Comparison of accurate methods for the integration of hyperbolic equations

By HEINZ-OTTO KREISS, *Department of Computer Science, University of Uppsala, and* JOSEPH OLIGER, *International Meteorological Institute in Stockholm and National Center for Atmospheric Research,*[1] *Boulder, Colorado, USA*

(Manuscript received November 29, 1971; revised version January 25, 1972)

ABSTRACT

Historically, second order accurate difference methods have been used for computations in dynamic meteorology and oceanography. We investigate more accurate difference methods and show that fourth order methods are optimal in some sense. This method is then compared with a variant of the Fourier technique.

1. Introduction

Historically, second—or lower, order accurate difference methods have been most commonly used for computations in dynamic meteorology and oceanography. More recently some investigators have looked at more accurate methods, e.g., Arakawa (1966), Burstein & Mirin (1970), Grammeltvedt (1969), Orzag (1971), Roberts & Weiss (1966), Rusanov (1967) and Sundström (1969). Grammeltvedt has compared fourth and second order methods for the shallow water equations. However, his comparisons are clouded by the use of inappropriate boundary conditions.

The equations used in dynamic meteorology and oceanography are not, in general, hyperbolic, but the behavior of these systems is essentially hyperbolic in regions of their space-time domains.

In section 2, we examine the phase errors of second, fourth and sixth order centered schemes when used to compute approximate solutions to linear first order hyperbolic equations. Several investigators have examined the phase errors of various difference methods, e.g., Crowley (1967), Kurihara (1965), Matsuno (1966), Okland (1958), Orszag (1971), and Roberts & Weiss (1966). These investigations have been, for the most part, limited to the computation of phase errors and perhaps the tabular or graphical illustration of their different behavior. More importantly, with the exception of the works of Roberts and Weiss and Orszag, the implication of these results has not been clear. We derive conditions such that the error for the computation of a wave structure of given complexity for a given length of time is less than some given bound. These are then used to indicate the computational advantage of the fourth and sixth order schemes.

In section 3 we compute the asymptotic phase error of centered finite difference schemes as the order of accuracy is increased. We conclude that at least two points per wave length are required for a finite difference scheme of any order.

The computational work, however, increases proportionally with the order of the method. Therefore, one does not gain much by considering difference methods of higher order than four. To decrease the number of points without increasing the necessary work too much, one must use other techniques.

It has been known for some time that a more efficient representation of these problems can be obtained by Fourier transforming the differential equations. Recently, S. Orszag (1971) has developed efficient techniques for implementing these transform methods. In section 4 we investigate an alternate method where we use the Fourier transform to compute very

[1] The National Center for Atmospheric Research is sponsored by the National Science Foundation.

accurate approximations to derivatives. This has the advantage that the convolution sums which are required by Orszag's methods, when the differential equations themselves are transformed, are not needed.

In section 5 we look at the nonlinear equation $u_t = uu_x$. We first show that a quasi-conservative scheme is unstable when leap-frog time differencing is used, unless a dissipative operator is added. We then show that, for the solutions of the fourth order centered scheme, a quasi-bound of the same type as for the quasi-conservative schemes holds if we add some—but very little, dissipation. Computational results indicate that this scheme is truely stable.

In the last section several computational examples are shown which illustrate the superiority of high-accuracy methods. We include a comparison of second and fourth order schemes for the shallow water equations to illustrate the relevance of the linear theory to a nonlinear problem typical of those in dynamic meteorology and oceanography.

2. Second, fourth and sixth order schemes

We restrict our attention to the scalar equation

$$u_t = -cu_x \tag{2.1}$$

where subscripts denote partial differentiation. It is easy to extend these results to strictly hyperbolic systems of the form $u_t = Au_x$, where u is an s-dimensional vector and A is a constant, non-singular, $s \times s$ matrix which can be diagonalized and has real eigenvalues. The extension to higher space dimensions is also straightforward.

Consider the problem of computing the solution of (2.1) on the interval $0 \leqslant x \leqslant 1$, $0 \leqslant t$, with boundary conditions, $u(0, t) = u(1, t)$, and the initial function

$$u(x, 0) = f(x) = e^{i2\pi wx} \tag{2.2}$$

This problem has the solution

$$u(x, t) = e^{i2\pi w(x-ct)} \tag{2.3}$$

We now approximate (2.1) by difference methods. We ignore any errors due to discretization in time, i.e., we consider the differential-difference equation

$$\frac{\partial}{\partial t} v(x, t) = -cD_0(h)\, v(x, t) \tag{2.4}$$

where

$$D_0(h)\, v(x, t) = \frac{v(x+h, t) - v(x-h, t)}{2h}$$

which has local truncation error $O(h^2)$.

If $v(x, 0) = e^{i2\pi wx}$ then (2.4) has the solution

$$v(x, t) = e^{i2\pi w(x - c_1(w)t)} \tag{2.5}$$

where

$$c_1(w) = c\left(\frac{\sin 2\pi wh}{2\pi wh}\right) \tag{2.6}$$

The phase error, e_1, is

$$e_1(w) = 2\pi wt(c - c_1(w)) \tag{2.7}$$

A fourth order approximation is

$$\frac{\partial}{\partial t} v(x, t) = -c(\tfrac{4}{3} D_0(h) - \tfrac{1}{3} D_0(2h))\, v(x, t) \tag{2.8}$$

If, as before, $v(x, 0) = e^{i2\pi wx}$, then (2.8) has the solution

$$v(x, t) = e^{i2\pi w(x - c_2(w)t)} \tag{2.9}$$

where

$$c_2(w) = c\left(\frac{8 \sin 2\pi wh - \sin 4\pi wh}{12\pi wh}\right) \tag{2.10}$$

The phase error, e_2, is

$$e_2(w) = 2\pi wt(c - c_2(w)) \tag{2.11}$$

We now look for conditions that the solutions (2.5) and (2.9) satisfy

$$e_1(w) \leqslant e \tag{2.12}$$

$$e_2(w) \leqslant e \tag{2.13}$$

for $0 \leqslant e \leqslant \frac{1}{2}$ and $0 \leqslant t \leqslant (j/wc)$. j denotes the number of periods we want to compute in time. It is easily seen from (2.6), (2.7), (2.10) and (2.11) that e_1 and e_2 are increasing functions of t. Therefore, (2.12) and (2.13) are satisfied for $0 \leqslant t \leqslant (j/wc)$ if we choose $N = (wh)^{-1}$ such that

$$e_1(w, j) = 2\pi j\left(1 - \frac{\sin (2\pi/N)}{2\pi/N}\right) = e \tag{2.14}$$

and

$$e_2(w,j) = 2\pi j\left(1 - \frac{8\sin(2\pi/N) - \sin(4\pi/N)}{12\pi/N}\right) = e \tag{2.15}$$

N denotes the number of points per wave length.

We develop the left-hand sides of (2.14) and (2.15) in power series in $(2\pi/N)$ and retain only the terms of lowest order. Then we have

$$e_1(j, N_1) \sim \frac{(2\pi)^3}{6}\, jN_1^{-2} \tag{2.16}$$

and

$$e_2(j, N_2) \sim \frac{(2\pi)^5}{30}\, jN_2^{-4} \tag{2.17}$$

Consider N_1 and N_2 as functions of j. Let e be the maximum phase error allowed. Utilizing (2.16) and (2.17) we have

$$N_1(j) \sim 2\pi(2\pi/6e)^{1/2} j^{1/2} \tag{2.18}$$

and

$$N_2(j) \sim 2\pi(2\pi/30e)^{1/4} j^{1/4} \tag{2.19}$$

A similar computation for the sixth order scheme

$$v_t = -c(\tfrac{3}{2}D_0(h) - \tfrac{3}{5}D_0(2h) + \tfrac{1}{10}D_0(3h))\,v(t) \tag{2.20}$$

yields

$$N_3(j) \sim 2\pi(72\pi/7!\,e)^{1/6} j^{1/6} \tag{2.21}$$

If $e = 0.1$ then

$$\left.\begin{aligned} &N_1(j) \sim 20\, j^{1/2}\\ &N_2(j) \sim 7\, j^{1/4}\\ &N_3(j) \sim 5\, j^{1/6}\\ &\text{and if } e = 0.01 \text{ then}\\ &N_1(j) \sim 64\, j^{1/2}\\ &N_2(j) \sim 13\, j^{1/4}\\ &N_3(j) \sim 8\, j^{1/6} \end{aligned}\right\} \tag{2.22}$$

Observe that the operation count of the sixth order method is approximately 3/2 times that of the fourth order method. The fourth order method has approximately twice the operation count of the second order method. The table above clearly illustrates the superiority of the fourth and sixth order schemes over the second order scheme. The superiority is much more pronounced for smaller errors. However, considering the additional effort the sixth order method requires over the fourth order method, the table above illustrates that little is gained by using the sixth order scheme. However, it should be noted that if we only allow errors much smaller than $e = 0.01$ then the sixth order scheme is definitely superior to the fourth order scheme. This is a rare situation. The superiority of the higher order methods is even greater when the computations are extended over long time intervals since N_1 grows like $j^{1/2}$, N_2 like $j^{1/4}$ and N_3 like $j^{1/6}$. For large j the sixth order scheme is again superior to the fourth order scheme.

