Simple albedo feedback models of the icecaps

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ABSTRACT

A series of simple models of the albedo feedback mechanism and its effect on the global climate are solved analytically. All of the models are similar to one considered by Budyko. The seasonal variation in incident solar radiation is ignored. Emphasis is placed on the parameter dependence of the models' sensitivity to changes in the solar constant. It is found in all cases that increasing the efficiency of the poleward transport of energy increases this sensitivity. It is also suggested that knowledge of the partitioning of the transport between the atmosphere and the oceans is of considerable importance for estimating sensitivity. The stability of equilibrium states is determined from the properties of small perturbations away from equilibrium. It is observed that relaxation times of perturbations can be increased considerably by the albedo feedback mechanism. The effect of variations in the obliquity of the planet's orbit on sensitivity and stability is also analyzed. The results indicate that albedo feedback may increase the significance of obliquity variations on Mars, as well as on the Earth.

1. Introduction

The fact that ice and snow have much larger albedos than bare soil or water must play an important role in any theory of ice ages. The sensitivity of the global climate to changes in the solar constant, orbital parameters of the Earth, atmospheric composition, or transport efficiency of the ocean-atmosphere system is enhanced by a feedback mechanism produced by this difference in albedos. The mechanism, which has been discussed frequently in the past (for example, Croll, 1897; Budyko, 1968; Eriksson, 1968; Kukla, 1972; and Budyko, 1972), works essentially as follows:

larger icecap ⇒ less solar radiation absorbed by the Earth,
⇒ cooler temperatures,
⇒ favorable conditions for further growth of icecap.

Budyko (1969) has considered a very simple model of this effect which he finds to be remarkably sensitive to changes in the solar constant. In this paper, several models similar to Budyko's are analyzed in some detail to understand this sensitivity more fully.

Despite the fact that seasonal variation may be of crucial importance for questions of climatic sensitivity, incident solar radiation has been given its annual mean values in this paper.

In our opinion, quantitative results on the sensitivity of climate derived from models as crude as those below have little significance.

2. The model

Budyko's model expresses the zonally averaged energy balance at the top of the atmosphere in terms of the zonally averaged surface temperature. When made time dependent in the most straighforward way, it takes the form:

$$C\frac{\partial T(\theta, t)}{\partial t} = Qs(\theta)\,\mathcal{A}(T(\theta, t)) - (A + BT(\theta, t)) + \gamma(\bar{T}(t) - T(\theta, t)) \quad (1)$$

$T(\theta, t)$ is the surface temperature (°C) at latitude θ. If the system is taken to be an atmosphere over a zero heat capacity surface, then C is the heat capacity at constant pressure of an atmospheric column of unit cross section. If C is increased to include some of the heat capacity

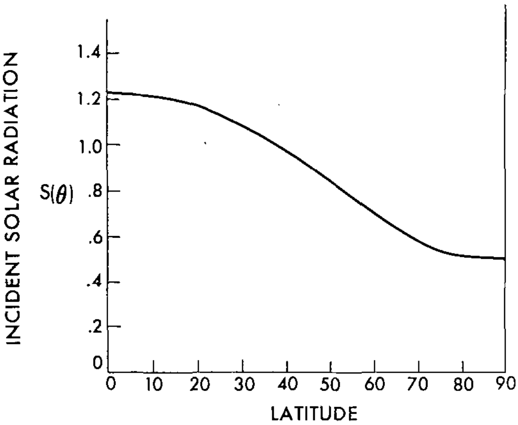

Fig. 1. Normalized, annual mean solar radiation at the top of the atmosphere at an obliquity of 23.5°.

of the oceans, all time scales are simply increased proportionately.

The first term on the right-hand side of eq. (1) is the net incoming solar radiation:

$\mathcal{A}(T) \equiv$ 1-planetary albedo,

$Qs(\theta) \equiv$ annual mean incoming solar radiation per unit area at the top of the atmosphere,

$Q \equiv$ global average of incoming radiation, $\equiv 1/4$ of the solar constant,

so that

$$\int_0^{\pi/2} s(\theta)\cos(\theta)\,d\theta = 1$$

$s(\theta)$ is symmetric about the equator, and one need consider only a single hemisphere, as above. The function $s(\theta)$ is shown in Fig. 1 for the present obliquity[1] of the earth's orbit.

The second term is the outgoing infrared radiation linearized about $T = 0°C$. A and B are constants. The third term is the energy gained due to meridional transport. γ is a constant. $\bar{T}(t)$ is defined to be the global average of surface temperature:

$$\bar{T}(t) \equiv \int_0^{\pi/2} T(\theta, t)\cos(\theta)\,d\theta$$

[1] The obliquity is the angle between the axis of rotation and the normal to the orbital plane.

Note that $\bar{T}$ is controlled primarily by temperatures in low latitudes, so that heating due to transport at high latitudes is roughly proportional to the temperature difference between high and low latitudes. This transport term provides for interaction between latitudes in a very simple way. Some such interaction, along with a temperature dependent albedo, is required for the feedback mechanism to be present in the model.

$1/B$ is a measure of the sensitivity of surface temperatures to changes in incident radiation, in the absence of albedo variations and transport. γ is a measure of the efficiency of the model in transporting energy poleward. We shall see that the sensitivity of the model is strongly dependent on the size of the dimensionless ratio $\delta \equiv \gamma/B$. Budyko (1969) chose parameters equivalent to $\delta \sim 2.4$. If one plots annual mean, zonally averaged infrared emission at different latitudes in the northern hemisphere (from Nimbus 3 measurements, Raschke et al. (1973)) versus annual mean, zonally averaged 1 000 mb temperatures (from Oort & Rasmusson (1971)), one obtains $B \sim 4.2$ ly day^{-1} °C^{-1}. From a similar plot of zonal radiation deficit versus 1 000 mb zonal mean temperatures one obtains $\gamma \sim 8.8$ ly day^{-1} °C^{-1}, or $\delta \sim 2.1$. In Section 5 we shall discuss alternate ways of evaluating δ.

Averaging eq. (1) over the hemisphere, one obtains

$$C\frac{d\bar{T}}{dt} = -(A + B\bar{T}(t)) + Q\int_0^{\pi/2} \mathcal{A}(T(\theta, t))\,s(\theta)\cos(\theta)\,d\theta \qquad (2)$$

If one fixes the albedos, perturbations in $\bar{T}$ will decay exponentially with a time constant equal to C/B, the radiative relaxation time scale. Returning to eq. (1) and holding $\bar{T}$ and the albedos fixed, perturbations in $T(\theta)$ will decay with a time constant equal to $C/(B+\gamma)$, the "energy redistribution time scale". In this time dependent framework, δ is the ratio of the dynamic and radiative contributions to the decay of local temperature perturbations.

For simplicity, we choose the albedo to be a discontinuous function of temperature:

$$\mathcal{A}(T) = \alpha \equiv 1 - \text{ice albedo}, \quad \text{for } T < T_0,$$
$$= \beta \equiv 1 - \text{ice-free albedo}, \quad \text{for } T > T_0$$

The sensitivity and stability of the resulting model are discussed in Section 3. In Section 4, we attempt to remove some unrealistic features of this model by adding a small diffusive transport term to the energy balance. The importance of the parameter δ is stressed in Section 5, where we discuss a "two-level" model which is mathematically equivalent to the model of Section 3 but which leads to different estimates of δ. The extent to which sensitivity estimates are affected by the assumption of constant "transport efficiency" is examined in Section 6, where models with linear and with non-linear diffusive transport are compared. The possible significance of obliquity variations for the size of icecaps on Earth and on Mars is discussed in Section 7.

3. Equilibrium states and perturbation analysis

We consider only equilibrium states for which $T(\theta)$ is a monotonic function of θ. Therefore,

$$T(\theta) < T_0 \text{ for } \theta > \theta_0$$
$$\text{and} \quad T(\theta) > T_0 \text{ for } \theta < \theta_0$$

for some θ_0.

In equilibrium, from eq. (2),

$$A + B\bar{T} = Q \int_0^{\pi/2} \mathcal{A}(T(\theta))\, s(\theta) \cos(\theta)\, d\theta$$
$$= \alpha Q \varphi(\vartheta_0) + \beta Q(1 - \varphi(\vartheta_0)) \qquad (3)$$

where

$$\varphi(\theta) \equiv \int_\theta^{\pi/2} s(\xi) \cos(\xi)\, d\xi.$$

Solving eq. (3) for $\bar{T}$, substituting into eq. (1) with $\partial T/\partial t \equiv 0$ and rearranging terms,

$$\frac{(1+\delta)}{Q}(A + BT(\theta)) = \mathcal{A}(T(\theta))\, s(\theta) + \delta[\alpha\varphi(\theta_0) + \beta(1 - \varphi(\theta_0))]$$

$T(\theta)$ must be discontinuous at θ_0, much like the solid line in Fig. 2.

