Gravity turbulence connected with interfaces

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ABSTRACT

The spectral distributions of turbulence, as generated by the gravity waves on the interface between two fluids, are investigated. Both stable and unstable surfaces are considered. An unstable surface refers to the early development of turbulence from the Taylor instability. A stable surface may refer to the sea-surface. A repeated-cascade method is used to close the hierarchy of correlations at their fourth order, and to determine the eddy transport property through a memory chain of eddy relaxations. The production, inertia and dissipation subranges of spectral distributions on an unstable surface with friction are found to follow the laws k^{-2}, k^{-3}, and k^{-3} for the kinetic energy, and $k^{-3.5}$, k^{-1}, and k^{-5} for the surface elevation. The inertia, eddy dissipation by gravity, and molecular dissipation subranges on a stable surface with friction are found to follow the law k^{-3} for the kinetic energy, and the laws k^{-1}, k^{-5} and k^{-5} for the surface elevation, respectively. The effects of surface tension are also investigated.

I. Introduction

The gravity turbulence can appear on a stable or an unstable surface, which separates a heavy liquid below or above a light one. For example, the sea surface is a stable surface. For the turbulent motions on a stable surface to be maintained, an external exciting agent, e.g., a surface wind, may be required. On the other hand, an unstable surface has its own source of energy supply, and may eventually develop into singular fingers, broken boundaries and droplets, which cease to constitute a continuous surface. Therefore, the gravity turbulence on an unstable surface can only refer to the early stage of its development.

In view of the complexity of the hydrodynamic equations of turbulence, dimensional methods had been used to predict the spectral structure of turbulence in a homogeneous fluid (Kolmogoroff, 1941; Heisenberg, 1948) and in a stratified medium (Shur, 1962). With the use of the acceleration of gravity g as a parameter in his dimensional analysis, Phillips (1966) proposed the following formulas of turbulent spectra in the spaces of wavenumber k and frequency ω on a sea surface:

(i) The spectrum of kinetic energy is

$$F = \text{const}\ gk^{-2} \tag{1}$$

and (ii) the spectra of surface elevation are

$$H(k) = \text{const}\ k^{-3} \tag{2}$$

$$H(\omega) = \text{const}\ g^2\omega^{-5} \tag{3}$$

in the spaces of wavenumber k and frequency ω, respectively.

It is obvious that the spectral structure could not be equally valid on both stable and unstable surfaces. Since the dimensional analysis only recognizes the parameter g, it is not able to distinguish between the stable and unstable conditions. In view of the above difficulty, it is necessary to advance here an analytic theory based upon the method of repeated-cascade (Tchen, 1973). It enables closing the herarchy of correlations to their fourth order, and determining an eddy transport property through a memory chain of eddy relaxations.

II. Basic equations for the moving surface

A moving surface is characterized by its velocity $\mathbf{u}$ and surface ζ. The governing equations are written in the space of wave vector $\mathbf{k} = (k_1, k_2)$, as follows:

$$\frac{\partial u_i(\mathbf{k})}{\partial t} + \int_{-\infty}^{\infty} d\mathbf{k}' i k_j' u_j(\mathbf{k}-\mathbf{k}') u_i(\mathbf{k}') + \gamma u_i(\mathbf{k}) = E_i(\mathbf{k}) - \nu k^2 u_i(\mathbf{k}) \tag{4}$$

$$\frac{\partial \zeta(\mathbf{k})}{\partial t} + \int_{-\infty}^{\infty} d\mathbf{k}' i k_j' u_j(\mathbf{k}-\mathbf{k}') \zeta(\mathbf{k}') = -\xi i k^{-1} k_j u_j(\mathbf{k}) - \lambda k^2 \zeta(\mathbf{k}) \tag{5}$$

with

$$E_i(\mathbf{k}) = -ig^*(\mathbf{k}) k_i \zeta(\mathbf{k}) \tag{6a}$$

$$g^*(k) = (1 + Tk^2/\varrho g) g \tag{6b}$$

$$\xi = 1, \text{ for a stable surface}$$
$$= -1, \text{ for an unstable surface} \tag{6c}$$

$i, j = (1, 2)$.

Here the convolution integrals represent the nonlinear inertia forces, γu_i is a frictional force, due, for example, to the wind friction on a sea surface, with a frictional coefficient γ. The density ϱ is constant. The molecular viscosity ν and diffusivity λ contribute to molecular dissipations $\nu k^2 u_i(\mathbf{k})$ and $\lambda k^2 \zeta(\mathbf{k})$. Finally, E_i is a driving force due to the stabilizing gravitational pull of the surface elevation, and g^* consists of the acceleration of gravity g added with the surface tension T. E_i serves as a coupling term in (4). The corresponding coupling term, $-\xi i k^{-1} k_j u_j(\mathbf{k})$ in the kinematic equation (5), represents the vertical pull of the surface elevation, and is obtained by integrating the equation of continuity of a volume element of fluid in three dimensions. The values of ξ in (6c) guarantee that a surface wave decreases its amplitude vertically upward for an unstable surface and downward for a stable surface.

It will be convenient to introduce a speed of propagation

$$c(k) = (g^*/k)^{\frac{1}{2}} \tag{7}$$

and a drift velocity of the surface, called "potential drift", in $\mathbf{k}$ space

$$w(\mathbf{k}) = ck\zeta(\mathbf{k}) \tag{8}$$

so that we can write (6a) and (5) in the following alternative forms

$$E_i(\mathbf{k}) = -ick_i w(\mathbf{k}) \tag{9}$$

and

$$\frac{\partial w(\mathbf{k})}{\partial t} + \int_{-\infty}^{\infty} d\mathbf{k}' i k_j' (k/k') u_j(\mathbf{k}-\mathbf{k}') w(\mathbf{k}') = -i\xi c k_j u_j(\mathbf{k}) - \lambda k^2 w(\mathbf{k}), \tag{10}$$

respectively.

The alternative system of equations (4) and (10) has the advantage that the gravitational coupling $\mathbf{E}(\mathbf{k}) \cdot \mathbf{u}(-\mathbf{k})$ is of equal magnitude but opposite signs in the equation of evolution of kinetic energy and potential energy

$$\tfrac{1}{2}\mathbf{u}(\mathbf{k}) \cdot \mathbf{u}(\mathbf{k}), \quad \tfrac{1}{2} w(\mathbf{k}) w(\mathbf{k})$$

for the case of a stable configuration, i.e. $\xi = 1$, as to be expected.

In applications, we shall use both systems of equations, (4) and (5), or (4) and (10). The latter system will be used for a stable surface when the mechanism of turbulence is controlled by the gravitational coupling between the kinetic energy and the potential energy, while the former system will be used for an unstable surface where such a coupling does not come into play.

It is evident that the existence of the gravitational coupling in (5) and (10) requires that

$$k_j u_j(\mathbf{k}) \neq 0 \tag{11a}$$

In the following, we shall, however, introduce the approximation

$$k_j u_j(\mathbf{k}) = 0 \tag{11b}$$

in all other terms of the above two systems. That practice is recognized as the Rayleigh-Boussinesq approximation (Chandrasekhar, 1953). Finally, we may remark that the basic equations (4) and (5) of the surface are laminar in nature, therefore containing a laminar friction in (4), while the effects of turbulence will manifest themselves during some processes of average.