3. Asymptotic estimates for higher order methods

In this section we consider even higher order approximations to the differential operator $\partial/\partial x$.

We define the operators D_+ and D_- by

$$D_+v(x) = \frac{v(x+h) - v(x)}{h}$$

$$D_-v(x) = \frac{v(x) - v(x-h)}{h}$$

We begin with

Lemma 3.1. *The coefficients in the formal expansion*

$$\partial/\partial x = D_0(h)\sum_{\nu=0}^{\infty}(-1)^\nu \alpha_{2\nu}(h^2 D_+ D_-)^\nu \tag{3.1}$$

are given by the expansion

$$(\arcsin \tau)^2 = 2\tau^2 \sum_{\nu=0}^{\infty} \frac{\alpha_{2\nu}\, 2^{2\nu}}{2\nu+2}\tau^{2\nu} \tag{3.2}$$

Therefore,

$$\lim_{\nu\to\infty}(\alpha_{2\nu})\frac{1}{2\nu} = \tfrac{1}{2} \tag{3.3}$$

Proof. We apply (3.1) to the function $e^{i2\pi wx}$ and obtain

$$i2\pi w = \frac{i\sin 2\pi wh}{h}\sum_{\nu=0}^{\infty}\alpha_{2\nu}4^\nu \sin^{2\nu}\left(\frac{2\pi wh}{2}\right) \tag{3.4}$$

Introducing $\xi = 2\pi wh$ and observing that

$$\sin \xi = 2 \sin (\xi/2) \cos (\xi/2) = 2 \sin (\xi/2) (1 - \sin^2 (\xi/2))^{\frac{1}{2}}$$

we obtain

$$(\xi/2) (1 - \sin^2(\xi/2))^{-\frac{1}{2}} = \sin (\xi/2) \times \sum_{\nu=0}^{\infty} \alpha_{2\nu} 4^{\nu} \sin^{2\nu} (\xi/2)$$

Therefore, if $\tau = \sin (\xi/2)$,

$$\frac{1}{2} \frac{d}{d\tau} (\arcsin \tau)^2 = \arcsin \tau \, (1 - \tau^2)^{-\frac{1}{2}} = \tau \sum_{\nu=0}^{\infty} \alpha_{2\nu} 4^{\nu} \tau^{2\nu}$$

(3.2) follows easily. The radius of convergence of the power series for $(\arcsin \tau)^2$ is 1 and the coefficients one positive.

Therefore

$$\lim_{\nu \to \infty} (\alpha_{2\nu} 2^{2\nu}/2\nu) \frac{1}{2\nu} = 1$$

and (3.3) follows.

Let us now approximate the problem (2.1), (2.2) by

$$\frac{\partial v}{\partial t} = - cD^{[n]}(h) v, \; v(x, 0) = e^{i2\pi x w}$$

where

$$D^{[2m]}(h) = D_0(h) \sum_{\nu=1}^{m-1} (-1)^{\nu} \alpha_{2\nu} (h^2 D_+ D_-)^{\nu}$$

Remark. $D^{[2m]}(h)$ can also be written

$$D^{[2m]}(h) = \sum_{\nu=1}^{m} \gamma_{\nu} D_0(\nu h), \; \gamma_{\nu} = \frac{-2(-1)^{\nu}(m!)^2}{(m+\nu)!\,(m-\nu)!}$$

When $m = 1, 2, 3$ we have the second, fourth and sixth order schemes of section 2.

In this case the phase error is given by

$$e_n = 2\pi cwt \left[1 - \frac{\sin 2\pi wh}{2\pi wh} \sum_{\nu=0}^{n-1} \alpha_{2\nu} 2^{2\nu} \sin^{2\nu} \left(\frac{2\pi wh}{2}\right)\right] \quad (3.5)$$

From (3.4) and (3.5) we obtain

$$e_n = 2\pi cwt \left(\frac{\sin (2\pi wh)}{2\pi wh} \sum_{\nu=n}^{\infty} \alpha_{2\nu} 2^{2\nu} \sin^{2\nu} (\pi wh)\right)$$

We now estimate the sum on the right. Lemma 3.1 implies that for large n

$$\alpha_{2n} 2^{2n} = (1 + c_n)^{2n}, \; c_n \to 0$$

and hence

$$\sum_{\nu=n}^{\infty} \alpha_{2\nu} 2^{2\nu} \sin^{2\nu} (\pi wh) = \sum_{\nu=n}^{\infty} (1 + c_{\nu})^{2\nu} \sin^{2\nu} (\pi wh) \equiv \beta_n$$

We then have for $d_n = \max_{\nu \geq n} c_{\nu}$

$$\beta_n \leq n \sum_{\nu=n}^{\infty} (1 + d_n)^{2\nu} \sin^{2\nu} (\pi wh) \leq d (1 + d_n)^{2n} \times \sin^{2n} (\pi wh) (1 - (1 + d_n)^2 \sin^2 (\pi wh))^{-1}$$

and

$$\lim_{n \to \infty} \beta_n^{1/n} \leq \lim_{n \to \infty} (1 + d_n) \sin (\pi wh) = \sin (\pi wh)$$

Alternatively,

$$\beta_n \geq \sum_{\nu=n}^{\infty} (1 - d_n)^{2\nu} \sin^{2\nu} (\pi wh)$$

and

$$\lim_{n \to \infty} \beta_n^{1/n} \geq \sin (\pi wh)$$

Therefore,

$$\lim_{n \to \infty} \beta_n^{1/n} = \sin (\pi wh)$$

Consequently, letting $N_n = (wh)^{-1}$ denote the number of points per wave length as before, we have for every $e_n = e > 0$

$$1 = \lim_{n \to \infty} \sin \frac{\pi}{N_n}$$

and therefore that $N_n \to 2$ as $n \to \infty$. Thus we must always have at least 2 points per wave length.

Observe that the amount of work the above $2n$th order method requires is approximately n times the work of the second order method if the work is performed conventionally. (This factor can be reduced to $O(\log n)$ by using the FFT.) In light of (2.22) it is doubtful that difference methods of order greater than six have any practical advantage.

4. The Fourier method

Let N be a natural number, $h=1/(2N+1)$ and $x_\nu=\nu h$, $\nu=0,1,\ldots,2N$. Consider a 1-periodic function $v(x)$, i.e., $v(x)=v(x+1)$, whose values we know at the gridpoints x_ν, $v_\nu=v(x_\nu)$. A very accurate method of approximating $dv(x_\nu)/dx$ is to interpolate the function values $v(x_\nu)$ by a trigonometric polynomial

$$v(x)=\sum_{|w|\leqslant N}\hat{v}(w)\,e^{2\pi i w x},\quad x=x_\nu$$

$$\hat{v}(w)=h\sum_{\nu=0}^{2N}v(x_\nu)\,e^{-2\pi i w x_\nu} \tag{4.1}$$

and to differentiate this polynomial obtaining

$$\left.\frac{dv(x)}{dx}\right|_{x=x_\nu}=\sum_{|w|\leqslant N}2\pi i w\hat{v}(w)\,e^{2\pi i w x_\nu} \tag{4.2}$$

This can be achieved by two fast Fourier transforms (FFT) and $2N$ multiplications. We introduce the vectors $\mathbf{v}=(v_0,\ldots,v_{2N})'$ and $\mathbf{w}=(dv_0/dx,\ldots,dv_{2N}/dx)'$. Then we can write the above process in operator form

$$\mathbf{w}=S\mathbf{v}$$

where S is a $(2N+1)\times(2N+1)$ matrix. Let a scalar product and norm be defined by

$$(\mathbf{v},\mathbf{u})_N=\sum_{j=0}^{2N}\bar{v}_j u_j,\quad \|\mathbf{v}\|_N^2=(\mathbf{v},\mathbf{v})$$

where $\bar{v}_j$ denotes the complex conjugate of v_j.

We have

Lemma 4.1. S is skew-hermitian and $\|S\|_N=2\pi N$.

Proof. Let

$$\mathbf{e}_w=(1,e^{2\pi i w h},e^{2\pi i w(2h)},\ldots,e^{2\pi i w(2Nh)})',$$

$$w=0,\pm 1,\ldots,\pm N$$

It is obvious that

$$S\,\mathbf{e}_w=2\pi i w\,\mathbf{e}_w, \tag{4.3}$$

i.e., $2\pi i w$ are the eigenvalues and $\mathbf{e}_w$ the corresponding eigenfunctions of S.

Also

$$(e_j,e_k)=\sum_{\nu=0}^{2N}h\,e^{2\pi i(k-j)\nu h}=\begin{cases}1 & \text{if } k=j\\ 0 & \text{if } k\neq j\end{cases}$$

and, therefore, the eigenfunctions form an orthonormal basis. Observing that the eigenvalues are purely imaginary and their absolute values bounded by $2\pi N$ the lemma follows.

