Comparison of two methods of astrogeodetic geoid determination based on least squares prediction and collocation

By S. HEITZ, *Institute for Applied Geodesy, Frankfurt am Main, Federal Republic of Germany*, and C. C. TSCHERNING, *Geodetic Institute, Copenhagen, Denmark*

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ABSTRACT

Two least squares methods used for astrogeodetic geoid determinations are described and discussed with regard to the possibilities of their practical application. The first one is based on a least squares prediction model for the geoid, namely, a linear autocorrelation of the geoid heights. The deflections of the vertical derived from this model yield observation equations for the determination of free coefficients of the linear autocorrelation by an adjustment by the method of least squares. In the second case, collocation is used as introduced in physical geodesy by T. Krarup. By applying this method, the geoid heights are directly represented by their linear correlation with the deflections of the vertical in formal accordance with the least squares prediction. The required covariance functions of the geoid heights, the components of the deflections, and between these quantities can be derived from the covariance function or the reproducing kernel of the anomalous potential by means of linear functionals. The two methods have been used for determination of a part of the geoid in the Federal Republic of Germany. A comparison of the results shows good agreement.

Introduction

Recently, new astrogeodetic determinations of the geoid have been carried out by the Institute for Applied Geodesy in Frankfurt am Main for the area of the Federal Republic of Germany (Heitz, 1969) and by the Geodetic Institute in Copenhagen for the area of Denmark (Tscherning, 1970). In the first case the computation was founded on a least squares prediction model for the geoid, which may be considered as the first stage of the collocation used by the Danish Geodetic Institute (Krarup, 1969).

In the present paper a description of these two similar methods is given followed by a comparison of their theoretical foundations and possibilities of practical applications.

1. The methods of astrogeodetic geoid determination

1.1. *Least squares prediction model*

This method was first applied to a geoid determination for West Germany in 1968 (Heitz, 1969). In the following, a description is given of a somewhat more general procedure partly used for comparisons of geoid determinations by collocation as described in paragraph 1.2.

It is assumed that the covariance function C of the geoid heights is known and that it depends only on the distance s_{ij} between the considered points P_i and P_j:

$$C_{ij} = C(P_i, P_j) = C(s_{ij}).$$

C can be derived from the covariance function of the gravity anomalies as shown in Krarup (1969), in Moritz (1970), or by steps estimated as in Heitz (1969).

The geoid shall be represented by the heights h_j above a reference ellipsoid in a set of points P_j, $j = 1, 2 \dots n$, in general uniformly distributed over the considered area. Then, the heights $h_{j'}$ in any test points $P_{j'}$, $j' = 1, 2 \dots n'$, can be obtained by a linear autocorrelation

$$h_{j'} = \sum_{j=1}^{n} a_{j'j} h_j \tag{1}$$

where, based on the method of least squares prediction (Moritz, 1963), the coefficients $a_{j'j}$ are determined by the following linear equations:

$$\sum_{j=1}^{n} a_{j'j} C_{jk} = C_{j'k} \qquad (2)$$

$$k = 1, 2 \dots n.$$

For the practical computations it is better to solve the equations

$$\sum_{j=1}^{n} c_j C_{jk} = h_k \qquad (3)$$

instead of (2) so that the estimations h_j can be obtained by

$$h_{j'} = \sum_{j=1}^{n} c_j C_{j'j}. \qquad (4)$$

The h_j must satisfy the condition

$$\sum_{j=1}^{n} h_j = 0. \qquad (5)$$

The remaining unknowns of the problem are the n geoid heights h_j in (1) which are to be determined by a set of known components of deflections of the vertical $d_{i,j'}$ ($i=1$: longitude, $i=2$: latitude) in the points $P_{j'}$, now called source points. For the definition of the deflections, local cartesian coordinate systems $z_{i,j'}$, $i = 1, 2, 3$ are introduced.