For small ε we must have $T(\theta_0 + \varepsilon) < T_0$ and $\mathcal{A}(T(\theta_0 + \varepsilon)) = \alpha$, so that

$$\frac{(1+\delta)}{Q}(A + BT_0) > \frac{(1+\delta)}{Q}(A + BT(\theta_0 + \varepsilon))$$
$$= \alpha s(\theta_0 + \varepsilon) + \delta[\alpha\varphi(\theta_0 + \varepsilon) + \beta(1 - \varphi(\theta_0 + \varepsilon))]$$

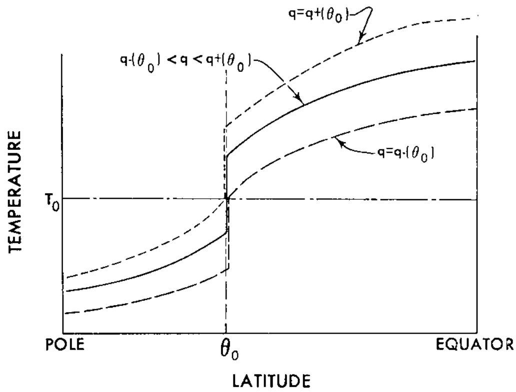

Fig. 2. Schematic representation of the latitudinal distribution of temperature for different values of q.

Similarly, $T(\theta_0 - \varepsilon) > T_0$ and $\mathcal{A}(T(\theta_0 - \varepsilon)) = \beta$, so that

$$\frac{(1+\delta)}{Q}(A + BT_0) < \frac{(1+\delta)}{Q}(A + BT(\theta_0 - \varepsilon))$$
$$= \beta s(\theta_0 - \varepsilon) + \delta[\alpha\varphi(\theta_0 - \varepsilon) + \beta(1 - \varphi(\theta_0 - \varepsilon))]$$

Taking the limit $\varepsilon \to 0$,

$$\frac{(1+\delta)}{\beta s(\theta_0) + \delta[\alpha\varphi(\theta_0) + \beta(1 - \varphi(\theta_0))]} \leqslant q$$
$$\leqslant \frac{(1+\delta)}{\alpha s(\theta_0) + \delta[\alpha\varphi(\theta_0) + \beta(1 - \varphi(\theta_0))]} \qquad (4)$$

where $q \equiv Q/(A + BT_0)$ is the normalized solar constant.

These inequalities give the range of q, $q_-(\theta_0) \leqslant q \leqslant q_+(\theta_0)$, for which equilibrium solutions to eq. (1) exist with the icecap extending down to latitude θ_0. When $q = q_-(\theta_0)$ or $q = q_+(\theta_0)$ the equilibrium temperature distribution has the form of the dotted lines labelled $q = q_-$ or $q = q_+$ in Fig. 2.

If the transport equals zero ($\gamma = \delta = 0$) all latitudes are independent of each other and no feedback is possible. Fig. 3 is a plot of $q_-(\theta_0)$ and $q_+(\theta_0)$ for $\delta = 0$, $\alpha = 0.4$, and $\beta = 0.7$. The two lines and the region between them correspond to equilibrium states of the system. For $q \geqslant 1/\beta s(\pi/2)$ an ice-free earth is a possible equilibrium state. For $q \leqslant 1/\alpha s(0)$ an ice-covered earth is a possible state. For

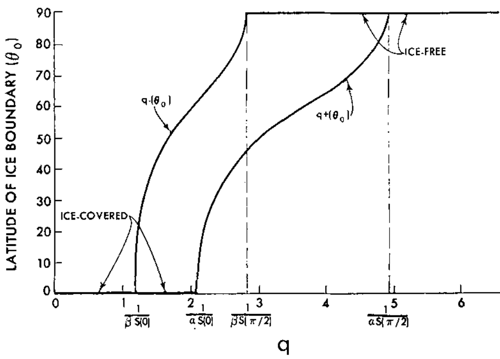

Fig. 3. Equilibrium diagram for the case with no transport $\delta = 0$ ($\gamma = 0$). The area between lines q_- and q_+ represents possible equilibrium states.

$$\frac{1}{\beta s(0)} < q < \frac{1}{\alpha s(\pi/2)}$$

there exist equilibrium states for a whole range of ice covers.

With $\delta \neq 0$, the feedback is allowed to operate. The diagram of equilibrium states for $\delta = 2.1$ (α and β as before) is shown in Fig. 4. Note that the lines q_- and q_+ turn around as functions of θ_0. For

$$q \geq \frac{(1+\delta)}{\beta\left(s\left(\frac{\pi}{2}\right) + \delta\right)}$$

the earth can be ice-free, while for

$$q \leq \frac{(1+\delta)}{\alpha(s(0)+\delta)}$$

the earth can be ice-covered.

Suppose the system is in equilibrium at some point P (or P') in Fig. 4 with an icecap extending down to $\hat{\theta}_0$, and then suppose q decreases. The icecap need not move until $q = q_-(\hat{\theta}_0)$. A further decrease in q forces the cap to search for a new equilibrium state. Budyko (1972) has argued on intuitive grounds that if

$$\left.\frac{dq_-}{d\theta_0}\right|_{\hat{\theta}_0} > 0,$$

a small perturbation of the kind described will result in a small increase in the icecap size, while if

$$\left.\frac{dq_-}{d\theta_0}\right|_{\hat{\theta}_0} < 0,$$

such a perturbation will result in unstable growth of the cap.

Similarly, for equilibrium states on the line $q = q_+(\theta_0)$, a small increase in q will result in a slight retreat of the cap if

$$\left.\frac{dq_+}{d\theta_0}\right|_{\hat{\theta}_0} > 0$$

and will result in an unstable retreat if

$$\left.\frac{dq_+}{d\theta_0}\right|_{\hat{\theta}_0} < 0.$$

These perturbations can be treated analytically and relaxation times calculated. As expected, the relaxation times become infinite at the critical latitudes where $dq_\pm/d\theta_0 = 0$.

We first consider the case in which q is decreased below $q_-(\hat{\theta}_0)$. The following set of coupled equations is derived for this case in Appendix A:

$$\frac{dT'}{d\tau} = -T' + \frac{Q(\beta-\alpha)\,s(\hat{\theta}_0)\cos(\hat{\theta}_0)}{B}\theta_0', \tag{5}$$

$$\frac{1}{(1+\delta)}\frac{d\theta_0'}{d\tau} = -\theta_0' - \frac{B\delta}{Q\beta\frac{ds}{d\theta}(\hat{\theta}_0)}T',$$

$$\tau \equiv t\Big/\left(\frac{C}{B}\right)$$

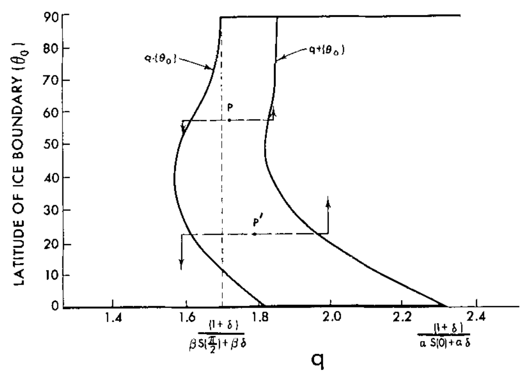

Fig. 4. A typical equilibrium diagram for the case of non-zero transport ($\delta = 2.1$). Unlike $\delta = 0$ (Fig. 3), there is a critical value of q (turn around point of line q_-) beyond which any decrease in q causes self-sustained growth of the ice to a completely ice-covered state.

where $T' \equiv T - \hat{T}$ and $\theta_0' \equiv \theta_0 - \hat{\theta}_0$ are perturbations which result from a decrease in the solar constant away from equilibrium states on the left-most line $q = q_-(\theta_0)$. The hatted quantities refer to the unperturbed equilibrium values. The interaction terms in eqs. (5) describe the feedback. In the absence of these interactions T' decays with the radiative equilibrium time scale, C/B, while θ_0' decays with the energy redistribution time scale $C/B(1+\delta)$. The response of T', to an advance of the icecap is proportional to the radiation lost to the system because of the change in albedo,

$$(\beta - \alpha) s(\hat{\theta}_0) \cos(\hat{\theta}_0) \theta_0'$$

while the response of the icecap to a decrease in $\hat{T}$ is proportional to $B\delta = \gamma$, and also is inversely proportional to the temperature gradient in front of the ice,

$$\frac{d\hat{T}}{d\theta} \propto \beta \frac{ds}{d\theta}$$

Eriksson (1968) obtains equations similar to eqs. (5) in his treatment of the ice age problem by assuming that $\hat{T}$ and θ_0 are the most important variables in the problem and working to obtain semi-empirical relations between them.

