, dr_1 \tag{2.4}$$

where $K(r_1, r_2)$ is the collection kernel describing collisions between droplets of radii r_1 and r_2. The functional behaviour of K for droplets in a uniform vertical shear of strain rate C can be determined from dimensional analysis

(Manton, 1974*c*). By comparing this behaviour with the results of Davis (1972) and Jonas & Goldsmith (1972), it is found that for small droplets—that is droplets with radii less than about 20 μm—the collection kernel may be approximated by

$$K(r_1, r_2) = \begin{cases} 180(g/\nu)\, r_1^2 r_2^2 |\ln(r_1/r_2)| & \text{for } C < C^*, \\ 5 \times 10^6 (g/\nu^2)\, r_1^3 r_2^3 C \\ \quad - 195(g/\nu)\, r_1^2 r_2^2 |\ln(r_1/r_2)| & \text{for } C > C^*, \end{cases} \quad (2.5)$$

where $C^* = 7.5 \times 10^{-5}\ (\nu/r_1 r_2)\,|\ln(r_1/r_2)|$ is the threshold strain rate below which a shear flow has no effect on K; g is the gravitational acceleration; ν is the kinematic viscosity of air. Eq. (2.5) is essentially valid for droplets in a homogeneous field of turbulence, provided that C is set equal to the root mean square strain rate of the turbulence (Manton, 1974*b*).

For a specified supersaturation S and strain rate C, eqs. (2.1)–(2.5) form a closed system for the spectral density $n(r, t)$. As the solution of this non-linear system is not obvious, we now normalise the equations and consider an appropriate asymptotic representation of the solution. If the initial spectral density n_0 is narrow and centred on the radius r_0, then the relevant normalised variables are

$$\begin{aligned} \eta &= n/n_0(r_0) \\ \varrho &= r/r_0 \\ \tau &= 2GSt/r_0 \end{aligned} \quad (2.6)$$

By putting (2.6) into (2.1) and (2.3)–(2.5), it is found that the normalised spectral density η is governed by the equation

$$2\frac{\partial \eta}{\partial \tau} + \frac{\partial}{\partial \varrho}\left(\frac{\eta}{\varrho}\right) = \varepsilon \left\{ \frac{1}{2}\int_0^{\varrho} \varrho^2 (\varrho^3 - \varrho_1^3)^{-2/3} k[(\varrho^3 - \varrho_1^3)^{1/3}, \varrho_1] \times \eta(\varrho_1, \tau)\, \eta[(\varrho^3 - \varrho_1^3)^{1/3}, \tau]\, d\varrho_1 - \eta(\varrho, \tau) \int_0^{\infty} k(\varrho, \varrho_1)\, \eta(\varrho_1, \tau)\, d\varrho_1 \right\} \quad (2.7)$$

where

$$k(\varrho_1, \varrho_2) = \begin{cases} 0.48\delta \varrho_1^2 \varrho_2^2 |\ln(\varrho_1/\varrho_2)| & \text{for } \varrho_1 \varrho_2 < \delta |\ln(\varrho_1/\varrho_2)| \\ \varrho_1^3 \varrho_2^3 - 0.52\delta \varrho_1^2 \varrho_2^2 |\ln(\varrho_1/\varrho_2)| & \text{for } \varrho_1 \varrho_2 > \delta |\ln(\varrho_1/\varrho_2)| \end{cases}$$

$$\delta = 7.5 \times 10^{-5} (\nu/Cr_0^2)$$

$$\varepsilon = 5 \times 10^6 g\, Cr_0^9 n_0(r_0)/\nu^2 GS$$

The parameter δ is the ratio of the magnitude of the rate of coalescence due to gravitational effects to the shear-induced rate of coalescence; while ε is the ratio of the rate of increase of droplets due to shear-induced coalescence to the rate of growth of droplets by condensation.