III. Outline of the cascade theory of turbulence

A. *Cascade decomposition*

A turbulent motion is a quasi-stationary process, having a continuous spectrum of coupled scales. The large scales form a "macroscopic background", prescribing the background conditions for the motion of small scales The smaller scales move more "randomly", and, upon statistical averages of fluctuations, provide eddy transport properties in the background medium. The above division into macroscopic and random variables are relative to any wave number of the spectrum. Thus we write, for a velocity $\mathbf{u}$ in the physical space or in the wave number space,

$$\mathbf{u} = \mathbf{u}^\circ + \mathbf{u}' \tag{12}$$

where $\mathbf{u}'$ can be subdivided into

$$\mathbf{u}' = \mathbf{u}^{(1)} + \mathbf{u}^{(2)} + \ldots + \mathbf{u}^{(N)} \tag{13}$$

so that

$$\mathbf{u} = \mathbf{u}^\circ + \ldots + \mathbf{u}^{(N)} \tag{14}$$

More generally, we can decompose $\mathbf{u}$ into

$$\mathbf{u} = \mathbf{V}^{(\alpha)} + \mathbf{u}^\alpha \tag{15}$$

instead of (12), with

$$\mathbf{V}^{(\alpha)} = \mathbf{u}^\circ + \ldots + \mathbf{u}^{(\alpha-1)} \tag{16}$$

$$\mathbf{u}^\alpha = \mathbf{u}^{(\alpha)} + \ldots + \mathbf{u}^{(N)} \tag{17}$$

The superscripts denote the ranks, with a higher rank possessing a higher degree of randomness.

The decompositions (12) and (14) will be called single and repeated cascades, respectively.

In the above picture of quasi-stationary turbulent process, where the large scale motions are considered relatively macroscopic and the smaller scales are more random, we can associate a high degree of randomness to high wavenumbers, by writing

$$\mathbf{u}^{(\alpha)}(\mathbf{x}) = \int_{k^{(\alpha-1)}}^{k^{(\alpha)}} d\mathbf{k}\, \mathbf{u}(\mathbf{k}) \exp(i\mathbf{k}\cdot\mathbf{x}) \equiv \int_{-\infty}^{\infty} d\mathbf{k}\, \mathbf{u}^{(\alpha)}(\mathbf{k}) \exp(i\mathbf{k}\cdot\mathbf{x}) \tag{18}$$

$$\mathbf{u}^{\alpha}(\mathbf{x}) = \int_{k^{(\alpha-1)}}^{\infty} d\mathbf{k}\, \mathbf{u}(\mathbf{k}) \exp(i\mathbf{k}\cdot\mathbf{x}) \equiv \int_{-\infty}^{\infty} d\mathbf{k}\, \mathbf{u}^{\alpha}(\mathbf{k}) \exp(i\mathbf{k}\cdot\mathbf{x}) \tag{19}$$

In the notation (18), $\mathbf{u}^{(\alpha)}(\mathbf{k})$ is understood to be truncated between the wave number interval $(k^{(\alpha-1)}, k^{(\alpha)})$. The truncation needs not be sharp, and, if necessary, can be regulated by a scaling distribution.

The cascade decomposition, which is described for the variable $\mathbf{u}$, is also valid for the variables ζ and w, and no specific elaboration will be repeated.

B. *Averaging rules*

A "cascade ensemble average", or "rank average", denoted by

$$\langle \ldots \rangle' \tag{20}$$

is expected to separate the two components in (12), by averaging over realizations under identical macroscopic background $\mathbf{u}^\circ$. After such an averaging procedure, the random component $\mathbf{u}'$ becomes macroscopically negligible, and the macroscopic component $\mathbf{u}^\circ$ comes out intact. Thus we have

$$\langle \mathbf{u} \rangle' = \mathbf{u}^\circ, \quad \langle \mathbf{u}^\circ \rangle' = \mathbf{u}^\circ, \quad \langle \mathbf{u}' \rangle' = 0 \tag{21}$$

Similarly a rank average

$$\langle \ldots \rangle^\circ \tag{22}$$

would annul $\mathbf{u}^\circ$, i.e.

$$\langle \mathbf{u}^\circ \rangle^\circ = 0 \tag{23}$$

The average (22) could correspond to a spatial average over a length interval X°. Eventually, X° may tend to infinity in a homogeneous turbulence.

More generally, we can introduce a rank average

$$\langle \ldots \rangle^\alpha \tag{24}$$

corresponding to a length interval $X^{(\alpha)}$. Then the following averaging rules apply:

$$\langle \mathbf{u}^{(\beta)} \rangle^\alpha = 0, \quad \text{if } \beta \geqslant \alpha \tag{25}$$

$$= \mathbf{u}^{(\beta)}, \quad \text{if } \beta < \alpha \tag{26}$$

as a generalization of (21) and (23).

C. Ranks and rank values of correlations

In view of the condition of quasi-stationarity of turbulence of rank α, we can write

$$\langle \mathbf{u}^{(\alpha)}(\mathbf{k}')\,\mathbf{u}^{(\alpha)}(\mathbf{k}'')\rangle^{\alpha} = \chi^{(\alpha)}\langle \mathbf{u}^{(\alpha)}(\mathbf{k}')\,\mathbf{u}^{(\alpha)}(-\mathbf{k}')\rangle\,\delta(k'+k'') \tag{27}$$

where

$$\chi^{(\alpha)} = (\pi/X^{(\alpha)})^{s} \tag{28}$$

is called a "scaling factor", and $s=2$ in two dimensions, giving the following relation between the velocity correlations in $\mathbf{k}$ space and $\mathbf{x}$ space:

$$\langle \mathbf{u}^{(\alpha)}(\mathbf{x})\,\mathbf{u}^{(\alpha)}(\mathbf{x})\rangle^{\alpha} = \int_{-\infty}^{\infty} d\mathbf{k}'\chi^{(\alpha)}\langle \mathbf{u}^{(\alpha)}(\mathbf{k}')\,\mathbf{u}^{(\alpha)}(-\mathbf{k}')\rangle^{\alpha} \tag{29}$$

In connection with the velocity correlation at two instants, we introduce the integrals

$$\eta_{ij}^{(\alpha)}(\mathbf{k}) = \int_{0}^{\infty} dt'\chi^{(\alpha)}\langle u_i^{(\alpha)}(t',\mathbf{k})\,u_j^{(\alpha)}(t,\mathbf{k})\rangle^{\alpha} \tag{30}$$

$$\eta_{ij}^{\alpha}(\mathbf{k}) = \int_{0}^{\infty} dt'\chi^{(\alpha)}\langle u_i^{\alpha}(t',\mathbf{k})\,u_j^{\alpha}(t,-\mathbf{k})\rangle^{\alpha} \tag{31}$$

with

$$\eta_{ij}^{(\alpha)} = \int_{-\infty}^{\infty} d\mathbf{k}\,\eta_{ij}^{(\alpha)}(\mathbf{k}) \tag{32}$$

and

$$\eta_{ij}^{\alpha} = \int_{-\infty}^{\infty} d\mathbf{k}\,\eta_{ij}^{\alpha}(\mathbf{k}) \tag{33}$$

If the turbulent motion of rank α is of a sufficiently small scale, the assumption of isotropy can be applied, giving

$$\eta_{ij}^{(\alpha)} = \eta^{(\alpha)}\delta_{ij} \tag{34}$$

$\eta^{(\alpha)}$ is called an eddy viscosity of the αth rank.