Requiring that T' and θ_0' are proportional to $e^{-\varkappa\tau}$ and solving the quadratic equation for $\varkappa$, one obtains

$$\varkappa_{\pm} = \left(1 + \frac{\delta}{2}\right) \pm \left[\left(1 + \frac{\delta}{2}\right)^2 - (1+\delta)(1-\mu_{\downarrow})\right]^{1/2} \tag{6}$$

where

$$\mu_{\downarrow} \equiv -\frac{(\beta-\alpha)}{\beta}\delta \frac{s(\hat{\theta}_0)\cos(\hat{\theta}_0)}{\frac{ds}{d\theta}(\hat{\theta}_0)} \tag{7}$$

Note that $\mu_{\downarrow} \geq 0$ so that $\varkappa_{\pm}$ are real.[1]

Also,

$$\mu_{\downarrow} < 1 \Rightarrow \varkappa_{\pm} > 0,$$

$$\mu_{\downarrow} = 1 \Rightarrow \varkappa_- = 0,$$

[1] $\mu_{\downarrow}(\hat{\theta}_0)$ is greater than zero for all $\hat{\theta}_0$ only if the obliquity is less than 45 degrees (see section 7).

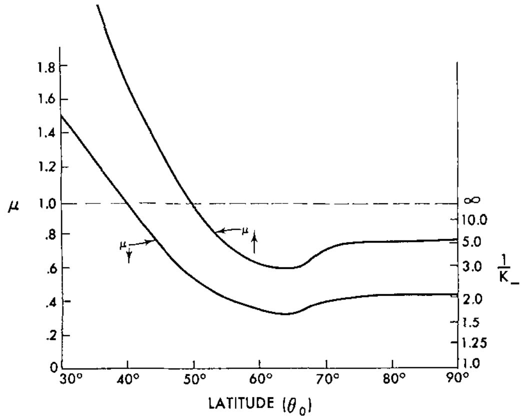

Fig. 5. The parameter $\mu_{\downarrow}$ and the relaxation time scale $1/k_-$ refer to perturbations which decrease q below q_- (Fig. 4), and $\mu_{\uparrow}$ to perturbation which increase q above line q_+ (Fig. 4).

and $\mu_{\downarrow} > 1 \Rightarrow \varkappa_- < 0 \Rightarrow$ instability.

From eq. (7) and eq. (4) we affirm that $(1-\mu_{\downarrow})$ and $dq_-/d\theta_0(\hat{\theta}_0)$ have the same sign. $\mu_{\downarrow}$ is, in fact, the product of the strengths of the two legs of the feedback mechanism: the effect of the larger icecap on the global mean temperature through the increased albedo, and the effect of lower global mean temperature on further growth of the icecap.

A plot of $\mu_{\downarrow}$ is shown in Fig. 5 for $\delta = 2.1$, $\alpha = 0.4$, and $\beta = 0.7$. We emphasize that the *shape* of this curve depends on none of these parameters but only on $s(\theta)$ which, in turn, depends only on the obliquity. As $\theta_0 \to 0$, $ds/d\theta \to 0$ and $\mu_{\downarrow} \to \infty$. Therefore, there will exist some unstable states for θ_0 sufficiently close to the equator as long as $(\beta - \alpha)\delta \neq 0$. We note also that $\mu_{\downarrow}(\hat{\theta}_0)$ has a shallow minimum at 65° independent of δ, α and β. Thus, in this simplest of models, the stablest icecap extends to 65° latitude; i.e., for an icecap of this size a decrease in solar constant results in a perturbation with the smallest possible relaxation time. As θ_0 retreats poleward, $\mu_{\downarrow}$ increases slightly and approaches a finite, non-zero limit as $\theta_0 \to 90°$. As shown in Appendix C, this limit is

$$\frac{(\beta-\alpha)}{\beta}\delta \frac{2\sin^2(\Phi)}{(1-2\sin^2(\Phi))}$$

where Φ is the obliquity. We see that the feedback mechanism can be significant even for a very small icecap, since temperature gradients are also very small near the pole.

The relaxation time, $1/\varkappa_-$, obtained from eq. (6) is also shown in Fig. 5. In the stablest region, for the parameters chosen, the time scale is increased above the radiative equilibrium time scale ($\equiv 1$) by a factor of 1.5–2. The enhancement of relaxation times by an order of magnitude occurs only within a few degrees of the critical latitude.

If the same analysis is performed for the equilibrium states on the line $q = q_+(\theta_0)$ by increasing, q, $\mu_\downarrow$ is replaced by

$$\mu_\uparrow \equiv -\frac{(\beta-\alpha)}{\alpha}\delta\frac{s(\hat{\theta}_0)\cos(\hat{\theta}_0)}{\dfrac{ds}{d\theta}(\hat{\theta}_0)},$$

since it is the temperature gradient poleward of the retreating icecap,

$$\frac{d\hat{T}}{d\theta} \propto \alpha \frac{ds}{d\theta}$$

which is now relevant. The new stability parameter, $\mu_\downarrow$, and the new relaxation times are also plotted in Fig. 5. Relaxation times are always larger for a retreating cap than for an advancing cap of the same size in this model.

In a preliminary experiment on the sensitivity of a general circulation model of the atmosphere to changes in the solar constant, Manabe and Wetherald (personal communication) have observed that the relaxation time of the model increases as the icecap is forced to advance into mid-latitudes. The simple arguments given above should help in understanding this enhancement.

4. Elimination of the region of equilibrium states

One can remove the unrealistic discontinuity in the equilibrium temperature distribution and the resulting "region" of neutrally stable equilibrium states by adding a small diffusive transport term to the energy balance. Other models such as those of Sellers (1969) and Faegre (1972), contain diffusive transport. Some of the effects of replacing the model of Section 3 with one containing only diffusive transport are considered in Section 6. In this section, however, we retain Budyko's parameterization of heating due to transport and include in addition a small amount of diffusive transport in order to consider the limit of zero diffusion—that is, in order to consider large temperature gradients at the ice boundary rather than discontinuous temperature distributions.

The energy balance equation is

$$C\frac{\partial T}{\partial t} = Qs(\theta)\,\mathcal{A}(T) - (A+BT) + \gamma(\bar{T}-T) + \frac{D}{\cos(\theta)}\frac{\partial}{\partial\theta}\left(\cos(\theta)\frac{\partial T}{\partial\theta}\right) \quad (8)$$

D is a constant. The boundary conditions are

$$\frac{\partial T}{\partial \theta}(0) = \frac{\partial T}{\partial \theta}(\pi/2) = 0$$

As shown in Appendix B, the equilibrium diagram is now determined by the equation

$$\frac{1}{q} = \sum_{\substack{l=0\\ l\ \text{even}}}^{\infty} \frac{\beta s_l - (\beta-\alpha)h_l(\theta_0)}{1+\delta(1-\Delta_{0l})+dl(l+1)} P_l(\sin(\theta_0)) \quad (9)$$

where $P_l(x)$ is the lth Legendre polynomial,

$$s_l \equiv (2l+1)\int_0^{\pi/2} s(\theta)\,P_l(\sin(\theta))\cos(\theta)\,d\theta,$$

$$h_l(\theta_0) \equiv (2l+1)\int_{\theta_0}^{\pi/2} s(\theta)\,P_l(\sin(\theta))\cos(\theta)\,d\theta,$$

$$d = D/B$$

and

$$\Delta_{l0} = \begin{cases} 1 & \text{if } l=0 \\ 0 & \text{if } l \neq 0 \end{cases}$$

The Legendre polynomials satisfy the boundary conditions naturally.