We now replace the differential equation

$$\partial u/\partial t=-c\,\partial u/\partial x \tag{4.4}$$

$$u(x,0)=f(x),\ u(0,t)=u(1,t)$$

by the system of ordinary differential equations

$$d\mathbf{v}/dt=-cS\mathbf{v}, \tag{4.5}$$

$$\mathbf{v}(x,0)=\mathbf{g}(x)$$

$\mathbf{g}$ is defined in the following way. Let

$$f(x)=\sum_w\hat{f}(w)\,e^{2\pi i w x}$$

and

$$g_\nu=g(x_\nu)=\sum_{|w|\leqslant N}\hat{f}(w)\,e^{2\pi i w x_\nu}$$

It follows from (4.3) that the solution of (4.5) is given by

$$\mathbf{v}(x,t)=\sum_{|w|\leqslant N}\hat{g}(w)\,e^{2\pi i w(x-ct)},\quad x=x_\nu$$

Thus the first $2N+1$ frequencies, $|w|\leqslant N$, are represented exactly.

Therefore, using this method we need only two points per wave length to represent the wave exactly, compared to seven points for the fourth order scheme allowing an error of 10%, and thirteen points allowing 1% error.

Now approximate (4.5) by the leap-frog scheme

$$\mathbf{v}(t+k)=\mathbf{v}(t-k)-2ckS\mathbf{v}(t) \tag{4.6}$$

It follows from lemma (4.1) that the approximation (4.6) is stable for $|2\pi Nck|<1$.

Since each FFT on $2N+1$ points requires approximately $N\log_2(2N)$ complex multiplications and $2N\log_2 2N$ complex additions, the number of operations per time step for (4.6) is approximately

$8N\log_2 2N$ real multiplications and

$$8N\log_2(2N)\text{ real additions} \tag{4.7}$$

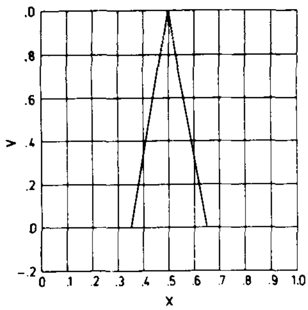

Fig. 1. $v(x, 0) = g(x)$.

The fourth order scheme requires $4N$ real multiplications and $6N$ real additions. We need approximately 4–7 times as many points for the fourth order scheme. Thus we must compare (4.7) with

$16N$–$28N$ real multiplications and

24–$42N$ real additions.

Therefore, the Fourier method is, in this case, at least as economical as a fourth order scheme as long as we compute no more than 16 wave numbers. The advantage of the Fourier method is more evident for longer time integrations. Furthermore, the storage requirements are reduced by a factor of 4–7 for every space dimension. The dissipation and data filtering problem is also much more easily handled by the Fourier method.

For equations with constant coefficients the above method is equivalent to the so-called spectral method. For equations with variable coefficients it is not. Therefore, some additional difficulties arise when this method is applied to equations with variable coefficients. Consider, for example, the equation

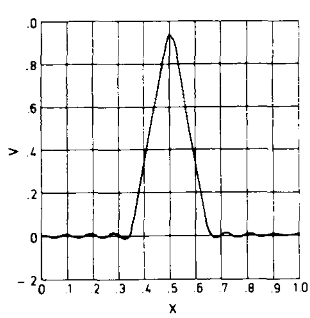

Fig. 2. $v(x, 0) = h(x)$.

$$(\partial u/\partial t) = c(x)\,\partial u/\partial x = Tu \tag{4.7}$$

Let the L_2 scalar product and norm be defined by

$$(u, v) = \int_0^1 \bar{u} v\,dx, \quad \|u\|^2 = (u, u) \tag{4.8}$$

Then (4.7) implies

$$\frac{\partial}{\partial t}\|u\|^2 = (u, Tu) + (Tu, u) = ((T + T^*)u, u)$$

$$= -\left(\mathbf{v}, \frac{\partial c}{\partial x}\mathbf{v}\right) \tag{4.9}$$

where $T^*u = -(\partial/\partial x)cu$ is the adjoint of T.

Therefore

$$(T + T^*)u = -\left(\frac{dc}{dx}\right)u$$

is a bounded operator. This is precisely the reason that the problem is well posed.

We approximate (4.7) by

$$\frac{d\mathbf{v}}{dt} = CS\mathbf{v} \tag{4.10}$$

where

$$C = \begin{pmatrix} c(x_0) & 0\ldots\ldots\ldots 0 \\ 0 & c(x_1)\ 0\ldots\ 0 \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ 0\ldots\ldots\ldots\ldots & 0\ c(x_{2N}) \end{pmatrix}$$

Then

$$\frac{d}{dt}\|\mathbf{v}\|_N^2 = (CS\mathbf{v}, \mathbf{v})_N + (\mathbf{v}, CS\mathbf{v})_N$$

$$= ((CS - SC)\mathbf{v}, \mathbf{v})_N$$

In general $CS - SC$ is not bounded independent of N. An easy calculation shows that

$$\|(CS - SC)\mathbf{v}\|_N = \|4\pi N e^{2\pi i N x} + 2\pi e^{2\pi i (N-1)x}\|_N > 4\pi N$$

if we choose $c(x) = 1 - 2\cos x$ and $\mathbf{v} = e^{2\pi i N x}$. Thus we cannot use (4.9). This difficulty is easily avoided. We write (4.7) in the form

$$\frac{\partial u}{\partial t} = \frac{1}{2}\left(c\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}(cu)\right) - \frac{1}{2}u\frac{dc}{dx}$$

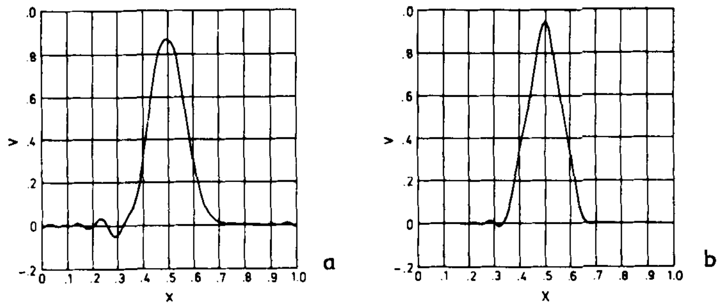

Fig. 3. Results with $v(x, 0) = g(x)$. (*a*) $v(x, 1)$ using 2nd order method; (*b*) $v(x, 1)$ using 4th order method.

and approximate it by

$$\frac{d\mathbf{v}}{dt} = \frac{1}{2}(CS + SC)\,\mathbf{v} - \frac{1}{2}\mathbf{v}\frac{d}{dx}C$$

$CS + SC$ is skew-hermitian and therefore

$$\frac{d}{dt}\|\mathbf{v}\|_N^2 = -\left(\mathbf{v}, \mathbf{v}\frac{dc}{dx}\right)$$

which is the same equality as (4.9).

There are other better ways to stabilize this method which will be discussed in a subsequent paper.

It is at the present time not clear what the accuracy of the Fourier method is for equations with variable coefficients, particularly when discontinuities are present. Some preliminary calculations have shown that if the solution is discontinuous, then the number of necessary frequencies must be increased substantially.

5. A nonlinear equation

In this section we look at a centered fourth order scheme for the quasi-linear equation

$$u_t - uu_x = 0 \qquad (5.1)$$

Quasi-conservative schemes, such as those developed by A. Arakawa (1966), are popular for the integration of equations of this type. Such schemes have been noted to be less prone to instabilities. The scheme

$$v_j(t+k) = v_j(t-k) + \frac{k}{3h}(v_{j+1}(t) + v_j(t) + v_{j-1}(t)) \times (v_{j+1}(t) - v_{j-1}(t)) \qquad (5.2)$$

with truncation error $0(h^2 + k^2)$ is of this type. The subscript j denotes the space coordinate $x_j = jh$. It can be shown that the differential-difference equation

$$v_t = \frac{1}{6h}(v_{j+1}(t) + v_j(t) + v_{j-1}(t))\,(v_{j+1}(t) - v_{j-1}(t)) \qquad (5.3)$$

conserves first and second moments.

If (5.3) is differenced in time using the implicit Crank–Nicolson method, the resulting difference method is conservative and stable. However, it should be noted that the scheme

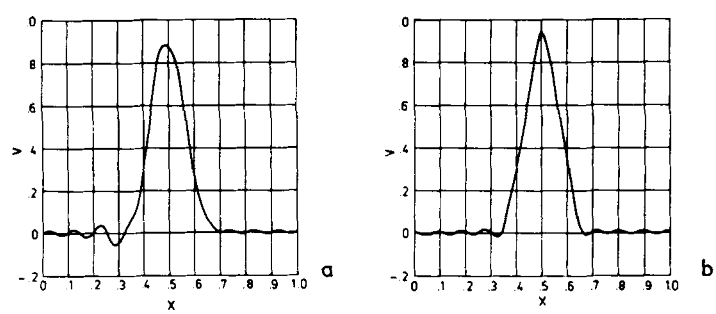

Fig. 4. Results with $v(x, 0) = h(x)$. (*a*) $v(x, 1)$ using 2nd order method; (*b*) $v(x, 1)$ using 4th order method.