In view of the averaging rule (26) and of the expected presence of

$$\nabla\langle \mathbf{u}^{(\alpha+1)}\mathbf{u}^{(\alpha+1)}\rangle^{(\alpha+1)} \tag{35}$$

in the equation describing the evolution of $\mathbf{u}^{(\alpha)}$, we deem that $\langle \mathbf{u}^{(\alpha+1)}\mathbf{u}^{(\alpha+1)}\rangle^{\alpha+1}$ has a rank value α. As a consequence, the eddy viscosity $\eta_{ij}^{(\alpha)}$, as obtained by a time integration which also amounts to a smoothing process, will have a rank value $\alpha-2$ or lower. Thus

$$\langle u_i^{\alpha} u_j^{\alpha}\rangle^{\alpha} = \langle u_i^{(\alpha)} u_j^{(\alpha)}\rangle^{\alpha} \quad \text{has a rank value } \alpha-1, \tag{36}$$

and

$$\eta_{ij}^{\alpha} = \eta_{ij}^{(\alpha)} \quad \text{has a rank value } \alpha-2 \tag{37}$$

IV. Dynamic equations in cascade representation

As mentioned in Section III, we will need a repeated cascade for the velocity and only a single cascade for the surface. Therefore, we write their respective dynamical equations as follows:

$$\begin{aligned}\frac{du_i^{(\alpha)}(\mathbf{k})}{dt} &\equiv \frac{\partial u_i^{(\alpha)}(\mathbf{k})}{\partial t} + \int_{-\infty}^{\infty} d\mathbf{k}' ik_j'\, V_j^{(\alpha+1)}(\mathbf{k}-\mathbf{k}')\,u_i^{(\alpha)}(\mathbf{k}') \\ &= -\int_{-\infty}^{\infty} d\mathbf{k}' ik_j'\,[u_j^{(\alpha)}(\mathbf{k}-\mathbf{k}')\,V_i^{(\alpha)}(\mathbf{k}') \\ &\quad + \langle u_j^{(\alpha+1)}(\mathbf{k}-\mathbf{k}')\,u_i^{(\alpha+1)}(\mathbf{k}')\rangle^{\alpha+1}] \\ &\quad + E_i^{(\alpha)}(\mathbf{k}) - \gamma u_i^{(\alpha)}(\mathbf{k}) - \nu k^2 u_i^{(\alpha)}(\mathbf{k})\end{aligned} \tag{38}$$

$$\begin{aligned}\frac{Du_i^{\alpha}(\mathbf{k})}{Dt} &\equiv \frac{\partial u_i^{\alpha}(\mathbf{k})}{\partial t} + \int_{-\infty}^{\infty} d\mathbf{k}' ik_j' u_j(\mathbf{k}-\mathbf{k}')\,u_i^{\alpha}(\mathbf{k}') \\ &= -\int_{-\infty}^{\infty} d\mathbf{k}' ik_j'\, u_j^{\alpha}(\mathbf{k}-\mathbf{k}')\,V_i^{(\alpha)}(\mathbf{k}') \\ &\quad + E_i^{\alpha}(\mathbf{k}) - \gamma u_i^{\alpha}(\mathbf{k}) - \nu k^2 u_i^{\alpha}(\mathbf{k})\end{aligned} \tag{39}$$

$$\begin{aligned}&\frac{\partial \zeta^{\circ}(\mathbf{k})}{\partial t} + \int_{-\infty}^{\infty} d\mathbf{k}' ik_j'\, u_j^{\circ}(\mathbf{k}-\mathbf{k}')\,\zeta^{\circ}(\mathbf{k}') \\ &= -\int_{-\infty}^{\infty} d\mathbf{k}' ik_j'\,\langle u_j'(\mathbf{k}-\mathbf{k}')\,\zeta'(\mathbf{k}')\rangle' \\ &\quad - \xi ik^{-1}k_j u_j^{\circ}(\mathbf{k}) - \lambda k^2 \zeta^{\circ}(\mathbf{k})\end{aligned} \tag{40}$$

$$\frac{\partial w^{\circ}(\mathbf{k})}{\partial t} + \int_{-\infty}^{\infty} d\mathbf{k}' ik_j'(k/k')\, u_j^{\circ}(\mathbf{k}-\mathbf{k}')\, w^{\circ}(\mathbf{k}')$$

$$= -\int_{-\infty}^{\infty} d\mathbf{k}' ik_j'(k/k')\, \langle u_j'(\mathbf{k}-\mathbf{k}')\, w'(\mathbf{k}')\rangle'$$

$$- \xi ick_j u_j^{\circ}(\mathbf{k}) - \lambda k^2 w^{\circ}(\mathbf{k}) \qquad (41)$$

$$\frac{D\zeta'(\mathbf{k})}{Dt} \equiv \frac{\partial \zeta'(\mathbf{k})}{\partial t} + \int_{-\infty}^{\infty} d\mathbf{k}' ik_j' u_j(\mathbf{k}-\mathbf{k}')\, \zeta'(\mathbf{k}')$$

$$= -\int_{-\infty}^{\infty} d\mathbf{k}' ik_j' u_j'(\mathbf{k}-\mathbf{k}')\, \zeta^{\circ}(\mathbf{k}')$$

$$- \xi ik^{-1} k_j u_j'(\mathbf{k})$$

$$+ \int_{-\infty}^{\infty} d\mathbf{k}' ik_j' \langle u_j'(\mathbf{k}-\mathbf{k}')\, \zeta'(\mathbf{k}')\rangle'$$

$$- \lambda k^2 \zeta'(\mathbf{k}) \qquad (42)$$

$$\frac{Dw'(\mathbf{k})}{Dt} \cong \frac{\partial w'(\mathbf{k})}{\partial t} + \int_{-\infty}^{\infty} d\mathbf{k}' ik_j'(k/k')\, u_j(\mathbf{k}-\mathbf{k}')\, w'(\mathbf{k}')$$

$$= -\int_{-\infty}^{\infty} d\mathbf{k}' ik_j'(k/k')\, u_j'(\mathbf{k}-\mathbf{k}')\, w^{\circ}(\mathbf{k}')$$

$$- \xi ick_j u_j'(\mathbf{k})$$

$$+ \int_{-\infty}^{\infty} d\mathbf{k}' ik_j'(k/k')\, \langle u_j'(\mathbf{k}-\mathbf{k}')\, w'(\mathbf{k}')\rangle'$$

$$- \lambda k^2 w'(\mathbf{k}) \qquad (43)$$

V. Langevin equation and Onsager's relation for turbulent motions

It is to be remarked that D/Dt in (39) represents a Lagrangian derivative, i.e., a rate of change following the path of a fluid element. The variable t in D/Dt can be treated as a one-dimensional variable in the Lagrangian representation, notwithstanding its four dimensions in time and three wavenumbers in the Eulerian representation. Under such a circumstance, and in analogy with the Brownian movements of molecules, eqs. (38), (39), (42) and (43) can be regarded as Langevin equations for turbulent motion, if $\mathbf{k}$ is taken to be a parameter. They will be useful to calculate fluxes or stresses and associated transport coefficients of turbulence. For this purpose, we make a formal integration of (39), giving

$$u_i^{\alpha}(t, \mathbf{k}) = \int_0^t dt' h_i^{\alpha}(t', \mathbf{k}) \exp[-\gamma(t-t')]$$

$$+ u_i^{\alpha}(0, \mathbf{k})\, e^{-\gamma t} \qquad (44)$$

where

$$h_i^{\alpha}(t, \mathbf{k}) = -\int_{-\infty}^{\infty} d\mathbf{k}' ik_s' u_s^{\alpha}(t, \mathbf{k}-\mathbf{k}')\, V_i^{(\alpha)}(t, \mathbf{k}')$$

$$+ E_i^{\alpha}(t, \mathbf{k}) \qquad (45)$$

It is to be noted that a term similar to that in $\langle\ldots\rangle'$ on the right hand side of (42) has not been carried over to (45), because it will not contribute to fluxes and transport properties. Also we have omitted the molecular dissipation.