For the parameters used in Fig. 4 and with $d = 0.006$ the equilibrium diagram has the form shown in Fig. 6. Note the destabilizing effect of diffusion for a small polar icecap. As $d \to 0$ this destabilization occurs in a smaller and smaller vicinity of the pole. In fact, it is shown in Appendix B that as $d \to 0$,

$$\frac{1}{q(\theta_0)} \to \frac{1}{2}\left[\frac{1}{q_-(\theta_0)} + \frac{1}{q_+(\theta_0)}\right]$$

or

$$q(\theta_0) \to q_m(\theta_0) \equiv \frac{(1+\delta)}{\frac{\alpha+\beta}{2} s(\theta_0) + \delta[\alpha\varphi(\theta_0) + \beta(1-\varphi(\theta_0))]} \tag{10}$$

In Appendix B it is also shown that the relaxation time $1/\varkappa$, is determined by the following relation

$$\frac{B\left|\frac{dT}{d\theta}(\hat{\theta}_0)\right|}{Qs(\hat{\theta}_0)\cos(\hat{\theta}_0)(\beta-\alpha)} = \sum_{\substack{l=0\\ l \text{ even}}}^{\infty} \times \frac{(2l+1)[P_l(\sin(\hat{\theta}_0))]^2}{1+\delta(1-\Delta_{l0})+dl(l+1)-\varkappa} \tag{11}$$

Of the many solutions for $\varkappa$, we are only interested in the smallest. From the results in the Appendix one can also show that

$$\frac{1}{q^2}\frac{dq}{d\theta_0} = \frac{B\left|\frac{dT}{d\theta}(\theta_0)\right|}{Q\cos(\theta_0)} - (\beta-\alpha)s(\theta_0)\sum_{\substack{l=0\\ l \text{ even}}}^{\infty}\frac{(2l+1)[P_l(\sin(\theta_0))]^2}{1+\delta(1-\Delta_{l0})+dl(l+1)}$$

Therefore, $\varkappa = 0$ if and only if $dq/d\theta_0 = 0$. Further (see the Appendix), $\varkappa \to 0$ as $d \to 0$, i.e., as the temperature gradient at θ_0 steepens.

As $d \to 0$, this perturbation theory is only valid for smaller and smaller displacements of θ_0 since perturbations in $T(\theta)$ for all θ must remain small. To obtain results analogous to those for $d \equiv 0$—such as small restoring forces to larger displacements of θ_0 within some well-defined region in the $q-\theta_0$ plane and an asymmetry of restoring forces to the left and right of this region—one would have to consider finite amplitude perturbations in the temperature field.

In any case, if d is very small, it is intuitively clear that within the "region" the system will slowly relax to equilibrium on a time scale determined by the diffusive term, while outside the "region" the time scales will more or less be those calculated in Section 3. If d is sufficiently small one can argue that there is no physically significant distinction between true equilibrium states and those states in the "region" which are almost in equilibrium, since natural fluctuations can easily overcome the small restoring forces, and large excursions from equilibrium can be expected.

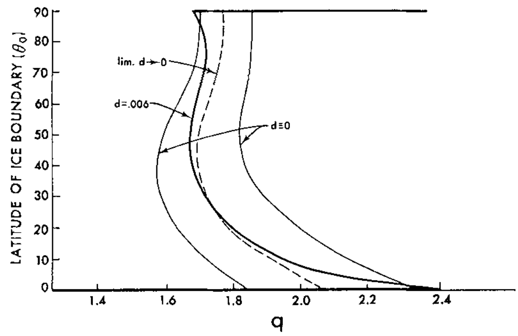

Fig. 6. Equilibrium diagram with a small diffusive transport ($d = 0.006$), the dashed line is the limit of zero diffusion, while the area between the light lines represents the equilibrium states of eq. (1) with the same parameters.

In the following discussion of sensitivity we consider only the limiting function, $q_m(\theta_0)$, defined in eq. (10).

5. Sensitivity and the parameter δ

The sensitivity of the icecap in the model discussed above is strongly influenced by the size of the parameter δ. Fig. 7 shows a plot of $q_m(\theta_0)$ for $\delta = 1.2$, 1.5, 1.8, 2.1 and 2.4. The effect of increasing δ is to reduce the amount of ice while at the same time making the icecap more sensitive to changes in solar constant. Note also that the parameter μ in Section 2 is proportional to δ; therefore, the larger δ the less stable the equilibrium states of the system. From Fig. 5 and eq. (7) we see that increasing δ moves the critical latitude poleward (the point where $\mu = 1$).

δ is a non-dimensional measure of the efficiency of the model in transporting energy poleward. The geophysical significance of this dependence of sensitivity on transport efficiency could conceivably be tested with numerical

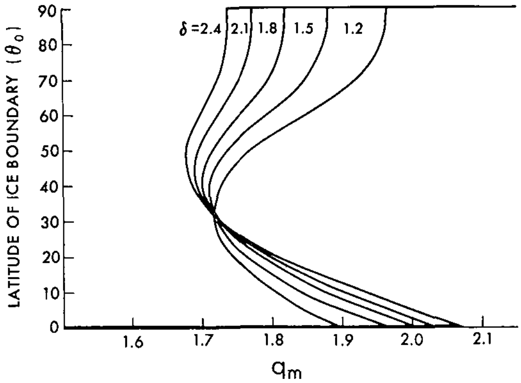

Fig. 7. The dependence of sensitivity on the parameter δ. In general an increase in δ at fixed q will cause the ice to receed and become more sensitive to changes in q. The lines plotted are $q_m(\theta_0)$ (see eq. (10)).

general circulation models of the coupled ocean–atmosphere–cryosphere system, such as Manabe (1969) and Bryan (1969). A model in which the atmospheric energy flux accounts for all of the poleward energy flux, and in which the oceans serve only as a source of water vapor, can be compared to a coupled ocean–atmosphere model in which the oceans contribute to the transport. The joint model should be more efficient transporting energy poleward and so should be more sensitive.

Although sensitivity experiments with such models have not been performed to date, examination of the one joint numerical experiment of Manabe and Bryan reveals that the situation is in reality a complex one. When oceanic transport is included in the model, warming occurs at high latitudes and cooling at low latitudes, as one would expect. However, the cooling at low latitudes extends throughout the troposphere while the warming at high latitudes is confined to the lowest few kilometers of the atmosphere. At 65° N the zonally averaged surface temperature increases 8°C while the zonally averaged 500 mb temperature increases 2°C. One suspects that the strength of the coupling of this atmospheric temperature (controlling the outgoing infrared flux) and the surface temperature (controlling the changes in albedo) will have an important effect on the sensitivity. Therefore, it is quite probable that the results from such experiments will not be understandable in terms of the simple one level model considered above.

In order to discuss this effect in the context of Budyko's model we consider the following set of two equations: the first, an energy balance of the atmosphere; the second, an energy balance at the surface,

$$0 = -(A_\uparrow + B_\uparrow T_a) - (A_\downarrow + B_\downarrow T_a) + (A_s + B_s T_s) + \gamma_a(\bar{T}_a - T_a) + H \qquad (12a)$$

$$0 = -(A_s + B_s T_s) + (A_\downarrow + B_\downarrow T_a) + Qs(\theta)\mathcal{A}(T_s) - H \qquad (12b)$$

Here

T_a	$=$ 500 mb temperature (°C)
T_s	$=$ surface temperature (°C)
$A_\uparrow + B_\uparrow T_a$	$=$ upward infrared flux at the top of the atmosphere,
$A_\downarrow + B_\downarrow T_a$	$=$ downward infrared flux reaching the surface,
$\gamma_a(\bar{T}_a - T_a)$	$=$ heating due to transport in the atmosphere
H	$=$ sensible and latent heat fluxes,
$A_s + B_s T_s$	$=$ upward infrared flux from the surface.

We take the linear relationship

$$H = \sigma(T_s - (T_a + \lambda)),$$

where λ is a constant temperature difference between 500 mb and the ground, and σ a constant of proportionality. This, of course, is not a good approximation for sensible and latent heat fluxes, but it allows us to solve the equations simply while including their effects qualitatively. The "window" radiation emitted at the ground and escaping to space, and the absorption and reflection of solar radiation by the atmosphere are ignored.