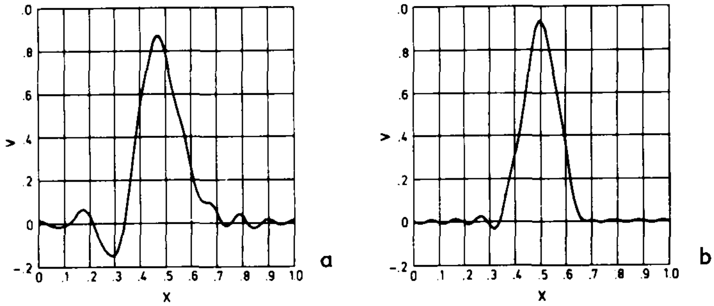

Fig. 5. Results with $v(x, 0) = h(x)$. (*a*) $v(x, 5)$ using 2nd order method; (*b*) $v(x, 5)$ using 4th order method.

(5.2) is not stable. Consider the following sections of the vectors $v(0)$ and $v(k)$:

$$\begin{array}{lrr|rrrr|} v(0): & d & -c & 0 & c & -d & 0 \\ v(k): & -b & -a & 0 & a & b & 0 \\ & & & j_0 & & & j_0+3 \end{array}$$

$$\begin{array}{|rrrrr} d & -c & 0 & c & -d \\ -b & -a & 0 & a & b \end{array}$$

If (5.2) is used to calculate $v(nk)$, $n = 2, 3, \ldots$, then $v_{j_0}(nk) = v_{j_0+3}(nk) = 0$ for $n = 2, 3, \ldots$ We can therefore restrict our attention to the points $v_{j_0+1}(nk)$ and $v_{j_0+2}(nk)$. Using (5.2) to calculate $v_{j_0+1}(nk)$ and $v_{j_0+2}(nk)$ for $n = 2, 3$ we find

$$\begin{array}{lcccc} v(2k): & 0 & c' & -d & 0 \\ v(3k): & 0 & a' & b' & 0 \\ & j_0 & j_0+1 & j_0+2 & j_0+3 \end{array}$$

where, letting $\lambda = k/3h$,

$$c' = c + \lambda b(a+b)$$
$$d' = d + \lambda a(a+b)$$
$$a' = a - \lambda d'(c'-d')$$
$$b' = b - \lambda c'(c'-d')$$

If we choose $d > c > 0$ and $a > b > 0$, then $d' > c' > 0$ and $a' > b' > 0$. Since $\lambda > 0$ we also have $c' > c$, $d' > d$, $a' > a$, and $b' > b$. It easily follows from this that $v_{j_0+1}(nk)$ and $v_{j_0+2}(nk)$ grow without bound as $n \to \infty$. Therefore, the scheme (5.2) is unstable.

This result shows that a smoothing operator of some sort must be used with the scheme (5.2).

We next show that a centered fourth order scheme with the addition of a small dissipative term satisfies the inequality $\partial/\partial t \|v\|^2 \leq 0$ where v is a solution to the corresponding differential-difference equation. $\|\cdot\|$ is the L_2-norm defined by (4.8). In fact, an obvious choice of this term will establish $\partial/\partial t \|v\|^2 = 0$, the quasi-conservative condition; however, this term is not dissipative.

Since

$$\tfrac{4}{3} D_0(h) - \tfrac{1}{3} D_0(2h) = D_0(h)\left(I - \frac{h^2}{6} D_+ D_-\right)$$

where I is the identity operator, we can write the fourth order centered scheme for (5.1) as

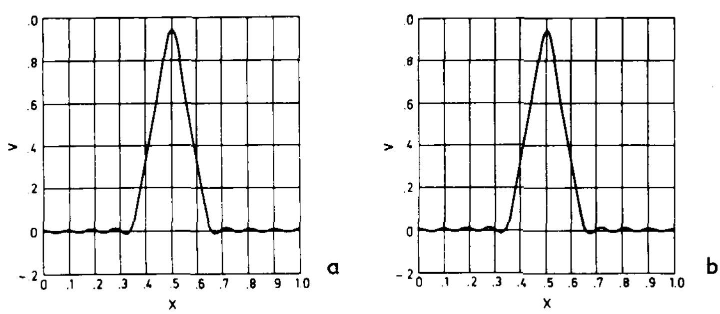

Fig. 6. Results with $v(x, 0) = h(x)$. (*a*) $v(x, 1)$ using Fourier method; (*b*) $v(x, 5)$ using Fourier method.

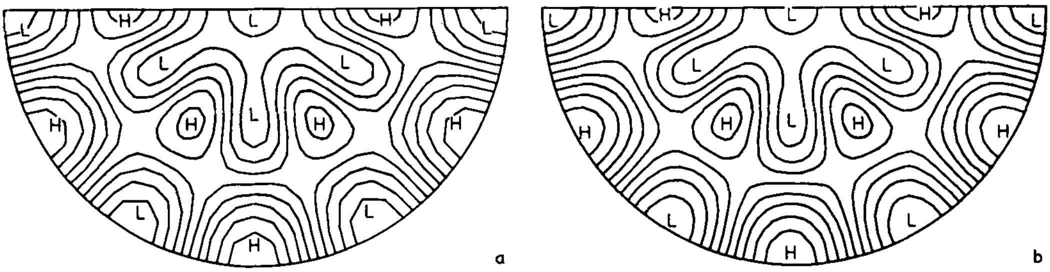

Fig. 7. u velocity component of the Haurwitz initial function. (a) 5° grid; (b) 2.5° grid.

$$v_t = vD_0\left(I - \frac{h^2}{6}D_+D_-\right)v - h^3 D_+D_-\gamma D_+D_- v$$

where γ is to be determined. We consider either the Cauchy problem or the periodic boundary problem. Using (5.4) we have

$$\frac{1}{2}\frac{\partial}{\partial t}\|v\|^2 = \left(v, v\left[D_0 - \frac{h^2}{6}D_0D_+D_-\right]v\right) - h^3(v, D_+D_-\gamma D_+D_- v)$$
$$= \left(v, v\left[D_0 - \frac{h^2}{6}D_0D_+D_-\right]v\right) - h^3(D_+D_-v, \gamma D_+D_- v)$$

Expanding the first term on the right yields

$$\left(v, v\left[D_0 - \frac{h^2}{6}D_0D_+D_-\right]v\right) = (v^2, D_0 v) - \frac{h^2}{6}(v^2, D_0D_+D_- v)$$

We now expand each of these terms

$$(v^2, D_0 v) = \tfrac{1}{3}(v^2, D_0 v) + \tfrac{2}{3}(v^2, D_0 v)$$
$$= -\tfrac{1}{3}(D_0 v^2, v) + \tfrac{2}{3}(v^2, D_0 v)$$
$$= -\frac{h^2}{3}([D_0 v][D_+D_- v], v)$$
$$= -\frac{h^2}{3}(vD_0 v, D_+D_- v)$$

and

$$-\frac{h^2}{6}(v^2, D_0D_+D_- v) = \frac{h^2}{3}(vD_0 v, D_+D_- v) + \frac{h^4}{6}(D_+D_- v, [D_0 v][D_+D_- v])$$

Combining these equalities we have

$$\frac{1}{2}\frac{\partial}{\partial t}\|v\|^2 = \frac{h^4}{6}(D_+D_- v, [D_0 v][D_+D_- v]) - h^3(D_+D_- v, \gamma D_+D_- v)$$
$$= h^3\left(D_+D_- v, \left[\gamma - \frac{h}{6}D_0 v\right]D_+D_- v\right)$$

We can conclude from this last equation that if

$$\gamma - \frac{h}{6}D_0 v \geq 0$$

then

$$\frac{\partial}{\partial t}\|v\|^2 \leq 0. \quad \gamma = \frac{h}{6}|D_0 v|$$

clearly satisfies this condition.

The work of Fornberg (1969) with the equation (5.1) using centered (leap-frog) time differencing as we are here, indicates that nonlinear instability will only occur in regions where u is nearly zero. For this reason we chose uniformly distributed sequences of random numbers between -0.1 and 0.1 as initial functions. We used the scheme (5.3) with centered time differencing, as used in (2.18) and (2.19), with periodic boundary conditions on the interval $0 \leq x \leq 1$ and $\gamma = h/6|D_0 v|$. This scheme was run to time $T = 10$ for 10 different random initial functions. In each instance the scheme was stable and the bounds on the perturbations about 0 were decreasing as the integrations progressed.

This dissipative term is quite small. The accuracy of the computations is not severely affected. Furthermore, the additional computations required by the averaging operators of quasi-conservative schemes are avoided.