Since a transport property is contributed by a correlation from a rank $\mathbf{u}^{\alpha}$ in a quasi-stationary background of rank $\mathbf{V}^{(\alpha)}(t, \mathbf{k})$ in the cascade (15), the upper limit t will belong to the quasi-stationary time scale which is much larger than the duration of that correlation. Therefore that upper limit can be replaced by ∞, and $V_i^{(\alpha)}(t', \mathbf{k}')$ can be replaced by $V_i^{(\alpha)}(t, \mathbf{k}')$. For the same reason, the initial value will not be correlated with any fluctuation at time t, thus simplifying (44) to

$$u_i^{\alpha}(t, \mathbf{k}) = \int_0^{\infty} dt' h_i^{\alpha}(t', \mathbf{k}) \exp[-\gamma(t-t')]$$

$$\cong \int_0^{\infty} d\tau\, h_i^{\alpha}(t-\tau, \mathbf{k}) \qquad (46)$$

Here $\tau = t - t'$, and

$$h_i^{\alpha}(t', \mathbf{k}) \cong -\int_{-\infty}^{\infty} dk'' ik_s'' u_s^{\alpha}(t, \mathbf{k}-\mathbf{k}'')\, V_i^{(\alpha)}(t, \mathbf{k}'')$$

$$+ E_i^{\alpha}(t', \mathbf{k}) \qquad (47)$$

We have also neglected the friction as being small compared to the eddy mixing process.

The expression (46) for the fluctuation $u_i^{(\alpha)}$ avails itself to find a flux

$$\langle u_j^\alpha(t, \mathbf{k}-\mathbf{k}')\, u_i^\alpha(t, \mathbf{k}')\rangle^\alpha$$

$$= -\int_{-\infty}^{\infty} d\mathbf{k}'' ik_s''\, V_i^{(\alpha)}(t, \mathbf{k}'')$$

$$\times \int_0^\infty d\tau \langle u_j^\alpha(t, \mathbf{k}-\mathbf{k}')\, u_s^\alpha(t-\tau, \mathbf{k}'-\mathbf{k}'')\rangle^\alpha$$

$$+ \int_0^\infty d\tau \langle u_j^\alpha(t, \mathbf{k}-\mathbf{k}')\, E_i^\alpha(t-\tau, \mathbf{k})\rangle^\alpha \qquad (48)$$

All the terms have the same rank value $\alpha-1$, except the last term which has a rank value up to $\alpha-2$. Such a disparate rank will not contribute to the flux, and therefore will be omitted. When we make use of the property (27) and the definition (31), we reduce the flux to the form

$$\langle u_j^\alpha(t, \mathbf{k}-\mathbf{k}')\, u_i^\alpha(t, \mathbf{k})\rangle^\alpha = -\eta_{js}^\alpha(\mathbf{k}-\mathbf{k}')\, ik_s\, V_i^{(\alpha)}(t, \mathbf{k}) \qquad (49\,\text{a})$$

We conclude that the statistical effect of the fluctuations of rank $\mathbf{u}'$ upon the evolution of $\mathbf{u}^\circ$ takes the form of a flux (49a), which is proportional to the background velocity gradient multiplied by an eddy viscosity. That a flux is proportional to the gradient of the quantity to be transported agrees with the Onsager relation. By repeating the method for the surface fluxes, we obtain

$$\langle u_j'(t, \mathbf{k}-\mathbf{k}')\, \zeta'(t, \mathbf{k}')\rangle' = -\eta_{js}'(\mathbf{k}-\mathbf{k}')\, ik_s\, \zeta^\circ(t, \mathbf{k}) \qquad (49\,\text{b})$$

$$\langle u_j'(t, \mathbf{k}-\mathbf{k}')\, w'(t, \mathbf{k}')\rangle' = -\eta_{js}'(\mathbf{k}-\mathbf{k}')\, ik_s\, w^\circ(t, \mathbf{k}) \qquad (49\,\text{c})$$

The relations (49a), (49b) and (49c) will be called the Onsager relations of turbulence.

VI. Transport of energy

By relying upon the Onsager relation (49a), we can transform the equation (38) for the momentum transport into the form

$$\left[\frac{d}{dt} + \omega'(\mathbf{k})\right] u_i^\circ(t, \mathbf{k}) = E_i^\circ(t, \mathbf{k}) - \gamma u_i^\circ(t, \mathbf{k}) \qquad (50)$$

where the relaxation frequency is

$$\omega'(\mathbf{k}) = \int_{-\infty}^{\infty} d\mathbf{k}' k_j' k_s' \eta_{js}'(\mathbf{k}-\mathbf{k}')$$

$$\cong \eta_{js}' \int_{-\infty}^{\infty} d\mathbf{k}' k_j' k_s' \,\delta(\mathbf{k}-\mathbf{k}')$$

$$= k_j k_s \eta_{js}' \qquad (51\,\text{a})$$

in an anisotropic medium, or

$$\omega'(k) = k^2 \eta' \qquad (51\,\text{b})$$

in an isotropic medium.

If we treat (50) as a Langevin equation, and integrate, we find

$$u_i^\circ(t, \mathbf{k}) = \int_0^\infty d\tau\, E_i^\circ(t-\tau, \mathbf{k})\, e^{-\omega'\tau} \qquad (52)$$

Here the initial value is omitted, because it does not contribute to any correlation. In addition, we have neglected γ as compared to ω'.

In a similar way, we reduce (40) and (41) to

$$\left[\frac{d}{dt} + \omega'(k)\right] \zeta^\circ(t, \mathbf{k}) = -\xi i k^{-1} \mathbf{k}\cdot\mathbf{u}^\circ(t, \mathbf{k}) \qquad (53)$$

and

$$\left[\frac{d}{dt} + \omega'(k)\right] w^\circ(t, \mathbf{k}) = -\xi i c \mathbf{k}\cdot\mathbf{u}^\circ(t, \mathbf{k}) \qquad (54)$$

or, in terms of E_i°,

$$\left[\frac{d}{dt} + \omega'(k)\right] E_i^\circ(t, \mathbf{k}) = -\xi c^2 k_i\, \mathbf{k}\cdot\mathbf{u}^\circ(t, \mathbf{k}) \qquad (55)$$

Upon multiplying (50), (53) and (54) by $u_i^\circ(-\mathbf{k})$, $\zeta^\circ(-\mathbf{k})$ and $w^\circ(-\mathbf{k})$, respectively, and taking an average, we find

$$\left(\frac{\partial}{\partial t} + \omega'\right) \langle u_i^\circ(\mathbf{k})\, u_i^\circ(-\mathbf{k})\rangle^\circ$$

$$= \langle E_i^\circ(\mathbf{k})\, E_i^\circ(-\mathbf{k})\rangle^\circ - \gamma \langle u_i^\circ(\mathbf{k})\, u_i^\circ(-\mathbf{k})\rangle^\circ$$

$$+ (\mathbf{k} \to -\mathbf{k}) \qquad (56)$$

$$\left(\frac{\partial}{\partial t} + \omega'\right) \langle \zeta^\circ(\mathbf{k})\, \zeta^\circ(-\mathbf{k})\rangle^\circ$$

$$= -\xi (g_* k)^{-1} \langle E_i^\circ(-\mathbf{k})\, u_i^\circ(\mathbf{k})\rangle^\circ + (\mathbf{k} \to -\mathbf{k}) \qquad (57)$$

$$\left(\frac{\partial}{\partial t} + \omega'\right) \langle w^\circ(\mathbf{k})\, w^\circ(-\mathbf{k})\rangle^\circ$$
$$= -\xi\langle E_i^\circ(-\mathbf{k})\, u_i^\circ(\mathbf{k})\rangle^\circ + (\mathbf{k}\to\mathbf{k}) \tag{58}$$

Here the convection terms do not contribute to the energy evolution in homogeneous turbulence, and are therefore omitted. The complex conjugate part is represented by $(\mathbf{k}\to -\mathbf{k})$, as obtained from replacing $\mathbf{k}$ by $-\mathbf{k}$.