To estimate the relative efficacies of the mechanisms operating on the droplet distribution, we consider typical values of the quantities δ and ε. An estimate of the root mean square strain rate C of the turbulence in a developing cumulus cloud is 10 sec^{-1} (Manton, 1974*a*). The spectral density near the base of such a cloud may be peaked at a radius $r_0 \sim 10$ μm, and so taking $\nu \sim 0.15$ cm^2 sec^{-1} we find that δ is of order unity. This implies that the gravitational and shear-induced coalescence terms in (2.7) are of comparable magnitude; but because the initial spectral density is assumed to be narrow, gravitational coalescence is expected to be ineffective. The droplet number density, which is of order $r_0 n_0(r_0)$, is typically 10^2 cm^{-3}. The supersaturation of the air is generally a few tenths of a percent (Warner, 1969) and so we set $S \sim 0.25$. Thus, taking $G \sim 10^{-8}$ cm^2 sec^{-1} and $g \sim 980$ cm sec^{-2} it is found that $\varepsilon \sim 8 \times 10^{-2}$. Therefore at this stage of development of the spectral density, condensation dominates the effects of coalescence. On the other hand ε is proportional to r_0^8 for $r_0 n_0(r_0)$ fixed, and this implies that shear-induced coalescence rapidly becomes of increasing importance as r_0 increases from 10 μm. Moreover ε is so small for r_0 less than 10 μm that coalescence clearly has a negligible influence upon the growth process for droplets with radii much less than 10 μm.

The scale chosen in (2.6) to normalise the time t corresponds to the response time of the droplet distribution due to condensation, and because ε is small the choice is appropriate. The time scale for the early development of the spectral density is therefore $r_0^2/2GS$, which is of order 200 sec when $S \sim 0.25$ and $r_0 \sim 10$ μm. Hence it would seem that the combined effects of condensation and shear-induced coalescence could broaden an initially narrow droplet spectrum in a time of about two hundred seconds, which is not a large fraction of the lifetime of a cloud.

3. Development of narrow spectral density

It is shown in § 2 that coalescence has a negligible effect on the droplet distribution for droplets much less than 10 μm in radius, but that shear-induced coalescence becomes increasingly important as the radius increases above about 10 μm. We consider therefore the development of a narrow spectral density initially centred at a radius r_0, of order 10 μm, such that the parameter ε is small. Thus the asymptotic solution of (2.7) as ε approaches zero is sought subject to the initial condition

$$\eta = \delta(\varrho - 1) \quad \text{at } \tau = 0 \tag{3.1}$$

where $\delta(x)$ is the Dirac delta function. The condition (3.1) represents the limiting case of a narrow spectrum, and so any dispersion of the spectral density found for this case is a lower limit on the dispersion of a more realistic initial distribution.

To solve (2.7), the spectral density is expanded in the asymptotic power series

$$\eta(\varrho, t; \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n \eta_n(\varrho, t) \tag{3.2}$$

which is substituted into the equation, and then a sequence of first order partial differential equations for the set $\{\eta_n\}$ is obtained by equating the coefficients of like powers of ε. Hence the leading term in the expansion (3.2) is given by

$$2\frac{\partial \eta_0}{\partial \tau} + \frac{\partial}{\partial \varrho}\left(\frac{\eta_0}{\varrho}\right) = 0 \tag{3.3}$$

with $\eta_0 = \delta(\varrho - 1)$ at $\tau = 0$.

The system (3.3) is readily solved to yield

$$\eta_0(\varrho, \tau) = \varrho(\varrho^2 - \tau)^{-1/2}\delta((\varrho^2 - \tau)^{1/2} - 1) \tag{3.4}$$

Using the result

$$\delta(f(x)) = \sum_{n=1}^{m} \left|\frac{df}{dx}(x_n)\right|^{-1} \delta(x - x_n) \tag{3.5}$$

where $x = x_n (n = 1, 2, ..., m)$ are the roots of $f(x) = 0$, we find that (3.4) may be written in the form

$$\eta_0(\varrho, \tau) = \delta(\varrho - (1 + \tau)^{1/2}) \tag{3.6}$$

The zeroth order solution (3.6) for the spectral density accounts only for the effects of condensation. Hence the density maintains its initial shape as the peak radius increases such that the total number of droplets, given by the integral of η, is conserved.