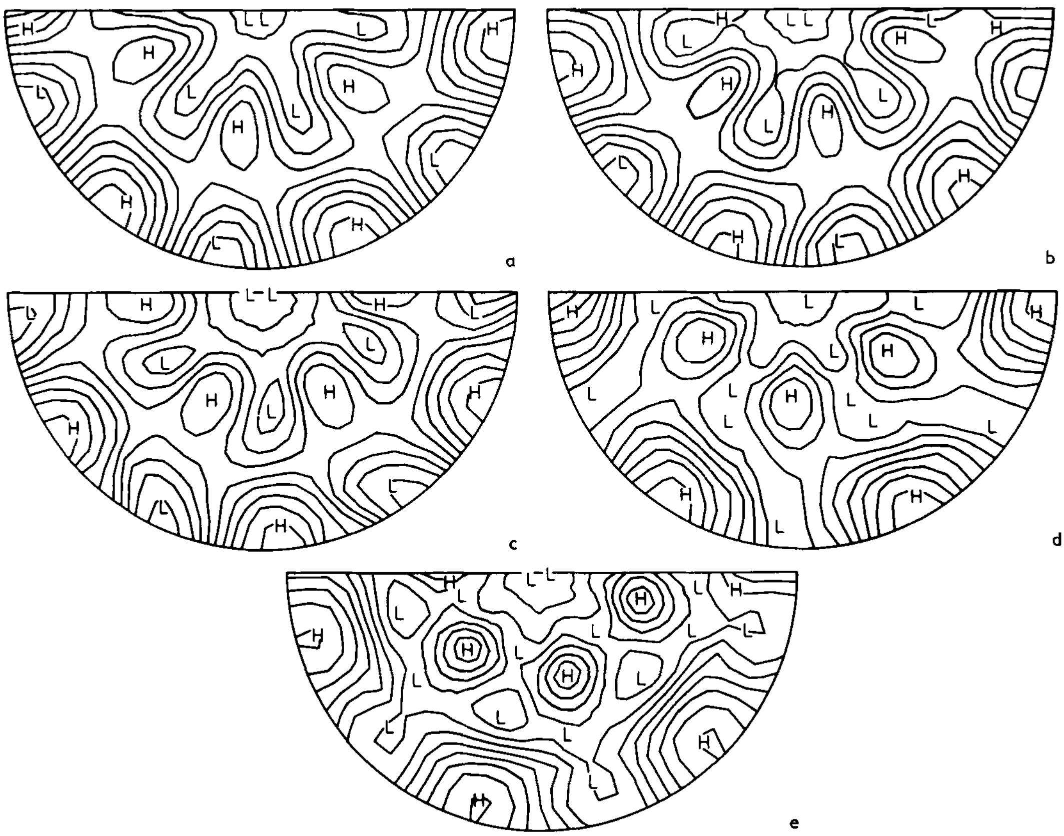

Fig. 8. u field computed using the 2nd order method on the 5° grid with $\nu = 4 \times 10^9$. (*a*) $t = 1$ day; (*b*) $t = 2$ days; (*c*) $t = 3$ days; (*d*) $t = 4$ days; (*e*) $t = 5$ days.

6. Numerical results

The color problem

To illustrate the phenomena discussed in section 2, we have used second and fourth order centered schemes to compute approximate solutions to the one-dimensional color problem. The problem:

$$u_t = -u_x, \quad 0 \leqslant x \leqslant 1, \quad 0 \leqslant t \leqslant T$$

$$u(0,t) = u(1,t)$$

$$u(x,0) = g(x)$$

$$g(x) = \begin{cases} 0 & 0 \leqslant x \leqslant 0.35 \\ \dfrac{2}{0.3}(x - 0.35) & 0.35 \leqslant x \leqslant 0.5 \\ -\dfrac{2}{0.3}(x - 0.5) + 1 & 0.5 \leqslant x \leqslant 0.65 \\ 0 & 0.65 \leqslant x \leqslant 1 \end{cases} \qquad (6.1)$$

Fig. 1 is a plot of $g(x)$. Consider the finite Fourier expansion (4.1) of $g(x)$ with $N = 50$ and truncate this series at $N' = 10$. Let $h(x)$ denote this truncated series for future reference. Fig. 2 is a plot of $h(x)$

We use the scheme

$$v(x, t+k) = v(x, t-k) - 2k\, D_0 v(x, t) + 2\varepsilon kh\, D_+ D_- v(x, t-k) \qquad (6.2)$$

for $x = jh$, $j = 0, 1, \ldots, 99$; $h = 10^{-2}$; $t = k, 2k, \ldots$; $k = \Delta t = 10^{-3}$; $v(0,t) = v(1,t)$. Equation (6.2) has local truncation error $O(h^2 + k^2)$ and corresponds to (2.4). The scheme corresponding to (2.8) with local truncation error $O(h^4 + k^2)$ is

$$v(x, t+k) = v(x, t-k) - 2k\left[\tfrac{4}{3} D_0(h) - \tfrac{1}{3} D_0(2h)\right] v(x,t) - 2\varepsilon kh^3 (D_+ D_-)^2 v(x, t-k) \qquad (6.3)$$

where x, h, t, k and $v(x,t)$ are defined as they were for the scheme (6.2.)

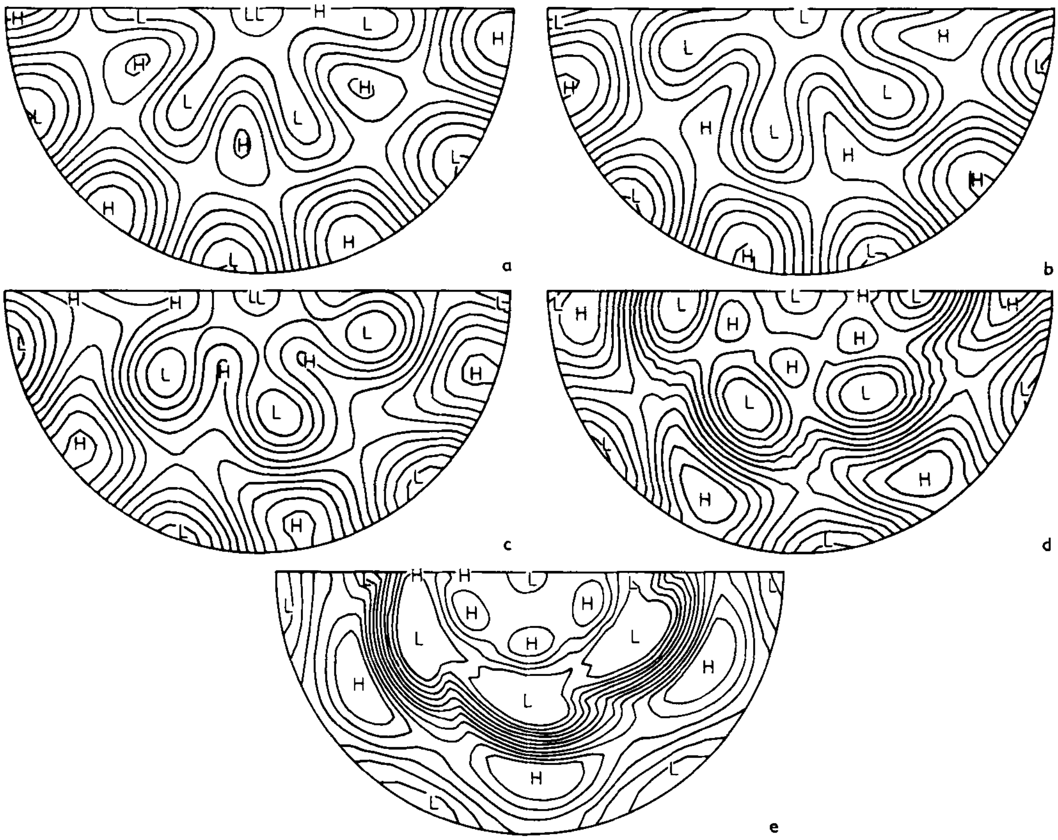

Fig. 9. u field computed using the 2nd order method on the 2.5° grid with $\nu = 1 \times 10^9$. (*a*) $t = 1$ day (*b*) $t = 2$ days; (*c*) $t = 3$ days; (*d*) $t = 4$ days; (*e*) $t = 5$ days.

Fig. 3 shows $v(x, 1)$ calculated using the schemes (6.2) and (6.3) with the initial function $g(x)$, $\varepsilon = 10^{-2}$. Fig. 4 shows $v(x, 1)$ calculated using the schemes (6.2) and (6.3) with the initial function $h(x)$, $\varepsilon = 0$. Fig 5 shows $v(x, 5)$ for the schemes (6.2) and (6.3) with the initial function $h(x)$, $\varepsilon = 0$.

To illustrate the Fourier method of section 4, we use the scheme of that section to calculate an approximate solution to the problem (6.1). Let S denote the Fourier differentiation operator of section 4. The scheme is

$$v(x, t + k) = v(x, t - k) - 2k\, S\, v(x, t) \qquad (6.4)$$

Fig. 6 shows the results of this integration using $N = 10$, $k = 10^{-3}$ and the initial function $h(x)$.

Shallow water wave equations

We compare second and fourth order schemes for the nonlinear shallow water wave equations to illustrate the relevance of the conclusions obtained using linear theory in a typical nonlinear setting.

Our integrations are carried out in spherical polar coordinates on a quarter-sphere, $0 \leqslant \lambda \leqslant \pi$, $0 \leqslant \theta \leqslant \pi/2$, where λ is longitude positive eastward and θ is latitude positive northward.