When we multiply (56), (57) and (58) by a scaling factor χ°, and integrate with respect to $\mathbf{k}$, as prescribed by (29), we obtain the equations of energy balance

$$\frac{\partial}{\partial t}\int_0^k dk' F(k') \equiv \frac{1}{2}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} d\mathbf{k}'\chi^\circ\langle u_i^\circ(\mathbf{k}')\, u_i^\circ(-\mathbf{k}')\rangle^\circ$$
$$= \Gamma^\circ - T^\circ - \nu J^\circ - \psi^\circ \tag{59 a}$$

$$\frac{\partial}{\partial t}\int_0^k dk' H(k') \equiv \frac{1}{2}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} d\mathbf{k}'\chi^\circ\langle \zeta^\circ(\mathbf{k}')\, \zeta^\circ(-\mathbf{k}')\rangle^\circ$$
$$= -\xi\Gamma_\zeta^\circ - T_\zeta^\circ - \lambda J_\zeta^\circ \tag{59 b}$$

$$\frac{\partial}{\partial t}\int_0^k dk' G(k') \equiv \frac{1}{2}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} d\mathbf{k}'\chi^\circ\langle w^\circ(\mathbf{k}')\, w^\circ(-\mathbf{k}')\rangle^\circ$$
$$= -\xi\Gamma^\circ - T_w^\circ - \lambda J_w^\circ \tag{59 c}$$

with the following transport functions:

(*a*) For the kinetic energy

$$T^\circ = \eta' J^\circ \tag{60 a}$$

$$J^\circ \equiv 2\int_0^k dk' k'^2 F(k')$$
$$= \int_{-\infty}^{\infty} dk' k'^2 \chi^\circ\langle \mathbf{u}^\circ(\mathbf{k}')\cdot\mathbf{u}^\circ(-\mathbf{k}')\rangle^\circ \tag{60 b}$$

$$\psi^\circ \equiv 2\gamma\int_0^k dk' F(k')$$
$$= \gamma\int_{-\infty}^{\infty} d\mathbf{k}'\chi^\circ\langle \mathbf{u}^\circ(\mathbf{k}')\cdot\mathbf{u}^\circ(-\mathbf{k}')\rangle^\circ \tag{60 c}$$

$$\Gamma^\circ \equiv \int_{-\infty}^{\infty} d\mathbf{k}'\chi^\circ\langle \mathbf{E}^\circ(-\mathbf{k}')\cdot\mathbf{u}^\circ(-\mathbf{k}')\rangle^\circ \tag{60 d}$$

$$\tfrac{1}{2}\langle \mathbf{u}^{\circ 2}\rangle^\circ \equiv \int_0^k dk' F(k') \tag{60 e}$$

(*b*) For the surface elevation

$$T_\zeta^\circ = \eta' J_\zeta^\circ \tag{61 a}$$

$$J_\zeta^\circ \equiv 2\int_0^k dk' k'^2 H(k')$$
$$= \int_{-\infty}^{\infty} d\mathbf{k}' k'^2 \chi^\circ\langle \zeta^\circ(\mathbf{k}')\, \zeta^\circ(-\mathbf{k}')\rangle^\circ \tag{61 b}$$

$$\Gamma_\zeta^\circ \equiv \int_{-\infty}^{\infty} dk' (g^* k')^{-1}\chi^\circ\langle \mathbf{E}^\circ(-\mathbf{k}')\cdot\mathbf{u}^\circ(\mathbf{k}')\rangle^\circ \tag{61 c}$$

$$\tfrac{1}{2}\langle \zeta^{\circ 2}\rangle^\circ \equiv \int_0^k dk' H(k') \tag{61 d}$$

(*c*) For the potential energy

$$T_w^\circ = \eta' J_w^\circ \tag{62 a}$$

$$J_w^\circ \equiv 2\int_0^k dk' k'^2 G(k')$$
$$= \int_{-\infty}^{\infty} d\mathbf{k}' k'^2 \chi^\circ\langle w^\circ(\mathbf{k}')\, w^\circ(-\mathbf{k}')\rangle^\circ \tag{62 b}$$

$$\tfrac{1}{2}\langle w^{\circ 2}\rangle^\circ \equiv \int_0^k dk' G(k') \tag{62 c}$$

The functions T°, T_ξ° and T_w° are called transfer functions, they govern the cascade transfer of energy across each individual spectrum. The terms νJ°, λJ_ζ° and λJ_w° are dissipation functions, proportional to the vorticity functions J°, J_ζ° and J_w°, with the molecular viscosity ν and diffusivity λ as coefficients. Finally Γ° and Γ_ζ° are called gravitational exchange functions, since they govern the exchange between E° and u°.

VII. Gravitational exchange

The gravitational exchange (60d) can be calculated by means of (50) and (55), giving

$$\langle \mathbf{E}^\circ(t,\mathbf{k})\cdot u^\circ(t, -\mathbf{k})\rangle^\circ$$
$$= \int_0^\infty d\tau\, \langle \mathbf{E}^\circ(t,\mathbf{k})\cdot\mathbf{E}^\circ(t-\tau,\mathbf{k})\rangle^\circ e^{-\omega' t} \tag{63}$$

and

$$\left(\frac{d}{dt} + \omega'\right) \langle E_i^\circ(t, \mathbf{k}) E_i^\circ(t', -\mathbf{k})\rangle^\circ$$

$$= -\xi c^2 k_i k_j \langle u_j^\circ(t, \mathbf{k}) E_i^\circ(t', \mathbf{k})\rangle^\circ \quad (64)$$

for an isotropic turbulence.

Assuming

$$kc < \omega' \quad (65)$$

as valid in the spectral subrange dominated by a gravitational pull, we obtain the approximate solution of (63) and (64) to be

$$\langle \mathbf{E}^\circ(t\ \mathbf{k}) \cdot \mathbf{u}^\circ(t, \mathbf{k})\rangle^\circ = \frac{1}{2\omega'} \langle \mathbf{E}^\circ(t, \mathbf{k}) \cdot \mathbf{E}^\circ(t, -\mathbf{k})\rangle^\circ \quad (66)$$

It follows:

$$\Gamma^\circ \equiv \int_{-\infty}^{\infty} d\mathbf{k}'(2\omega')^{-1}\chi^\circ \langle \mathbf{E}^\circ(\mathbf{k}') \cdot \mathbf{E}^\circ(-\mathbf{k}')\rangle^\circ$$

$$= \int_0^k dk' \frac{[k'c(k')]^4 H(k')}{\omega'(k')} \quad (67)$$

$$\Gamma_\xi^\circ \equiv \int_{-\infty}^{\infty} d\mathbf{k}'[2\omega'(k')\, g^*(k')\, k']^{-1}\chi^\circ$$

$$\times \langle \mathbf{E}^\circ(\mathbf{k}') \cdot \mathbf{E}^\circ(-\mathbf{k}')\rangle^\circ$$

$$= \int_0^k dk' \frac{[k'c(k')]^2 H(k')}{\omega'(k')} \quad (68)$$

VIII. Eddy viscosity and relaxations

As an extension of (50), we can rewrite (38) as

$$\left(\frac{d}{dt} + \omega^{\alpha+1}\right) u_i^{(\alpha)}(t, \mathbf{k})$$

$$= -\int_{-\infty}^{\infty} d\mathbf{k}' i k_j' u_j^{(\alpha)}(t, \mathbf{k}-\mathbf{k}')\, V^{(\alpha)}(t, \mathbf{k}')$$