Integrating eq. (12a) over the hemisphere one can solve for $\bar{T}_a$ in terms of $\bar{T}_s$; substitution into eq. (12a) gives T_a in terms of T_s and $\bar{T}_s$. Further substituting into eq. (12b) and arranging terms gives the model of the preceeding section:

$$0 = Qs(\theta)\,\mathcal{A}(T_s) - (A + BT_s) + \gamma(\bar{T}_s - T_s)$$

where

$$A \equiv (A_s - A_\downarrow - \sigma\lambda) + \frac{(B_\downarrow + \sigma)(A_\uparrow + A_\downarrow - A_s + \sigma\lambda)}{(B_\uparrow + B_\downarrow + \sigma)},$$

$$B \equiv \frac{B_{\uparrow}(B_s+\sigma)}{(B_{\uparrow}+B_{\downarrow}+\sigma)}$$

$$\gamma \equiv \frac{\gamma_a(B_s+\sigma)(B_{\downarrow}+\sigma)}{(B_{\uparrow}+B_{\downarrow}+\sigma)(B_{\downarrow}+B_{\uparrow}+\gamma_a+\sigma)} \tag{13}$$

and so

$$\delta \equiv \frac{\gamma}{B} = \frac{\gamma_a}{B_{\uparrow}} \frac{(B_{\downarrow}+\sigma)}{(B_{\uparrow}+B_{\downarrow}+\gamma_a+\sigma)} \tag{14}$$

Note that, although eqs. (12) reduce to eq. (1), the physical interpretation of the terms given in Section 2 and under eqs. (12) result in physically distinct models. For example, while in eq. (1) we let $A + BT_s$ represent the outgoing infrared radiation at the top of the atmosphere and chose A and B accordingly, in eqs. (12) $A_{\uparrow} + B_{\uparrow} T_a$ is the outgoing infrared radiation. From the definitions (13) it is easy to verify that

$$A_{\uparrow} + B_{\uparrow} T_a = A + BT_s + \frac{B_{\uparrow}}{B_{\downarrow}} (\bar{T}_s - T_s)$$

so that part of what is now outgoing infrared radiation was previously included in the transport. We may view eqs. (12) as a distinct and yet physically meaningful way of choosing the constants in eq. (1).

More physically, the important difference between eqs. (12) and eq. (1) is that in eqs. (12) we have slightly decoupled the outgoing radiation and the surface albedo by making them functions of different temperatures, connected only by the infrared radiation and the heat flux H.

One expects the major portion of the outgoing infrared flux to be better expressed as a function of the 500 mb temperature than of the surface temperature. This is clear from Fig. 8, where, in the upper half, we plot seasonally varying outgoing infrared flux versus seasonally varying surface temperature (both zonally averaged) at different latitudes in the Northern Hemisphere. If the data at each latitude is separately fitted by a straight line, the slopes are strongly dependent on latitude. In the lower half of the figure 500 mb temperatures are used rather than surface temperatures, and the data is well fitted by one line with slope 7.0 ly day^{-1}

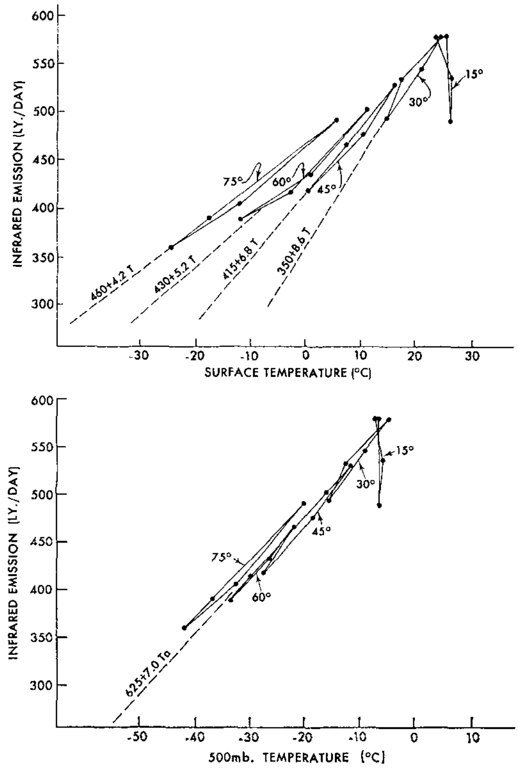

Fig. 8. Note the strong latitudinal dependence of the parameters A and B when the outgoing infrared radiation is plotted versus surface temperature (upper figure). The dependence disapears when the same values are plotted versus 500 mb temperature (lower figure). The plotted points for each latitude are zonal means at four times of the year. (Data from Raschke et al. (1973) and Oort & Rasmusson (1971).)

°C^{-1}. $B_{\uparrow}$ is therefore given this value.[1] (Data is again taken from Raschke et al. (1973) and Oort & Rasmusson (1971).) γ_a can be estimated from a plot of annual mean radiation deficit versus 500 mb temperature. We obtained $\gamma_a \sim 11$ ly day^{-1} °C^{-1}. Assuming blackbody radiation and linearizing about 0°C gives $B_s \sim 9.5$ ly day^{-1} °C^{-1}. Finally, from Smagorinsky's (1963) radiation model we may estimate $B_{\downarrow}/B_{\uparrow} \sim 1.6$.

From eq. (14) we see that in the limit $\sigma \to \infty$ ($T_s - T_a$ = const.), that is, when the coupling between the two temperatures is maximum,

[1] That this linear dependence (as opposed to the fourth power dependence in the Stefan-Boltzmann law) is a very good approximation must partly be due to the increase in absolute humidity with increasing temperature, as discussed by Manabe and Wetherald (1967).

$\gamma_a/B_\uparrow \sim 1.6$—a somewhat smaller value than that obtained in Section 2 using surface temperatures only. Decreasing σ decreases δ; therefore, in the present model, sensitivity of the icecap to solar constant variations decreases as the coupling between T_s and T_a is reduced. Trying to estimate σ makes little sense because of the crudeness of the parameterization. Without this estimate we can only say that (without including oceanic transport) the appropriate value of δ is probably less than 1.6. The difference in sensitivity of a model with Budyko's value of $\delta = 2.4$ and the value $\delta = 1.5$ is appreciable, as the reader can see from Fig. 7.

Also from eq. (14) we can see that increasing γ_a does not increase δ, or the sensitivity, indefinitely, but that δ tends to the limit

$$\delta \to \frac{B_\downarrow + \sigma}{B_\uparrow}$$

(It is important to realize that it is the non-dimensional equilibrium temperatures, $BT(\theta)/(A + BT_0)$ which are functions of the non-dimensional parameters, q, α, and β; so one can more or less fit a given temperature distribution with different values of δ as long as one varies B appropriately.)

The sharply decreased sensitivity of the two level model with atmospheric transport only is due to the inefficiency of the atmospheric transport in changing the ground temperature. Part of the transported energy is radiated away before it can affect the surface heat budget and the albedos.

To see the effectiveness of transport done at the ground in increasing the sensitivity of the model, we can include the term $\gamma_s(\bar{T}_s - T_s)$ in the surface energy balance, eq. (12b),

$$0 = -(A_s + B_s T_s) + (A_\downarrow + B_\downarrow T_a) + Qs(\theta)\mathcal{A}(T_s) - H + \gamma_s(\bar{T}_s - T_s)$$

In this case the parameter δ of the equivalent one level model is:

$$\delta = \frac{\gamma_s(B_\uparrow + B_\downarrow + \sigma)}{(B_s + \sigma) B^\uparrow} + \frac{\gamma_a}{B_\uparrow} \frac{(B_\downarrow + \sigma)}{(B_\uparrow + B_\downarrow + \gamma_a + \sigma)}$$

$\gamma_s(\bar{T}_s - T_s)$ may be viewed as a crude way to include the effects of oceanic transport in the surface energy balance. Since for realistic conditions $B_\uparrow + B_\downarrow > B_s$, an increased coupling will decrease the sensitivity due to the surface transport while increasing the sensitivity due to the atmospheric transport. Also, by increasing the transport efficiency at the surface one may increase the sensitivity indefinetely, as in the original one level model.

The coupling between "radiative" and "albedo" temperatures is also important in considering the sensitivity of the ice extent to variations in meridional transport (rather than to solar constant variations as discussed thus far). A case where the two are relatively decoupled (as the polar latitudes of the Manabe–Bryan model seem to be) would have an ice boundary more sensitive to variations in surface (i.e. oceanic) transport and less sensitive to variations in atmospheric transport than a case where the two temperatures are more strongly coupled.

These observations suggest that knowledge of the partitioning of transport between the atmosphere and oceans may be necessary in estimating the strength of the albedo feedback.