The second order scheme we use is the centered scheme used by J. Gary (1969), patterned after the scheme of W. Washington and A. Kasahara (1970). We refer the reader to Gary's paper for details. Our integrations differ slightly from those of Gary. Firstly, we have integrated the equations in their dimensional form, while Gary non-dimensionalized the equations. Due to this we include the dimensional form, of the differential and difference equations here. Secondly, we have used a slightly different grid. We use a grid which, like the grid Gary used, is not uniform in $\Delta\lambda$. However, we have altered

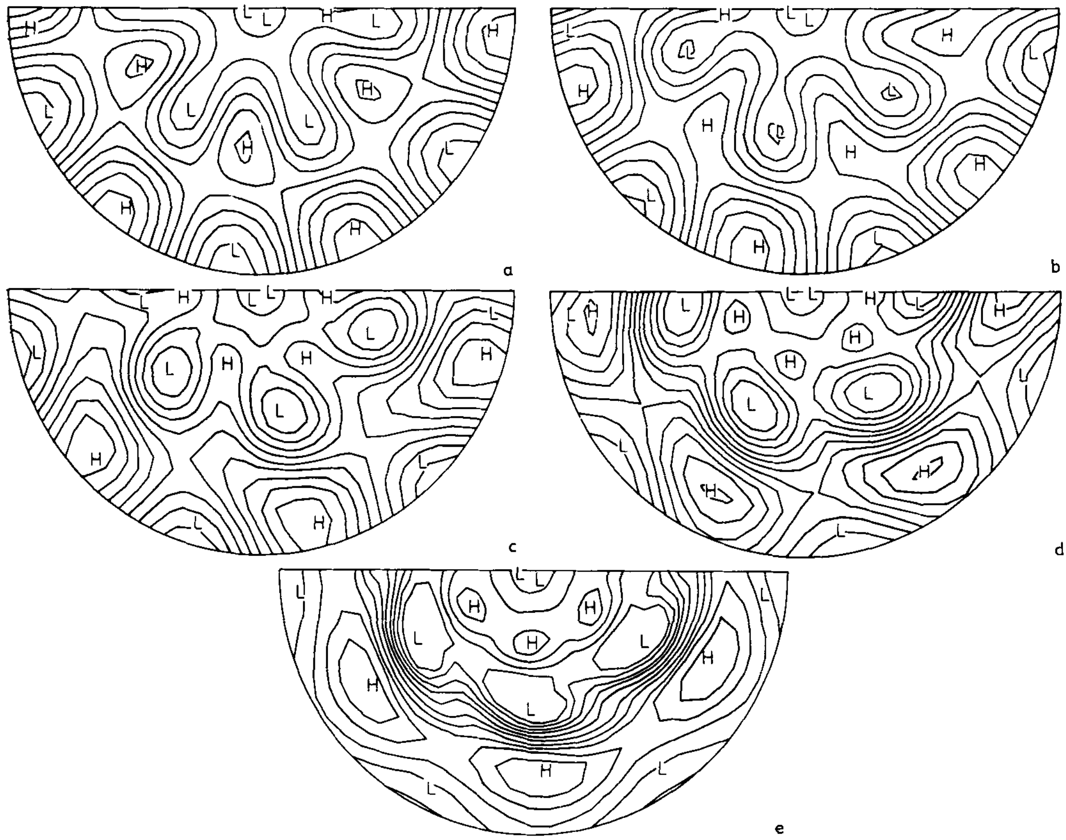

Fig. 10. u field computed using the 4th order method on the 5° grid with $\nu = 4 \times 10^9$. (*a*) $t = 1$ day; (*b*) $t = 2$ days; (*c*) $t = 3$ days; (*d*) $t = 4$ days; (*e*) $t = 5$ days.

Table 1. *Values of $N(j)$ for the two grids used for the shallow water wave computations*

Latitude	j	$N(j)$
5° grid		
90°	18	1
85°	17	4
80°	16	8
75°	15	16
70°	14	26
65°	13	36
60°	12	36
.	.	.
.	.	.
.	.	.
0°	0	36

Latitude	j	$N(j)$
2½° grid		
90°	36	1
87.5°	35	4
85°	34	8
82.5°	33	12
80°	32	16
77.5°	31	24
75°	30	32
72.5°	29	42
70°	28	52
67.5°	27	62
65°	26	72
62.5°	25	72
.	.	.
.	.	.
.	.	.
0°	0	72

the number of points on the latitude circles near the pole so that the grid is more nearly uniform in $d_j = a\,\Delta\lambda_j \cos\theta_j$, $\Delta\lambda_j = \pi/(N_j - 1)$ and N_j is the number of points on the latitude circle at $\theta_j = j\Delta\theta$. The functions N_j we use are given in Table 1.

The differential equations are

$$\frac{\partial}{\partial t}(hu) = -\frac{1}{a\cos\theta}\left[\frac{\partial}{\partial\lambda}(hu^2) + \frac{\partial}{\partial\theta}(huv\cos\theta)\right] + \left(f + \frac{u}{a}\tan\theta\right)hv - \frac{g}{a\cos\theta}\frac{\partial}{\partial\lambda}\frac{h^2}{2} \quad (6.5)$$

$$\frac{\partial}{\partial t}(hv) = \frac{1}{a\cos\theta}\left[\frac{\partial}{\partial\lambda}(huv) + \frac{\partial}{\partial\theta}(hv^2\cos\theta)\right] - \left(f + \frac{u}{a}\tan\theta\right)hu - \frac{g}{a}\frac{\partial}{\partial\theta}\frac{h^2}{2} \quad (6.6)$$

and

$$\frac{\partial}{\partial t}h = -\frac{1}{a\cos\theta}\left[\frac{\partial}{\partial\lambda}(hu) + \frac{\partial}{\partial\theta}(hv\cos\theta)\right] \quad (6.7)$$

h denotes the free surface height, and u and v are the velocity components in the λ and θ directions, respectively. The coriolis parameter, f, is given by $f = 2\Omega\sin\theta$. We have used the following constants: $\Omega = 7.292 \times 10^{-5}$ radians/sec, $a = 6.371 \times 10^{8}$ cm, $g = 980.6$ cm/sec^2.

We write the difference equations, using the difference operators defined previously, with the addition of superscripts to indicate the coordinate direction in which the operator acts. We also use superscripts and subscripts to denote particular net points, $(\cdot)^n_{i,j}$ denotes the variable $(\cdot)$ at the point $(i\Delta\lambda, j\Delta\theta, n\Delta t)$ where $\Delta\lambda$, $\Delta\theta$, Δt, denote increments in the coordinates λ, θ, and t, respectively. We write $D_0^\lambda = D_0^\lambda(\Delta\lambda)$, $D_0^\theta = D_0^\theta(\Delta\theta)$ and $D_0^t = D_0^t(\Delta t)$. We use the notations $\tilde{u}$, $\tilde{v}$ and $\tilde{h}$ to denote the solutions of the difference equations as opposed to the solutions of the differential equations.

The second order difference equations are

$$D_0^t(\tilde{h}\tilde{u})^n_{i,j} = -\frac{1}{a\cos\theta_j} \times [D_0^\lambda(\tilde{h}\tilde{u}^2)^n_{i,j} + D_0^\theta(\tilde{h}\tilde{u}\tilde{v}\cos\theta)^n_{i,j}] + \left(f_j + \frac{\overline{\tilde{u}^n_{i,j}}}{a}\tan\theta_j\right) \times \left(\frac{(\tilde{h}\tilde{v})^n_{i,j} + (\tilde{h}\tilde{v})^{n-1}_{i,j}}{2}\right) - \frac{g}{a\cos\theta_j}D_0^\lambda\left(\frac{\tilde{h}^2}{2}\right)^n_{i,j} + F_\lambda \quad (6.8)$$

$$D_0^t(\tilde{h}\tilde{v})^n_{i,j} = \frac{1}{a\cos\theta_j} \times [D_0^\lambda(\tilde{h}\tilde{u}\tilde{v})^n_{i,j} + D_0^\theta(\tilde{h}\tilde{v}^2\cos\theta)^n_{i,j}] - \left(f_j + \frac{\overline{\tilde{u}^n_{i,j}}}{a}\tan\theta_j\right) \times \left(\frac{(\tilde{h}\tilde{u})^{n+1}_{i,j} + (\tilde{h}\tilde{u})^{n-1}_{i,j}}{2}\right) - \frac{g}{a}D_0\left(\frac{\tilde{h}^2}{2}\right)^n_{i,j} + F_\theta \quad (6.9)$$

and

$$D_0^t\tilde{h}^n_{i,j} = -\frac{1}{a\cos\theta_j} \times [D_0^\lambda(\tilde{h}\tilde{u})^n_{i,j} + D_0^\theta(\tilde{h}\tilde{v}\cos\theta)^n_{i,j}] + H \quad (6.10)$$

where $\overline{\tilde{u}^n_{i,j}} = \frac{1}{4}(\tilde{u}^n_{i+1,j} + \tilde{u}^n_{i-1,j} + \tilde{u}^n_{i,j+1} + \tilde{u}^n_{i,j-1})$.