$$+ E_i^{(\alpha)}(t, \mathbf{k}) \quad (69)$$

where

$$\omega^{\alpha+1} = k^2\eta^{\alpha+1} \quad (70)$$

and the frictions are neglected. Equation (69) serves as a basis of calculating the eddy viscosity η^α. For this purpose, we remark that the first term on the right hand side of (69) will not contribute, in view of the presence of a macroscopic variable $V^{(\alpha)}$ which may change its sign; also we remark that the last term of (69) will give a negligible contribution under the condition (65). By omitting the details of calculations, we find the expression

$$\eta^\alpha = \frac{1}{2}\int_{-\infty}^{\infty} d\mathbf{k} \frac{\chi^{(\alpha)}\langle \mathbf{u}^\alpha(\mathbf{k}) \cdot \mathbf{u}^\alpha(-\mathbf{k})\rangle^\alpha}{\omega^{\alpha+1}} \quad (71)$$

which can be rewritten more explicitly in the form of the following memory-chain of relaxations

$$\eta' \equiv \eta'(x/k) = \int_k^\infty dk' \frac{F(k')}{k'^2\eta''(x/k')}$$

$$\eta''(x/k') = \int_{k'}^\infty dk'' \frac{F(k'')}{k''^2\eta'''(x/k'')}$$

$$\vdots \quad (72)$$

For an isomeric structure of the spectrum for all links of the chain, we find the solution of the infinite sequence (72), to be

$$\eta' = \left[2\int_k^\infty dk' k'^{-2} F(k')\right]^{\frac{1}{2}} \quad (73)$$

It is to be remarked that the nonlinearity of the basic equations (4) and (5) of the interface generates a hierarchy of correlations. The method of single cascade decomposition (12) degenerates the fourth correlations into products of two double correlations, as achieved in the transfer functions (60a), (61a) and (62a), and hence closes that hierarchy. However, an eddy viscosity η' is involved in the above functions, and requires a repeated-cascade to describe its intrinsic memory-chain (72). The hypothesis of isomeric spectrum in all links of the chain enables to close the chain too.

IX. Universal range

The universal range holds at sufficiently large wave numbers, such that the time derivatives in (59) become independent of k. Noting that

$$T^\circ(k=\infty)=0, \quad T^\circ_\zeta(k=\infty)=0$$

we can rewrite (59) in new forms for $k=\infty$, and subtract the new forms from the original equations, yielding

(i) *for an unstable surface* ($\xi=1$)

$$\begin{aligned} -\Gamma^\circ + T^\circ + \psi^\circ + \nu J^\circ &= -\Gamma + \psi + \varepsilon \\ \xi\Gamma^\circ_\zeta + T^\circ_\zeta + \lambda J^\circ_\zeta &= \xi\Gamma_\zeta + \varepsilon_\zeta \end{aligned} \tag{74}$$

(ii) *for a stable surface* ($\xi=-1$)

$$\begin{aligned} -\Gamma^\circ + T^\circ + \psi^\circ + \nu J^\circ &= -\Gamma + \psi + \varepsilon \\ \xi\Gamma^\circ + T^\circ_w + \lambda J^\circ_w &= \xi\Gamma + \varepsilon_w \end{aligned} \tag{75}$$

with the notations

$$\Gamma = \Gamma^\circ(k=\infty), \quad \Gamma_\zeta = \Gamma^\circ_\zeta(k=\infty), \quad \psi = \psi^\circ(k=\infty)$$

$$J = J^\circ(k=\infty), \quad J_\zeta = J^\circ_\zeta(k=\infty), \; J_w = J^\circ_w(k=\infty)$$

$$\varepsilon = \nu J, \quad \varepsilon_\zeta = \lambda J_\zeta, \quad \varepsilon_w = \lambda J_w \tag{76}$$

We refer the configuration of a heavy liquid above a lighter one as unstable ($\xi=-1$), and the reverse configuration, e.g., sea-surface, as stable ($\xi=1$). Their distinction lies in the roles of the gravitational pull: in the unstable configuration, the gravitational pull is an energy source for both the surface elevation and the velocity which endows it, while in the stable case, the gravitational pull serves to produce the kinetic energy at the expense of the potential energy, so that the two energies must balance.

In the quasi-stationary turbulence, the time t does not appear as an explicit variable, but the spectral distributions may vary slowly with time through the physical parameters Γ, Γ_ζ, J, J_ζ, J_w.

X. Inertia and dissipation subranges for stable and unstable surfaces

The gravitational pull is absent in the present subranges, so that the equation (74) governing the spectral distributions reduce to

$$T^\circ + \psi^\circ + \nu J^\circ = \psi + \varepsilon \tag{77a}$$

$$T^\circ_\zeta + \lambda J^\circ_\zeta = \varepsilon_\zeta \tag{77b}$$

We shall discuss the solutions for various subranges. The simplest case is the inertia subrange described by

$$T^\circ = \varepsilon, \quad T^\circ_\zeta = \varepsilon_\zeta \tag{78}$$

as a degenerate form of (77), with solutions

$$\begin{aligned} F &= 0.83\varepsilon^{2/3}k^{-5/3}, \\ H &= 0.83\varepsilon_\zeta\varepsilon^{-1/3}k^{-5/3} \end{aligned} \tag{79}$$

in agreement with the Kolmogoroff (1941) law. However, in the presence of the friction or dissipation, the vorticity J° attains practically its saturated value

$$J^\circ \cong J \tag{80}$$

so that the drop of the F-spectrum can be calculated from the differential form of (77a), which is

$$\frac{d\eta'}{dk}J + \nu\frac{dJ^\circ}{dk} + 2\gamma F = 0 \tag{81}$$

with the aid of (80), and with the approximation

$$\nu + \eta' \cong \nu \tag{82}$$

The solution of (81) is

$$F = \tfrac{1}{2}J^2(\gamma + 2\nu k^2)(\gamma + \nu k^2)^{-3}k^{-3} \tag{83}$$

A substitution of (83) into (77b) gives the solution

$$H = \frac{JJ_\zeta k^{-5}}{2\lambda}\left[1 + \frac{Jk^{-2}}{2\lambda(\nu+\gamma k^2)}\right]^{-2}\frac{\gamma + 2\nu k^2}{\gamma + \nu k^2} \tag{84}$$

The general solutions (83) and (84) cover an interpolation over the following three subranges (*a*)–(*c*):

(*a*) *Inertia subrange with friction* ($\nu=0$, $\lambda=0$)

$$F = \tfrac{1}{2}(J/\gamma)^2k^{-3} \tag{85a}$$

$$H = 2(\varepsilon_\zeta\gamma/J)k^{-1} \tag{85b}$$

(*b*) *Diffusion subrange with friction* ($\nu=0$)

When the F-spectrum falls by friction according to (85a), the H-spectrum is dissipated by molecular diffusivity and becomes

$$H = \tfrac{1}{2}(JJ_\zeta/\gamma\lambda)k^{-5} \tag{86}$$

(c) *Dissipation subrange without friction* ($\gamma = 0$)

$$F = (J/\nu)^2 k^{-7},$$
$$H = (JJ_\zeta/\nu\lambda) k^{-7} \quad (87)$$

in agreement with the Heisenberg (1948) law.