6. Linear and non-linear diffusive models

A common criticism of simple "climate" models with constant transport efficiency (i.e., constant δ in Budyko's model) is that they overestimate sensitivity by not taking into account the negative feedback inherent in the transport of energy by large scale eddies in the Earth's atmosphere (Stone, 1973). The transport efficiency of the large eddies increases as the meridional temperature gradient increases, so that meridional temperature gradients should be less sensitive to perturbations in external parameters than the results of models with constant transport efficiency suggest. We investigate the importance of this effect by comparing a model with linear diffusive transport

$$C\frac{\partial T(\theta, t)}{\partial t} = Qs(\theta)\,\mathcal{A}(T(\theta, t)) - (A + BT(\theta, t)) + \frac{1}{\cos(\theta)}\frac{\partial}{\partial\theta}\left(\cos(\theta)\,D\frac{\partial T(\theta, t)}{\partial\theta}\right), \quad (15)$$

$$D = \text{const.},$$

with a model with non-linear diffusive transport —the same as eq. (15) but with $D = D^*|dT/d\theta|$. Neither model can simulate the transport of heat due to the mean meridional circulation in the tropics.

The equilibrium diagram and the relaxation rates of perturbations away from equilibrium for the linear m odel can be obtained from eq. (9) and eq. (11) of Section 4 by setting $\delta = 0$. The sensitivity of the model is again effectively determined by the size of the "transport efficiency", $d = D/B$; the larger d the more sensitive the model. The more common "Austausch coefficient" used in the literature is

$$d \times \frac{(\text{Radius of earth})^2}{(\text{Radiative relaxation time of atmosphere})}$$

$d = 0.28$ gives a reasonable pole-to-equator temperature difference of 38.8°C when θ_0 is 65°. The equilibrium diagram for $d = 0.28$ (with $\alpha = 0.4$ and $\beta = 0.7$) is shown in Fig. 9. Note the existence of two critical latitudes, one below which the icecap grows unstably until covering the whole Earth, and the other above which the icecap recedes unstably until disappearing completely.

The non-linear diffusion model is solved by a straightforward numerical iteration procedure. The equilibrium states found are plotted in Fig. 9 for the case $d^* = D^*/B = 0.01$, which yields essentially the same pole-to-equator temperature difference, 38.5°C at $\theta_0 = 65°$, as the linear case with $d = 0.28$. In both cases, $B = 4.3$ ly day^{-1} °C^{-1}.

Except for very small caps, the two models have essentially the same sensitivity. This similarity is due to the fact that the sensitivity of these models is determined by the strength of the albedo feedback mechanism. As discussed in Section 3, the response of the icecap to perturbations in solar constant should be dependent only on the size of the icecap and the unperturbed temperature gradient at the ice boundary, the later being controlled by the strength of the transport. Very small icecaps are stabilized in the non-linear case because the local strength of diffusion is relatively small (since $dT/d\theta$ is small). For $\theta_0 \sim 50°$, the local strength of diffusion at the ice boundary is larger than in the linear case ($d^* |dT/d\theta| \sim 0.37$) and the equilibrium state is, in fact, somewhat more sensitive than the corresponding state in the linear model. Thus, non-linear diffusion affects the sensitivity primarily through its effect on the unperturbed temperature gradients.

These results suggest that for models in which the ice and snow albedo feedback controls sensitivity, the negative feedback in the transport of heat by large scale eddies does not have an important stabilizing effect, except, possibly, for very small caps. Whether the sensitivity of the Earth's climate is controlled by this albedo feedback mechanism is an open question.

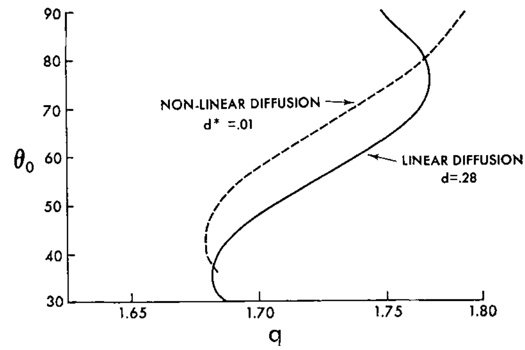

Fig. 9. The equilibrium diagrams for models with comparable magnitudes of linear and non-linear diffusion. Note that the non-linear diffusion stabilizes the polar latitudes where the temperature gradient is small.

7. Obliquity variations

Calculations indicate that the Earth's obliquity varies $\sim \pm 1.5°$ from its present value of 23.5° on time scales of the order of 40 000 years (for example, Vernekar, 1968). Since

$$s\left(\frac{\pi}{2}\right) = \frac{4}{\pi} \sin(\Phi)$$

where Φ is the obliquity, an increase in Φ of 1.5° from its present value results in an increase of $\sim 6\%$ in the radiation incident at the pole.

The strength of the albedo feedback mechanism for a small polar icecap is sensitive to the curvature of $s(\theta)$, or to the value of $s(\theta)\cos(\theta)/(ds/d\theta)$ near the pole. (See the discussion of μ in Section 3.)

In Appendix C we show that

$$-\frac{s(\theta)\cos(\theta)}{ds/d\theta} \to \frac{2\sin^2(\Phi)}{1 - 2\sin^2(\Phi)} \quad \text{as} \quad \theta \to \frac{\pi}{2}$$

From this result one can show that increasing Φ by 1.5° from 23.5° increases μ at the pole by 19%—which would seem to be a substantial decrease in stability.

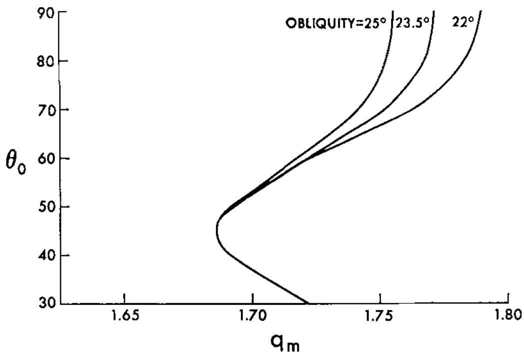

Fig. 10. Dependence of the equilibrium states on the obliquity. Like the dependence on δ an increase in obliquity at fixed q causes the ice to receed and become more sensitive to q. Again the lines plotted are $q_m(\theta_0)$ (see eq. (10)).

The effect of obliquity variations on the equilibrium diagram, $q_m(\theta_0)$ for the discontinuous model ($\delta = 2.1$, $\alpha = 0.4$, $\beta = 0.7$) is shown in Fig. 10. Changes in θ_0 due to changes in Φ (at fixed solar constant) become larger as the icecap boundary nears the pole. The changes in critical latitude are negligible.

Recent calculations by Ward (1973) indicate that the obliquity of Mars varies between the extremes of 15° and 35°, the present value being 25.1°. For $\Phi \sim 35.0°$, the curvature of $s(\theta)$ at the pole will be severely reduced, and any icecap which exists should be less stable than the present one. In an attempt to be a bit more quantitative, we refer to calculations by Stone (1972) which indicate that on Mars

$$\frac{\left.\dfrac{dT}{d\theta}\right|_{\text{no dynamics}}}{\left.\dfrac{dT}{d\theta}\right|_{\text{dynamics}}} \sim 1.3 - 1.4$$

In the models used above, this ratio equals $(1 + \delta)$ over a constant albedo surface. For arguments sake, we choose $\delta = 0.35$, $\alpha = 0.4$, $\beta = 0.85$. Then, with $\Phi = 35°$, $\mu_\downarrow = 0.31$ and $\mu_\uparrow = 0.65$ at the pole—values comparable to those on Earth.

Of course there will be important differences between models of the Martian icecaps and models of the icecaps on Earth, mostly due to the fact that the condensate on Mars is the major constituent of the atmosphere (Leighton & Murray, 1966). The considerations above are meant only to suggest that the albedo feedback mechanism may, at times, be as important on Mars as it is on Earth, despite the smaller value of δ.

Conclusions

We emphasize a few of the points brought out by the detailed analyses of the several models:

The sensitivity of the icecaps to changes in solar constant is strongly increased by the albedo feedback mechanism. The strength of this effect is dependent on the parameter δ(or d), a non-dimensional measure of the efficiency of the system in transporting energy poleward—the greater this efficiency the more sensitive the model.

A critical latitude beyond which an icecap will grow unstably until the earth is ice-covered always exists and moves poleward as the parameter δ (or d) increases.

Stability of equilibrium states is conveniently discussed in terms of relaxation times of perturbations away from equilibrium. Relaxation times are invariably increased by the feedback mechanism, particularly as one approaches the "critical latitude".