We have added the dissipative terms F_λ, F_θ and H which are defined by

$$F_\lambda = \nu D(\tilde{h}\tilde{u})^n_{i,j}$$
$$F_\theta = \nu D(\tilde{h}\tilde{v})^n_{i,j} \quad (6.11)$$
$$H = \nu D(\tilde{h})^n_{i,j}$$

where

$$Dw^n_{i,j} = \frac{1}{a^2\cos\theta_j(\Delta\theta)^2}\left[\cos\theta_{j+\frac{1}{2}}w^n_{i,j+1} - 2\cos\theta_j\cos\frac{\Delta\theta}{2}w^{n-1}_{i,j} + \cos\theta_{j-\frac{1}{2}}w^n_{i,j-1}\right] + \frac{1}{a^2\cos^2\theta(\Delta\lambda_j)^2} \times [w^n_{i+1,j} - 2w^{n-1}_{i,j} + w^n_{i-1,j}] \quad (6.12)$$

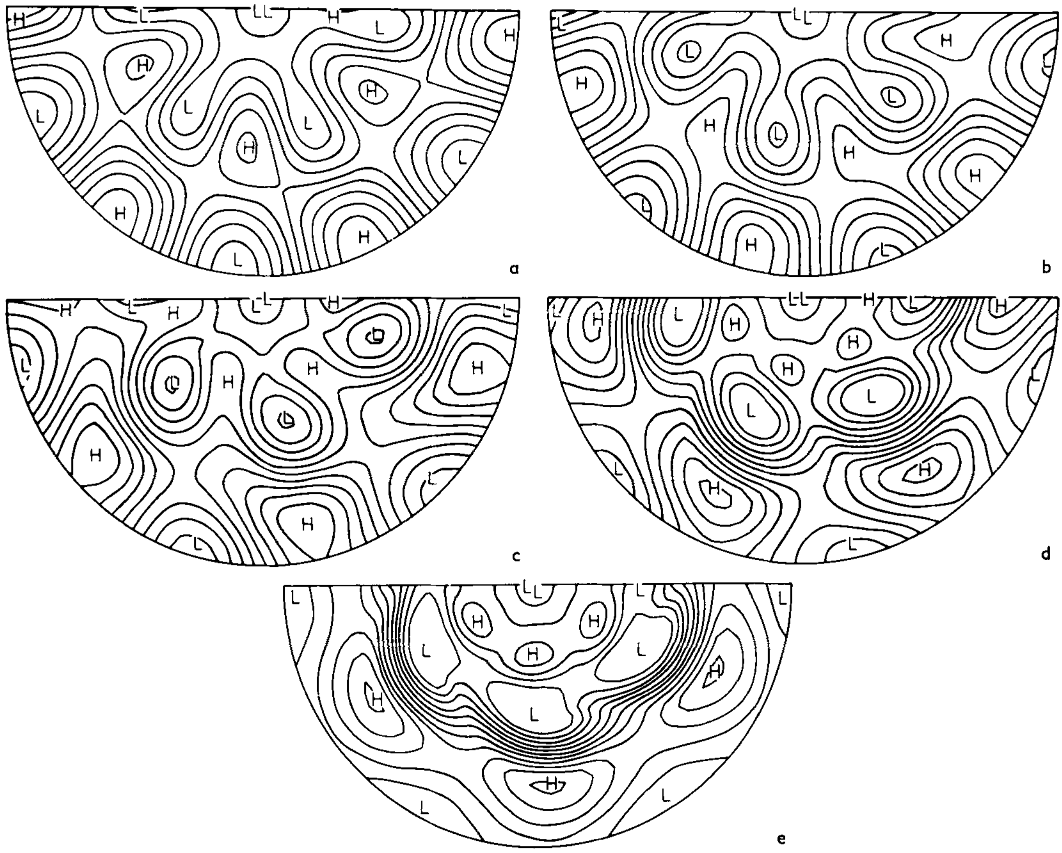

Fig. 11. u field computed using the 4th order method on the 2.5° grid with $\nu = 1 \times 10^9$. (*a*) $t = 1$ day; (*b*) $t = 2$ days; (*c*) $t = 3$ days; (*d*) $t = 4$ days; (*e*) $t = 5$ days.

D is consistent with the operator Δ defined by

$$\Delta = \frac{1}{a \cos^2 \theta} \frac{\partial}{\partial \theta} \cos \theta \frac{\partial}{\partial \theta} + \frac{1}{a^2 \cos^2 \theta} \frac{\partial^2}{\partial \lambda^2} \quad (6.13)$$

as $\Delta\lambda$, $\Delta\theta$, $\Delta t \to 0$. We note that the operator D is different than that used by Gary.

For the integrations shown here we have used linear interpolation to fill our computational stencil in the irregular portion of the grid as Gary did. The scheme is obviously only first order in this region. However, we have performed the same integrations on a net which is not staggered in time with quadratic interpolation, and the results differed only very slightly. We also performed these integrations on a grid uniform in $\Delta\lambda$ with the very small time step necessary for stability and found no significant difference. This is probably due to the fact that our initial function has a very simple structure in the polar region.

We have used the same wave number six Haurwitz wave, Haurwitz (1940), for our initial function that Gary used (see Fig. 7).

The results of our computations using the second order centered scheme are shown in Figs. 8 and 9. Fig. 8 shows the u-field of an integration using the 5° grid with $\Delta t = 300$ sec, $\nu = 4.0 \times 10^9$ cm^2 sec^{-1}. Gary concluded that $\nu = 4.0 \times 10^9$ cm^2 sec^{-1} was the smallest coefficient for which this scheme was stable for an extended integration, using this initial function. Fig. 9 shows the u-field using the $2\frac{1}{2}$° grid, $\Delta t = 150$ sec., $\nu = 1.0 \times 10^9$ cm^{2}sec^{-1}. We notice that for up to three days the approximate solutions are quite similar. However, at four days the two approximations differ considerably in a fundamental way. Instead of having

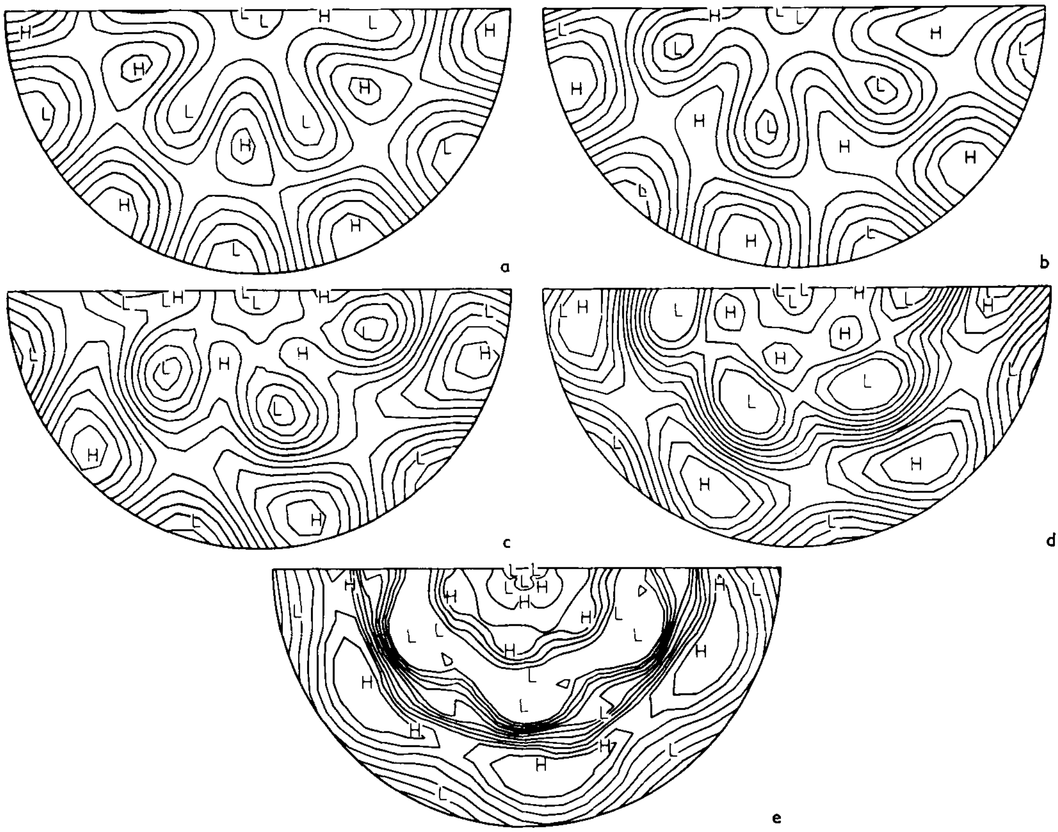

Fig. 12. u field computed using the 4th order method on the 2.5° grid with $\nu = 0$. (*a*) $t = 1$ day; (*b*) $t = 2$ days; (*c*) $t = 3$ days; (*d*) $t = 4$ days; (*e*) $t = 5$ days.

a dominant system of lows at midlatitude, the 5° case has a dominant band of highs. We believe this to be a striking departure from the solution to this problem.