XI. Generation of turbulence by gravitational instability ($\zeta = -1$)

In the unstable configuration of a heavy liquid above a lighter one (i.e., $\xi = -1$), a turbulent interface is generated by the gravitational instability. The molecular dissipations can be neglected, reducing the equation of energy balance to

$$-\int_0^k dk' \frac{[c(k')]^4 k'^2 H(k')}{\eta'(x/k')} + \eta' J^\circ + \psi^\circ = \Gamma + \varepsilon + \psi,$$
$$-\int_0^k dk' \frac{[c(k')]^4 H(k')}{\eta'(x/k')} + \eta' J_\zeta^\circ = \Gamma_\zeta + \varepsilon_\zeta \quad (88)$$

The flow of energy from the gravitational instability into the wave transfer by inertia can be more explicitly demonstrated from a differential form of (88),

$$-\frac{k^2 c^4 H}{\eta'} + \eta' 2k^2 F + J^\circ \frac{d\eta'}{dk} + \gamma F = 0,$$
$$-\frac{c^2 H}{\eta'} + \eta' 2k^2 H + J_\zeta^\circ \frac{d\eta'}{dk} = 0 \quad (89)$$

The gravitational pull in the unstable configuration provides a natural source of energy, for maintaining both spectra in their respective subranges of production, and for keeping them from being disintegrated into dissipation. Under those circumstances, the vorticity functions J° and J_ζ° controlling the eddy dissipations, can be neglected, reducing (89) to

$$-c^4 k^2 H + 2\eta'^2 k^2 F + \gamma \eta' F = 0,$$
$$-c^2 H + 2\eta'^2 k^2 H = 0 \quad (90)$$

We find the solutions

$$F = A_1 g k^{-2} \left(1 + \frac{T}{3\varrho g} k^2\right) \quad (91\,a)$$

$$H = A_1 k^{-3} \frac{1 + (T/3\varrho g) k^2}{1 + (T/\varrho g) k^2} \times \{1 + (k_\gamma/k)^{\frac{1}{2}} [1 + (T/\varrho g) k^2]^{-\frac{1}{2}}\} \quad (91\,b)$$

with

$$A_1 = \tfrac{3}{4} \quad (92\,a)$$

and

$$k_\gamma = C_1 \gamma^2 / g, \quad C_1 = \tfrac{1}{2} \quad (92\,b)$$

Here k_γ is a frictional transition wave number separating the following two regimes:

(a) *Non-frictional*, $k \gg k_\gamma$, and $T = 0$

$$H = A_1 k^{-3} \quad (93)$$

(b) *Frictional*, $k \ll k_\gamma$, and $T = 0$

$$H = A_1 k_\gamma^{\frac{1}{2}} k^{-3.5} \quad (94)$$

XII. Sea-surface turbulence

In a stable configuration ($\xi = 1$) like the sea-surface, the effect of gravity is to pull down the surface elevation, and, during this course of action, to disperse the surface liquid by raising the velocity of dispersal. In order to subscribe to the above mechanism, the same gravitational exchange function, which represents the pull, should not only play the role of building up the kinetic energy as an acceleration, but also of depleting the potential energy as a stabilizing force for an equal amount.

By neglecting the friction and dissipation we reduce (75) to

$$-\Gamma^\circ + \eta' J^\circ = -\Gamma + \varepsilon,$$
$$\Gamma^\circ + \eta' J_w^\circ = \Gamma + \varepsilon_w \quad (95)$$

In order to discern the energy flows more conveniently, we rewrite (95) in the differential form

$$-\frac{(gk)^2 H}{\omega'} + \eta' \frac{dJ^\circ}{dk} + J^\circ \frac{d\eta'}{dk} = 0,$$
$$\frac{(gk)^2 H}{\omega'} + \eta' \frac{dJ_w^\circ}{dk} + J_w^\circ \frac{d\eta'}{dk} = 0 \quad (96)$$

The spectrum of kinetic energy is in its early stage of development in k-space, in view of its regime of production, permitting the approximation $J^\circ \cong 0$. On the other hand, the spectrum of potential energy is in its later stage of development in k space, in view of its stabilization by gravity, bringing its vorticity to saturation $J_w^\circ \cong J_w$, and at the same time rendering η' negligible. As a result, the system of equations (96) simplifies to

$$-\frac{(gk)^2 H}{\omega'} + \eta' \frac{dJ^\circ}{dk} = 0 \tag{97 a}$$

and

$$\frac{(gk)^2 H}{\omega'} + J_w \frac{d\eta'}{dk} = 0 \tag{97 b}$$

yielding the solutions

$$F = J_w k^{-3} \tag{98}$$

$$H = (J_w/g^*)^2 k^{-5} \tag{99}$$

XIII. Similarity theory

We shall summarize the results of the above analytical theory in Table 1. This gives us an opportunity of outlining their fundamental mechanisms and introducing a similarity theory. For the sake of abbreviation, we shall omit the surface tension and the numerical coefficients.

We distinguish the following subranges:

A. *Production by gravitational acceleration*

This subrange exists for unstable surfaces and is absent for stable surfaces.

The frictionless case is governed by the parameter g, giving the spectra

$$F = gk^{-2}, \quad H = k^{-3} \tag{100}$$

The frictional case has the frictional wavenumber

$$k_\gamma = \gamma^2/g \tag{101}$$

as a second parameter. Since the surface elevation is opposed by the friction γ to its first power, we find

$$F = gk^{-2}, \quad H = (k_\gamma/k)^{\frac{1}{2}} k^{-3} \tag{102}$$

B. *Inertia*

The spectral laws in the inertia subrange are independent of the gravitational effects, and are, therefore, common to stable and unstable surfaces.

The frictionless laws are governed by the parameters ε and ε_ζ, according to

$$F = \varepsilon^{2/3} k^{-5/3}, \quad H = \varepsilon_\zeta \varepsilon^{-1/3} k^{-5/3} \tag{103}$$

If the friction is dominant, the kinetic energy is transferred across the spectrum to secure a balance between the friction and the saturated vorticity. The governing parameter is

$$\Omega_\gamma = J/\gamma \tag{104}$$

giving a spectrum

$$F = \Omega_\gamma k^{-3} \tag{105}$$

Since the surface spectrum should be proportional to ε_ζ according to (77b), a dimensional analysis using the parameters Ω_γ and ε_ζ yields

$$H = (\varepsilon_\zeta/\Omega_\gamma) k^{-1} \tag{106}$$

C. *Frictionless eddy dissipation by gravitational pull*

This mechanism of wave dissipation controls a stable surface only. The governing parameters are the potential vorticity J_w and the gravity g. Since the build-up of the kinetic energy occurs at the expense of the potential energy with a vorticity J_w, the spectrum of the kinetic energy is

$$F = J_w k^{-3} \tag{107}$$

By balancing the gravitational acceleration with the nonlinear eddy transfer in (97a), we find

$$\zeta = u^2/g \tag{108}$$

With the substitution of (107), we transform (108) into

$$H = (J_w/g)^2 k^{-5} \tag{109}$$

D. *Molecular dissipation*

The frictionless laws are found to be

$$F = u_\nu^2 k^{-7},$$

$$H = \mu_\lambda \mu_\nu k^{-7} \tag{110}$$

Table 1. *Spectral distributions F and H for stable and unstable surfaces*

Surfaces	Subranges: Production by gravitational acceleration — Frictionless	Production by gravitational acceleration — Frictional	Inertia — Frictionless	Inertia — Frictional	Eddy dissipation by gravitational pull (frictionless and inviscid)
Unstable surface	$F = gk^{-2}$	$F = gk^{-2}$			
Turbulence from			$F = \varepsilon^{2/3}k^{-5/3}$	$F = \Omega_\gamma^2 k^{-3}$	Absent
Taylor instability	$H = k^{-3}$	$H = (k/k_\gamma)^{-\frac{1}{2}}k^{-3}$	$H = \varepsilon_\zeta \varepsilon^{-1/3}k^{-5/3}$	$H = (\varepsilon_\zeta/\Omega_\gamma)k^{-1}$	
Stable surface					$F = J_w k^{-3}$
Sea surface turbulence	Absent				$H = k_g^2 k^{-5}$
Controlling parameters F	g	g	ε	$\Omega_\gamma = J/\gamma$	J_w
Controlling parameters H		$k_\gamma = \gamma^2/g$	$\varepsilon, \varepsilon_\zeta$	$\varepsilon_\zeta, \Omega_\gamma$	$k_g = J_w/g$

with

$$\mu_\nu = J/\nu, \quad \mu_\lambda = J_\zeta/\lambda \tag{111}$$

and agree with the Heisenberg theory (1948) for viscous dissipation.