The parameter μ, which determines the stability of the icecap in the simple model of Section 3, has an appealing physical interpretation as the product of the strengths of the two legs of the feedback mechanism. The importance of the temperature gradient in front of an advancing cap or behind a retreating cap is thus emphasized.

The models which allow a discontinuous equilibrium temperature distribution have the intriguing property that, for a given solar constant, the system is capable of being in equilibrium for a whole range of ice extents. We conjecture that this "region" of neutrally stable states is replaced by a "quasi-region" of almost neutrally stable states when the temperature discontinuity is replaced by a sharp temperature gradient.

Because surface temperatures are less strongly coupled to atmospheric temperatures in polar latitudes than in lower latitudes, it appears that the vertical distribution of heating due to meridional transport, in particular, the partitioning of transport between ocean and atmosphere, must be considered when estimating the importance of albedo feedback.

Partly for this reason, estimates of the sensitivity of the terrestrial climate based on models of the type discussed above should be treated with extreme scepticism. However, the models do help us understand the albedo feedback mechanism—a mechanism which we feel must be considered when analyzing the sensitivity of the Earth's climate, and possibly the climate of Mars.

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Appendix A: Perturbation theory with a discontinuous albedo

Start from an equilibrium state with $Q/(A+BT_0) = q = q_-(\hat{\theta}_0)$ and denote the corresponding equilibrium temperatures by $\hat{T}(\theta)$:

$$0 = -(A + B\hat{T}(\theta)) + \gamma(\hat{\bar{T}} - \hat{T}(\theta)) + Qs(\theta)\mathcal{A}(\hat{T}(\theta)) \quad \text{(A 1)}$$

Now decrease Q slightly to $Q - \Delta Q$ and define $T'(\theta, t) \equiv T(\theta, t) - \hat{T}(\theta)$, $\bar{T}'(t) \equiv \bar{T}(t) - \hat{\bar{T}}$, and $\theta_0'(t) \equiv \theta_0(t) - \hat{\theta}_0$.

From eq. (1) and eq. (A 1) one has

$$c\frac{dT'}{dt} = -BT' + \gamma(\bar{T}' - T') + Qs(\theta)[\mathcal{A}(T) - \mathcal{A}(\hat{T})] - \Delta Qs(\theta)\mathcal{A}(T) \quad \text{(A 2)}$$

Averaging over the hemisphere and keeping only first order terms in the small quantities ΔQ and θ_0', one finds

$$c\frac{d\bar{T}'}{dt} = -B\bar{T}' + Q(\beta - \alpha)s(\hat{\theta}_0)\cos(\hat{\theta}_0)\theta_0' - \Delta Q[\alpha\varphi(\hat{\theta}_0) + \beta(1 - \varphi(\hat{\theta}_0))] \quad \text{(A 3)}$$

We now derive an equation for $d\theta_0'/dt$.

It is clear from eq. (A 2) that all temperatures must decrease initially—i.e., the ice must advance. Evaluate eq. (A 2) at a latitude slightly equatorward of the instantaneous position of the ice-boundary, say at $(\theta_0(t) - \varepsilon)$. The third term on the right-hand side of eq. (A 2) then equals zero since

$$\mathcal{A}(T((\theta_0(t) - \varepsilon), t)) = \mathcal{A}(\hat{T}(\theta_0(t) - \varepsilon)) = \beta$$

But we know that as the ice advances

$$\frac{d\theta_0}{dt} = -\frac{\partial T}{\partial t}(\theta_0, t) \Big/ \frac{\partial T}{\partial \theta}(\theta_0, t) \quad \text{(A 4)}$$

where

$$\left.\frac{\partial T}{\partial \theta}\right|_{\theta_0}$$

is the temperature gradient equatorward of the advancing ice. Eq. (A 4) can be derived from a simple graphical construction or from the two identities

$$\frac{dT}{dt}(\theta_0(t), t) \equiv \frac{\partial T}{\partial \theta}(\theta_0, t)\frac{d\theta_0}{dt} + \frac{\partial T(\theta_0, t)}{\partial t}$$

and

$$T(\theta_0(t), t) \equiv T_0$$

Eq. (A 4) implies, to first order in small quantities, that

$$\frac{d\theta_0'}{dt} = -\frac{\dfrac{\partial T'}{\partial t}(\hat{\theta}_0, t)}{\dfrac{\partial \hat{T}(\hat{\theta}_0)}{\partial \theta}}$$

where, from eq. (A 1),

$$\frac{d\hat{T}(\hat{\theta}_0)}{d\theta} = \frac{Q}{(1+\delta)B}\beta\frac{ds}{d\theta}(\hat{\theta}_0) \quad \text{(A 5)}$$

is the equilibrium temperature gradient equatorward of the ice boundary. Thus

$$c\left(\frac{-Q}{(1+\delta)B}\beta\frac{ds}{d\theta}(\hat{\theta}_0)\right)\frac{d\theta_0'}{dt} = -BT'(\theta) + \gamma(\bar{T}' - T'(\theta_0)) - \Delta Qs(\theta_0)\beta$$

But

$$T'(\theta_0) = T(\theta_0) - \hat{T}(\theta_0)$$
$$= T_0 - \hat{T}(\theta_0)$$
$$\approx -\frac{d\hat{T}}{d\theta}(\hat{\theta}_0)\,\theta_0'$$

where $d\hat{T}(\hat{\theta}_0)/d\theta$ is again the equatorward gradient given by eq. (A 5).

Thus,

$$C\left(\frac{-Q}{(1+\delta)B}\,\beta\frac{ds}{d\theta}(\hat{\theta}_0)\right)\frac{d\theta_0'}{dt} = Q\beta\frac{ds(\hat{\theta}_0)}{d\theta}\,\theta_0' + \gamma\bar{T}' - \Delta Q s(\hat{\theta}_0)\beta \quad \text{(A 6)}$$

The homogeneous part of the eqs. (A 3) and (A 6) reduce to the pair of eqs. (5) of Section 3, after introducing $\tau = t/(C/B)$.

Note that we have not assumed that $T'(\theta)$ is a small quantity. In fact, it is large over the (small) area which is covered with ice by the perturbation.

Appendix B: The diffusive model

(a) Equilibrium states

Write

$$s(\theta)\mathcal{A}(T) = \beta s(\theta) - (\beta-\alpha)s(\theta)H(\theta-\theta_0)$$

where

$$H(\theta-\theta_0) \equiv 1 \quad \text{if} \quad \theta > \theta_0,$$
$$\equiv 0 \quad \text{if} \quad \theta < \theta_0$$

Expanding everything in eq. (8) in Legendre polynomials (we need consider only even l)

$$s(\theta) = \sum_{l=0}^{\infty} s_l P_l(\sin(\theta)),$$

$$s(\theta)H(\theta-\theta_0) = \sum_{l=0}^{\infty} h_l(\theta_0)P_l(\sin(\theta)),$$

$$T(\theta) = \sum_{l=0}^{\infty} T_l P_l(\sin(\theta)),$$

$$\frac{1}{\cos(\theta)}\frac{\partial}{\partial\theta}\left(\cos(\theta)\frac{\partial T}{\partial\theta}\right) = -\sum_{l=0}^{\infty} l(l+1)T_l P_l(\sin(\theta))$$

Substituting into eq. (8), with $\partial T/\partial t = 0$, and solving for T_l,

$$T_l = \frac{\beta Q s_l - (\beta-\alpha)Qh_l(\theta_0) - A\Delta_{l0}}{B+\gamma(1-\Delta_{l0}) + Dl(l+1)},$$

where

$$\Delta_{l0} = 1 \quad \text{if} \quad l = 0$$
$$= 0 \quad \text{if} \quad l \neq 0$$

Therefore,

$$T_0 = \sum_{l=0}^{\infty} T_l P_l(\sin(\theta_0))$$
$$= \frac{1}{B}\sum_{l=0}^{\infty}\left[\frac{\beta Q s_l - (\beta-\alpha)Qh_l(\theta_0) - A\Delta_{l0}}{1+\delta(1-\Delta_{l0}) + dl(l+1)}\right]P_l(\sin(\theta_0))$$

where

$$d = D/B.$$

or

$$\frac{A+BT_0}{Q} = \sum_{l=0}^{\infty}\frac{\beta s_l - (\beta-\alpha)h_l(\theta_0)}{1+\delta(1-\Delta_{l0}) + dl(l+1)}P_l(\sin(\theta_0)),$$

which is eq. (9).