We have differenced the so-called advective form of the equations (6.5) and (6.6) for our fourth order scheme. These equations are

$$\frac{\partial u}{\partial t} = -\frac{u}{a\cos\theta}\frac{\partial u}{\partial\lambda} - \frac{v}{a}\frac{\partial u}{\partial\theta} + \left(f + \frac{u}{a}\tan\theta\right) \times v - \frac{g}{a\cos\theta}\frac{\partial h}{\partial\lambda} \qquad (6.14)$$

and

$$\frac{\partial v}{\partial t} = -\frac{u}{a\cos\theta}\frac{\partial v}{\partial\lambda} - \frac{v}{a}\frac{\partial v}{\partial\theta} - \left(f + \frac{u}{a}\tan\theta\right) \times u - \frac{g}{a}\frac{\partial h}{\partial\theta} \qquad (6.15)$$

The fourth order difference equations corresponding to (6.14), (6.15) and (6.3) are

$$D_0^t \tilde{u}^n_{i,j} = -\frac{\tilde{u}^n_{i,j}}{a\cos\theta_j} d^\lambda \tilde{u}^n_{i,j} - \frac{\tilde{v}^n_{i,j}}{a} d^\theta \tilde{u}^n_{i,j} + \left(f_j + \frac{\tilde{u}^n_{i,j}}{a}\tan\theta_j\right)\left(\frac{\tilde{v}^{n+1}_{i,j} + \tilde{v}^{n-1}_{i,j}}{2}\right) - \frac{g}{a\cos\theta_j} d^\lambda \tilde{h}_{i,j} + F'_\lambda \qquad (6.16)$$

$$D_0^t \tilde{v}^n_{i,j} = -\frac{\tilde{u}^n_{i,j}}{a\cos\theta_j} d^\lambda \tilde{v}^n_{i,j} - \frac{\tilde{v}^n_{i,j}}{a} d^\theta \tilde{v}^n_{i,j} - \left(f_j + \frac{\tilde{u}^n_{i,j}}{a}\tan\theta_j\right)\left(\frac{\tilde{u}^{n+1}_{i,j} + \tilde{u}^{n-1}_{i,j}}{2}\right) - \frac{g}{a} d^\theta \tilde{h}^n_{i,j} + F'_\theta \qquad (6.17)$$

and

$$D_0^t \tilde{h}_{i,j}^n = -\frac{1}{a\cos\theta_j} \times [d^\lambda(\tilde{h}\tilde{u})_{i,j}^n + d^\theta(\tilde{h}\tilde{v}\cos\theta)_{i,j}^n] + H' \quad (6.18)$$

where

$$d^\alpha = \tfrac{4}{3}D_0^\alpha(\Delta\alpha) - \tfrac{1}{3}D_0^\alpha(2\Delta\alpha) \quad (6.19)$$

The dissipative terms F'_λ, F'_θ, and H' are defined by

$$F'_\lambda = \nu D'\tilde{u}_{i,j}^n$$

$$F'_\theta = \nu D'\tilde{v}_{i,j}^n \quad (6.20)$$

and

$$H' = \nu D'\tilde{h}_{i,j}$$

where

$$D'w_{i,j}^n = \frac{1}{a^2\cos\theta_j}\frac{1}{12(\Delta\theta)^2}\{16[\cos\theta_{j+\frac{1}{2}} \times (w_{i,j+1}^n - w_{i,j}^{n-1}) - \cos\theta_{j-\frac{1}{2}} \times (w_{i,j}^{n-1} - w_{i,j-1}^n)] - [\cos\theta_{j+1} \times (w_{i,j+2}^n - w_{i,j}^{n-1}) - \cos\theta_{j-\frac{1}{2}} \times (w_{i,j}^{n-1} - w_{i,j-2}^n)]\}$$
$$+ \frac{1}{a^2\cos^2\theta_j}\frac{1}{12(\Delta\lambda)^2}\{-w_{i-2,j}^n + 16w_{i-1,j}^n - 30w_{i,j}^{n-1} + 16w_{i+1,j}^n - w_{i+2,j}^n\} \quad (6.21)$$

D' is consistent with the operator Δ of (6.13) and has fourth order truncation error in $\Delta\lambda$ and $\Delta\theta$.

In the regions where the grid is irregular we use essentially the same procedure as for the second order scheme, but use a quartic interpolating polynomial. There is an additional difficulty: Our scheme, as written above, requires five points in the θ-direction and the equations are not defined at the pole itself. The following procedure was worked out in collaboration with David Williamson: In a small neighborhood of the pole, we denote by

$$U(x, y, t),\ V(x, y, t),$$

where

$$x = a\left(\frac{2\cos\theta}{1+\sin\theta}\right)\cos\lambda, \quad y = a\left(\frac{2\cos\theta}{1+\sin\theta}\right)\sin\lambda$$

the velocity components in polar stereographic coordinates. Then it is obvious that $U(0,0,t) \equiv V(0,0,t) \equiv 0$ and

$$U = -u\sin\lambda - v\cos\lambda$$

$$V = u\cos\lambda - v\sin\lambda$$

in all other points. Therefore, we can use the relations

$$\frac{\partial u}{\partial\theta} = \frac{\partial}{\partial\theta}(-U\sin\lambda + V\cos\lambda)$$

$$\frac{\partial v}{\partial\theta} = \frac{\partial}{\partial\theta}(-U\cos\lambda - V\sin\lambda)$$

to derive the difference approximations in the neighborhood of the pole.

Fig. 10 shows the u field computed using the fourth order scheme on the 5° grid with $\nu = 4\times10^9$ cm²sec⁻¹, $\Delta t = 300$ sec. Fig. 11 shows the u field computed by the fourth order scheme using the $2\frac{1}{2}°$ grid, $\nu = 1\times10^9$ cm² sec⁻¹, $\Delta t = 150$ sec. A comparison of Figs. 9 and 10 shows that the fourth order scheme at 5° resolution is comparable to the second order scheme using $2\frac{1}{2}°$ resolution. We have also run the fourth order scheme at 5° resolution with $\nu = 0$ for 5 days. Fig. 12 shows the u field from these computations. These computations illustrate the phenomenon discussed in section 5, i.e., the fourth order centered schemes have very nice stability properties.

Acknowledgements

This work was begun when both authors were at the National Center for Atmospheric Research (NCAR) and all the computer runs were carried out there on the NCAR Control Data Corporation 6600 and 7600 machines.

The authors wish to thank several NCAR staff members for their assistance. Gerald Browning contributed exceptional programming assistance to our color problem computations. David Williamson helped design the shallow water wave computations and assisted with the experiments. Chester Ellis coded the input–output and graphics routines for the shallow water wave computations.

REFERENCES

Arakawa, A. 1966. Computational design for long-term numerical integration of the equations of fluid motion: two-dimensional incompressible flow, part 1. *J. Comp. Phys. 1*, 119–143.

Burstein, S. Z. & Mirin, A. A. 1970. Third order difference methods for hyperbolic equations. *J. Comp. Phys. 5*, 547–571.

Crowley, W. P. 1967. Second-order numerical advection. *J. Comp. Phys. 1*, 471–484.

Fornberg, B. 1969. A study of the instability of the leap-frog approximation of a nonlinear differential equation. *Report no. 22*, Dept. of Comp. Sci., Uppsala Univ., Uppsala, Sweden, 19 pp.

Gary, J. M. 1969. A comparison of two difference schemes used for numerical weather prediction *J. Comp. Phys. 4*, 279–305.

Grammeltvedt, A. 1969. A survey of finite-difference schemes for the primative equations for a barotropic fluid. *Mon. Weath. Rev. 97*, 384–404.

Haurwitz, B. 1940. The motion of atmospheric disturbances on the spherical earth. *J. of Marine Research 3*, 254–267.

Kurihara, Y. 1965. On the use of implicit and iterative methods for the time integration of the wave equation. *Mon. Weath. Rev. 1*, 33–46.

Matsuno, T. 1966. False reflection of waves at the boundary due to the use of finite differences. *J. Met. Soc. Japan 44*, 145–157.

Økland, H. 1958. False dispersion as a source of integration errors. *Sci. Rep. No. 1*, Det Norske Met. Inst., Oslo, Norway, 14 pp.

Orszag, S. A. 1971. Numerical simulation of incompressible flows within simple boundaries: accuracy. *J. Fluid Mech. 49*, 75–112.

Roberts, K. V. & Weiss, N. O. 1966. Convective difference schemes. *Math. Comp. 20*, 272–299.

Rusanov, V. V. 1968. Difference schemes of the third order accuracy for continuous computation of discontinuous solutions. *Sov. Math. Dokl. 9*, 771–777.

Sundström, A. 1969. A truncation error reducing scheme for balanced forecast models. *Mon. Weath. Rev. 97*, 150–154.

Washington, W. & Kasahara, A. 1970. A January simulation experiment with the two-layer version of the NCAR global circulation model. *Mon. Weath. Rev. 98*, 559–579.

СРАВНЕНИЕ МЕТОДОВ РАЗЛИЧНОГО ПОРЯДКА ТОЧНОСТИ ДЛЯ ИНТЕГРИРОВАНИЯ ГИПЕРБОЛИЧЕСКИХ УРАВНЕНИЙ

Исторически в динамической метеорологии и океанографии использовались разностные методы второго порядка точности. Мы проводим тщательное исследование более аккуратных разностных методов и показываем, что методы четвертого порядка точности являются в некотором смысле оптимальными. Этот последний метод сравнивается с некоторым вариантом спектрального метода.