It is found from (2.7), (3.1), (3.2), (3.5) and (3.6) that the first order term for the spectral density is governed by the equation

$$2\frac{\partial \eta_1}{\partial \tau} + \frac{\partial}{\partial \varrho}\left(\frac{\eta_1}{\varrho}\right) = \frac{1}{2}k(2^{-1/3}\varrho, 2^{-1/3}\varrho) \times \delta(\varrho - 2^{1/3}(1+\tau)^{1/2}) - k(\varrho, \varrho)\,\delta(\varrho - (1+\tau)^{1/2}) \tag{3.7}$$

with $\eta_1 = 0$ at $\tau = 0$.

It is clear from (2.7) and (3.7) that the first order correction η_1 is independent of gravitational coalescence because of the narrowness of the initial spectral density. The first term on the right hand side of (3.7) corresponds to the production of droplets by the shear-induced coalescence of droplets in the primary peak; while the second term represents the loss of the coalescing droplets from the primary peak. The first order partial differential equation (3.7) may be solved to yield

$$\begin{aligned}\eta_1(\varrho, \tau) = {} & \tfrac{1}{2} a_0^{-1} \varrho\, k([(\varrho^2 - \tau - 1)/a_0]^{1/2}, [(\varrho^2 - \tau - 1)/a_0]^{1/2}) \\ & \times \{H(2^{1/3}(1+\tau)^{1/2} - \varrho) - H(2^{1/3}(1 + 2^{-2/3}\tau)^{1/2} - \varrho)\} \\ & - \left\{\varrho \int_0^{\varrho} k(\varrho_1, \varrho_1)\, d\varrho_1 - (\varrho^2 - \tau)^{1/2} \times \int_0^{(\varrho^2-\tau)^{1/2}} k(\varrho_1, \varrho_1)\, d\varrho_1\right\} \\ & \times \delta(\varrho - (1+\tau)^{1/2})\end{aligned} \tag{3.8}$$

where $a_0 = 2^{2/3} - 1$ and $H(x)$ is the Heaviside unit step function. Eq. (3.8) shows that the main effect of shear-induced coalescence is to generate a secondary peak ahead of the primary peak and extending from a normalised radius of $2^{1/3}(1 + 2^{-2/3}\tau)^{1/2}$ to $2^{1/3}(1 + \tau)^{1/2}$.

Using (2.7), (3.2), (3.7) and (3.8), we find that the asymptotic representation of the spectral density η is given by

$$\eta(\varrho,\tau) = \{1 - \tfrac{1}{7}\varepsilon[(1+\tau)^4 - 1]\}\,\delta(\varrho - (1+\tau)^{1/2}) + \tfrac{1}{2}\varepsilon(2^{2/3}-1)^{-4}\varrho(\varrho^2-\tau-1)^3 \times \{H(2^{1/3}(1+\tau)^{1/2}-\varrho) - H(2^{1/3}(1+2^{-2/3}\tau)^{1/2}-\varrho)\} + 0(\varepsilon^2) \quad (3.9)$$

Thus η consists of two components. The primary peak moves towards larger radii under the influence of condensation, such that it is centred on $\varrho = (1+\tau)^{1/2}$, as shown in Fig. 1. Shear-induced coalescence between droplets in this peak gives rise to a secondary peak which increases with time in amplitude and in width (see Fig. 2); the latter effect is due to condensation which varies the radius of the coalescing droplets in the primary peak. The number of droplets per unit volume, normalised with respect to the initial number, is given by

$$N(\tau) = \int_0^\infty \eta(\varrho,\tau)\,d\varrho \quad (3.10)$$

By putting (3.9) into (3.10), it is seen that coalescence leads to a decrease in the number density; in particular,

$$N(\tau) = 1 - \frac{9}{112}\varepsilon\{(1+\tau)^4 - 1\} + 0(\varepsilon^2) \quad (3.11)$$

Thus Fig. 3 shows the fraction of the initial droplet number density lost by shear-induced coalescence, to the first order in ε.