Now,

$$\frac{A+BT_0}{Q} = \beta - (\beta-\alpha)\varphi(\theta_0) + \sum_{l=2}^{\infty}[\dots]$$

Letting $d \to 0$, and using the theorem (Kaplan, 1959, p. 428) that the Legendre expansion of $H(\theta-\theta_0)$ converges to the value 1/2 at the point $\theta = \theta_0$, we find

$$\frac{A+BT_0}{Q} = \beta - (\beta-\alpha)\varphi(\theta_0) + \frac{1}{1+\delta}\left[\beta s(\theta_0) - \frac{(\beta-\alpha)}{2}s(\theta_0) - \beta + (\beta-\alpha)\varphi(\theta_0)\right]$$

$$= \frac{1}{1+\delta}\left[\delta(\alpha\varphi(\theta_0)+\beta(1-\varphi(\theta_0))) + \left(\frac{\alpha+\beta}{2}\right)s(\theta_0)\right]$$

$$= \frac{1}{2}\left[\frac{1}{q_-(\theta_0)}+\frac{1}{q_+(\theta_0)}\right],$$

from eq. (4).

(b) Perturbation theory

Using the variable $x \equiv \sin(\theta)$, the perturbation equation becomes

$$C\frac{\partial T'}{\partial t} = -BT' + \gamma(\bar{T}' - T') + D\frac{\partial}{\partial x}\left((1-x^2)\frac{\partial T'}{\partial x}\right) + Qs(x)\frac{d\mathcal{A}}{dT}(\hat{T}(x))\,T' \qquad (\mathrm{B}\,1)$$

where

$$\frac{d\mathcal{A}}{dT}(\hat{T}(x)) = (\beta-\alpha)\,\delta_D(\hat{T}(x)-\hat{T}_0) = \frac{(\beta-\alpha)\,\delta_D(x-\hat{x}_0)}{\left|\frac{d\hat{T}}{dx}\right|_{\hat{x}_0}}$$

and $\delta_D(x)$ is the Dirac delta function. Now use the completeness relation,

$$\delta_D(x-\hat{x}_0) = \sum_{l\ \mathrm{even}} (2l+1)\,P_l(x)\,P_l(\hat{x}_0)$$

Expand eq. (B 1) in even Legendre polynomials, take $T'_l \propto e^{-\varkappa t}$, and solve for T'_l:

$$T'_l = \left[\frac{Q(\beta-\alpha)\,s(\hat{x}_0)}{B\left|\frac{d\hat{T}}{dx}\right|_{\hat{x}_0}}\right]\frac{T'(\hat{x}_0)\,(2l+1)\,P_l(\hat{x}_0)}{1+\delta(1-\Delta_{l0})+dl(l+1)-\varkappa}$$

But $T'(\hat{x}_0) = \sum_l T'_l\,P_l(\hat{x}_0)$, so, for consistency,

$$\frac{B\left|\frac{d\hat{T}}{dx}\right|_{\hat{x}_0}}{Qs(\hat{x}_0)\,(\beta-\alpha)} = \sum_{l\ \mathrm{even}} \frac{(2l+1)\,(P_l(x_0))^2}{1+\delta(1-\Delta_{l0})+dl(l+1)-\varkappa} \qquad (\mathrm{B}\,2)$$

which is eq. (11). (The series in eq. (B 2) does not converge at the pole, where $P_l(1) \equiv 1$.)

One can show, in a rather formal way, that $\varkappa \to 0$ as $d \to 0$. Both $|d\hat{T}/dx|_{\hat{x}_0}$ and the series in eq. (B 2) become infinite in this limit. From part (*a*) of this appendix one can show that

$$\frac{B\left|\frac{d\hat{T}}{dx}\right|_{\hat{x}_0}}{Qs(\hat{x}_0)\,(\beta-\alpha)} = \frac{-1}{s(\hat{x}_0)\,(\beta-\alpha)}\sum_{\substack{l=0\\ l\ \mathrm{even}}}^{\infty}\frac{\beta s_l-(\beta-\alpha)\,h_l(\hat{x}_0)}{1+\delta+dl(l+1)}\frac{dP_l}{dx}(\hat{x}_0)$$

Therefore,

$$\lim_{d\to 0}\frac{B\left|\frac{d\hat{T}}{dx}\right|_{\hat{x}_0}}{Qs(\hat{x}_0)\,(\beta-\alpha)} \sim \frac{1}{s(\hat{x}_0)}\sum_{\substack{l=0\\ l\ \mathrm{even}}}^{\infty}\frac{h_l(\hat{x}_0)}{1+\delta+dl(l+1)}\frac{dP_l}{dx}(\hat{x}_1)$$

$$\sim \frac{1}{(1-\delta)\,s(\hat{x}_0)}\frac{d}{dx}(s(x)\,H(x-\hat{x}_0))\big|_{x=\hat{x}_0}$$

$$\sim \frac{1}{1+\delta}\,\delta_D(x-\hat{x}_0)\big|_{x=\hat{x}_0}$$

While R.H.S. of eq. (B2) $\to$

$$\frac{1}{1+\delta-\varkappa}\,\delta_D(\hat{x}_0-\hat{x}_0) \quad \text{as } d\to 0.$$

Appendix C: Curvature of $S(\theta)$ near the pole

Using the standard climatological formulae found, for example, in Sellers (1965), and assuming that the ratio

$$\frac{\text{(length of day)}}{\text{(length of year)}} \to 0$$

we have

$$s(\theta) = \sin(\Phi)\sin(\theta)\int_{-\pi}^{\pi}(H(\tau) - \tan(H(\tau)))\sin(\tau)\,d\tau \qquad (\mathrm{C}\,1)$$

where H is the "half day length",

$$\cos(H) = -\tan(\theta)\tan(\omega)$$

and ω is the solar declination,

$$\sin(\omega) = \sin(\Phi) \sin(\tau)$$

More precisely, $H = \cos^{-1}(\tan(\theta)\tan(\omega))$ when $|\tan(\theta)\tan(\omega)| < 1$, $H = \pi$ (polar day) when $\tan(\theta)\tan(\omega) > 1$, and $H = 0$ (polar night) when $\tan(\theta)\tan(\omega) < -1$.

$\tau = 0$ is the vernal equinox in the Northern Hemisphere. To lowest order in $\theta' \equiv \pi/2 - \theta$, the polar day in the North occurs when

$$\frac{\theta'}{\sin(\Phi)} < \tau < \pi - \frac{\theta'}{\sin(\Phi)}$$

and the polar night when

$$-\pi + \frac{\theta'}{\sin(\Phi)} < \tau < -\frac{\theta'}{\sin(\Phi)}$$

One can split (C1) into two integrals—the first, over the polar day, in which $H - \tan(H) = \pi$, and the second, over the remainder of the year, in which one can assume $\omega \ll 1$.

Keeping terms quadratic in θ' in both integrals, straightforward manipulation leads to the result

$$s(\theta) \approx \frac{4}{\pi} \sin(\Phi) \left[1 + \frac{\theta'^2}{2}\left(\frac{1 - 2\sin^2(\Phi)}{2\sin^2(\Phi)}\right)\right]$$

Therefore,

$$-\frac{s(\theta)\cos(\theta)}{ds/d\theta} \to \frac{2\sin^2(\Phi)}{1 - 2\sin^2(\Phi)} \quad \text{as} \quad \theta \to \frac{\pi}{2}$$

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МОДЕЛИ ПОЛЯРНЫХ ШАПОК, ИСПОЛЬЗУЮЩИЕ АЛЬБЕДО КАК МЕХАНИЗМ ОБРАТНОЙ СВЯЗИ

Аналитически исследованы несколько простых моделей, использующих альбедо как механизм обратной связи. Рассмотрен эффект этого механизма на глобальный климат. Прототипом для всех моделей явилась модель Будыко. Сезонные изменения падающей солнечной радиации не учитываются. Ударение делается на параметрическую зависимость чувствительности модели к изменениям солнечной постоянной. Во всех случаях найдено, что возрастание эффективности переноса энергии к полюсу повышает чувствительность. Предполагается также, что для оценки чувствительности существенное значение имеет соотношение вкладов атмосферы и океана в этот перенос.

Из анализа малых возмущений определяется устойчивость состояния равновесия. Найдено также, что альбедо существенно повышает время релаксации возмущений. Анализируется эффект переменности наклона орбиты к плоскости эклиптики. Результаты показывают, что альбедо может усиливать значение переменности наклона орбиты к плоскости эклиптики как для Марса, так и для Земли.