In order to transform those viscous laws into inviscid but frictional laws, a wavenumber

$$k^* = (\gamma/\nu)^{\frac{1}{2}} \equiv (\mu_\nu/\Omega_\gamma)^{\frac{1}{2}} \tag{112}$$

has to be introduced, converting (110) into

$$F = \mu_\nu k^{-7}(k/k^*)^m,$$

$$H = \mu_\lambda \mu_\nu k^{-7}(k/k^*)^n \tag{113}$$

We see that we must have $m = 4$ and $n = 2$ to render (113) inviscid, entailing

$$F = \Omega_\gamma^2 k^{-3}, \quad H = \mu_\lambda \Omega_\gamma k^{-5} \tag{114}$$

We conclude that the above similarity considerations enable the reproduction of the results of the analytical theory.

XIV. Comparison with observations

Formula (99) of the present theory has predicted a sea-surface spectrum k^{-5} in the gravitational subrange. This result differs from the dimensional law (2) proposed by Phillips (1966).

With the use of a streaming velocity u_s of the surface as a reference velocity, we can introduce a frequency

$$n = ku_s \tag{115}$$

and convert (99) into a spectrum

$$H(n) = \beta g^2 n^{-5} \tag{116}$$

normalized to

$$\tfrac{1}{2}\langle\zeta^2\rangle = \int_0^\infty dn H(n) \tag{117}$$

with a dimensionless coefficient

$$\beta = (J_w u_s^2/g^2)^2 \tag{118}$$

The empirical law (3), as suggested by Phillips (1966), has a close appearance to the spectrum (116), but differs by the variable coefficient (118).

Observations of sea-surface fluctuations have been reported by Miyake et al. (1970), Phillips (1966), Schwartz et al. (1968) and Volkov (1968), as they are conveyed to the point of the probe by a streaming velocity u_s. If u_s is large in comparison with the phase velocity $c(k > k_m)$, their spatial fluctuations past the point of measurement will be observed as fluctuations in time, with a frequency (115) according to the Taylor hypothesis. Here k_m is the wavenumber corresponding to the peak of the spectrum preceding the gravitational subrange. Although the average phase velocity may be large, the small scales of the gravitational subrange may give a small enough $c(k > k_m)$ as to validate the Taylor hypothesis.

Molecular dissipation	
Frictionless	Frictional
$F = \mu_\nu^2 k^{-7}$	$F = \Omega_\gamma^2 k^{-3}$
$H = \mu_\lambda \mu_\nu k^{-7}$	$H = \mu_\lambda \Omega_\gamma k^{-5}$
$\mu_\nu = J/\nu$	Ω_γ
$\mu_\nu, \mu_\lambda = J_\zeta/\lambda$	$\mu_\lambda, \Omega_\gamma$

We can rewrite (99) in the form

$$\frac{dJ_\zeta^\circ}{dk} = (J_w/g)^2 k^{-3} \qquad (119)$$

with $T = 0$, suggesting a saturation vorticity

$$J_\zeta \cong (J_w/gk_m)^2 \qquad (120)$$

or equivalently

$$J_w = gk_m J_\zeta^{\frac{1}{2}} \qquad (121)$$

A further substitution of (121) reduces (118) to

$$\beta = \alpha J_\zeta \qquad (122a)$$

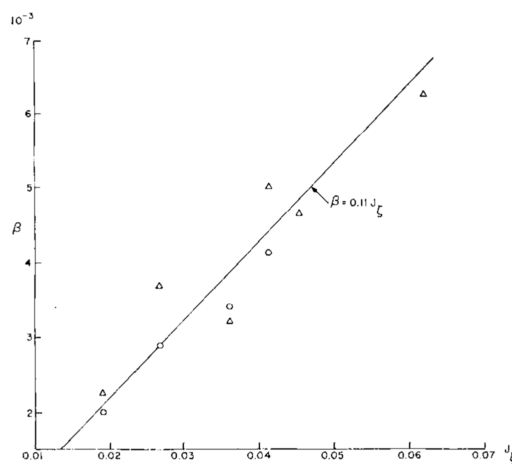

Fig. 1. Variation of coefficient β in eq. (118) with mean square slope J_ζ. The experimental points are obtained by using the spectra data reported by Volkov (1968). The solid line represents the theoretical prediction of eqs. (122a) and (122b).

with

$$\alpha = (k_m u_s^2/g)^2 \cong 0.11 \qquad (122b)$$

This numerical value is based upon empirical estimates and data by Keulegan (1951), Miyake et al. (1970) and Schwartz et al. (1968), with

$$k_m = g/u'^2, \quad u_s = 0.87u_*, \quad u' = 1.5u_* \qquad (123)$$

The mean square slope J_ζ has been measured by Wu (1972) and the spectrum (116) with the coefficient (122) has been reported by Volkov (1968) under various sea conditions. We have plotted β versus J_ζ in Fig. 1. The experimental data are found in a good agreement with the theoretical predictions (116) and (122).

XV. Conclusions

A stable surface has the k^{-5} law (99) in the subrange of eddy dissipation by gravity. The same power (86) holds in the subrange by molecular dissipation too. If the coefficients of the two formulas (86) and (99) do not differ much, the two laws will appear in a continuous succession. This explains why the 5th power law is so easily found on sea-surface turbulence.

Under the circumstances where the Taylor hypothesis is valid, the analytical law can be brought to a form (116) which has an appearance analogous to the dimensional law (3) proposed by Phillips (1966). The coefficient β, which was proposed as a universal constant in the dimensional theory, becomes a function of the dimensionless sea-surface slope, as predicted by (122). The agreement between the measured relation for β and the theoretical prediction (122) is shown to be satisfactory in Fig. 1.

The empirical laws (1) and (2), as proposed on a dimensional basis by Phillips (1966) for a sea-surface, cannot be justified analytically. On the contrary, they are justified for an unstable surface. This difficulty is inherent to the dimensional method, which does not distinguish between the disparate mechanisms of turbulence on stable and unstable surfaces.

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ГРАВИТАЦИОННАЯ ТУРБУЛЕНТНОСТЬ, СВЯЗАННАЯ С ПОВЕРХНО СТЯМИ РАЗДЕЛА

Изучаются спектральные распределения турбулентности, генерируемой гравитационными волнами на поверхности раздела между двумя жидкостями. Рассматриваются как устойчивые, так и неустойчивые поверхности. Неустойчивая поверхность связывается с ранней стадией развития турбулентности благодаря неустойчивости Тэйлора. Устойчивая поверхность может относиться к поверхности моря. Используется метод повторного каскада для замыкания иерархии корреляций в их четвертом порядке, а также для определения вихревого переноса с помощью цепочки с памятью вихревых релаксаций. Найдено, что в подынтервалах продукции, инерционном и диссипативном спектральные распределения энергии на неустойчивой поверхности с трением следуют законам κ^{-2}, κ^{-3} и κ^{-3}, а спектры возвышений поверхности — законам $\kappa^{-3.5}$, κ^{-1} и κ^{-5}. На устойчивой поверхности с трением найдено, что в подынтервалах инерционном, турбулентной диссипации путем работы против силы тяжести и молекулярной диссипации спектры кинетической энергии следуют закону κ^{-3} и законам κ^{-1}, κ^{-5} и κ^{-5} для возвышений поверхности, соответственно. Изучается также влияние эффектов поверхностного натяжения.