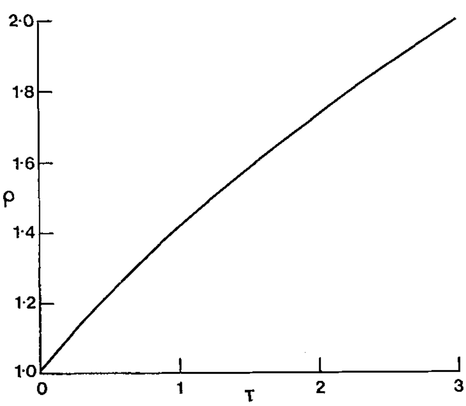

Fig. 1. Radius ϱ of droplets in primary peak of spectral density as a function of time τ, from eq. (3.9).

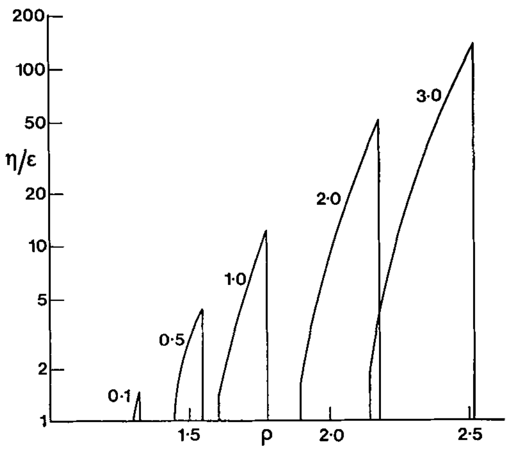

Fig. 2. Development of shear-induced secondary peak in the spectral density η as a function of radius ϱ and time τ, as calculated from eq. (3.9). Numbers refer to τ.

The asymptotic representations (3.9) and (3.11) should be valid provided that only a small fraction of the initial number of droplets has been involved in coalescence; that is provided that the first order correction term in (3.11) is small compared with unity. We found in § 2 that for a spectral density initially centred on $r_0 = 10$ μm the parameter ε is typically of order 8×10^{-2}. Hence it is seen from Fig. 3 that after the time $\tau = 2$ about one half of the initial number of droplets has been lost by coalescence, and so second order effects would be significant by this time. Figs. 1 and 2 show that at $\tau = 2$ the initially narrow spectrum is dispersed such that droplets lie

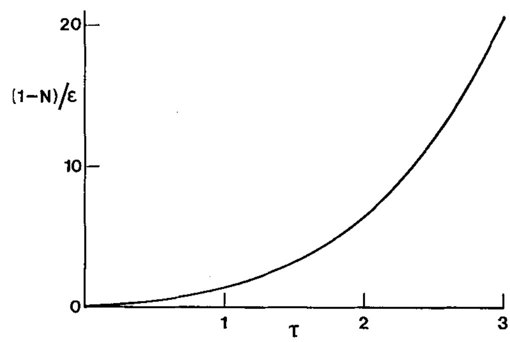

Fig. 3. Decrease in number of droplets per unit volume N due to shear-induced coalescence as a function of time τ, calculated from eq. (3.11)

in the interval $1.73 \leqslant \varrho \leqslant 2.18$ with maxima in the spectral density at the end points of the interval. Thus droplets are available at that time to coalesce with smaller droplets corresponding to a droplet radius ratio of about 0.8, which is small enough to produce significant gravitational coalescence. The time scale for the process, $r_0^2/2GS$, is about 200 sec for a supersaturation of 0.25. This implies that an initially narrow spectrum centred on $r_0 = 10$ μm broadens to the extent that droplets lie between 17.3 μm and 21.8 μm after a time of about 400 sec, which is not a large fraction of the expected life-time of a cloud. Because the condensation time scale for the early development of the spectrum is inversely proportional to the degree of supersaturation S, it would seem that S is an important parameter in determining whether a droplet distribution can develop fast enough to produce rain before the cloud itself is dissipated.

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О ДИСПЕРСИИ РАЗВИВАЮЩЕГОСЯ СПЕКТРА КАПЕЛЬ В ТУРБУЛЕНТНОМ ОБЛАКЕ

Показано, что совместный эффект конденсации и обусловленной сдвигом коагуляции может приводить к значительной дисперсии в узком начальном спектре облачных капель, центр которого приходится приблизительно на 10 *мкм*. При условии, что степень перенасыщения окружающего воздуха достаточно велика, временной масштаб этого процесса дисперсии мал в сравнении с ожидаемым временем жизни облака.