The origin of such a system is identical with the projection of $P_{j'}$ on the reference ellipsoid, $z_{3,j'} = h_{j'}$ is the coordinate in the direction of the outer normal of the ellipsoid, and $z_{1,j'}$ and $z_{2,j'}$ are the coordinates in eastern and northern direction so that

$$d_{i,j'} = -\partial h / \partial z_{i,j'}, \quad i = 1, 2. \qquad (6)$$

This applied to the expression (1) yields:

$$d_{i,j'} = \sum_{j=1}^{n} b_{i,j'j} h_j \qquad (7)$$

where

$$b_{i,j'j} = -\partial a_{j'j} / \partial z_{i,j'}.$$

The $b_{i,j'j}$ satisfy the equations (2) if the $C_{j'k}$ on the right hand sides are replaced by their negative derivatives to $z_{i,j'}$.

The number of the components $d_{i,j'}$ may be n'', for which the relation

$$n'' \leqslant 2n$$

holds, since it is not necessary that two components in each source point are given. Owing to condition (5) the number of unknown heights h_j is $n-1$. Therefore, the unknowns are in general uniquely determined by the linear equations (7) in the case $n'' = n-1$.

In the case $n'' > n-1$ the h_j can be determined by a least squares adjustment based on the n'' observation equations

$$v_{i,j'} = -d_{i,j'} + \sum_{j=1}^{n} b_{i,j'j} h_j \qquad (8)$$

and condition (5), where $v_{i,j'}$ are the corrections of the components of the deflections of the vertical.

If the source points with known deflections are well distributed, it is possible to use them as basis when choosing the points P_j for the determination of the identical geoid as done in Heitz (1969). Then, the number of redundancies is $n+1$, if in all source points both components of the deflection are given. The covariance function can be well represented by a sum of exponential functions as follows:

$$C_{ij} = \sum_{k=1}^{r} p_k \exp(q_k s_{ij}^2).$$

But it is also possible to derive the covariance function from expression (14) using Bruns formula $h = T/\gamma$

$$C_{ij} = \frac{1}{\gamma^2} \sum_{n=0}^{\infty} a_n \left(\frac{R^2}{r_i r_j}\right)^{n+1} P_n(\cos v).$$

1.2. Collocation

An estimate of the anomalous potential T may be determined by collocation, i.e. in such a way that the resulting analytic expression agrees exactly with given observations.

This is due the fact that T is an element of a reproducing kernel Hilbert space and that it may be approximated arbitrarily well by functions which are elements of such a space (Krarup, 1969; Meschkowski, 1962).

For example, T may be approximated by elements of the Hilbert space of functions harmonic in the space Ω outside a Bjerhammar-sphere (radius R, lying totally within the Earth) equipped with the norm

$$\|T\| = \int_{\Omega} (\nabla T)^2 d\Omega.$$

The corresponding reproducing kernel is in this case

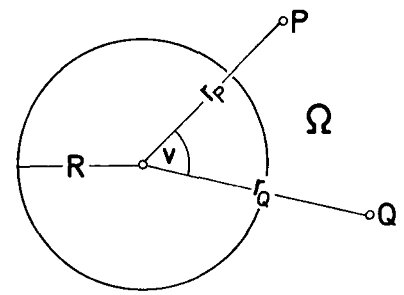

Fig. 1.

$$K(P, Q) = \sum_{n=0}^{\infty} \frac{2n+1}{n+1} \frac{R^{2n+1}}{(r_P r_Q)^{n+1}} P_n(\cos v),$$

(see Fig. 1).

An estimate $\tilde{T}$ agreeing with given observations $\{m_i\}$ and having the least norm will be expressed by

$$\tilde{T}(Q) = \sum_{i=1}^{n} a_i L_i(K(P_i, Q)) \tag{9}$$

where a_i are certain constants, n is the number of observations, L_i are the (linear) functionals corresponding to these with source points P_i.

If the observations are deflections of the vertical, the functional may in spherical approximation be expressed by

$$L_P T = -\frac{1}{r_P \gamma_P} \frac{\partial T}{\partial \varphi}\bigg|_P \quad \text{or} \quad -\frac{1}{\cos(\varphi) r_P \gamma_P} \frac{\partial T}{\partial \lambda}\bigg|_P \tag{10}$$

where γ_P is the reference gravity, φ the latitude and λ the longitude.

The constants a_i are determined by solving a system of linear equations

$$a_i = \{L_i L_j K(P_i, P_j)\}^{-1}\{m_j\} \tag{11}$$

The geoid heights $h(P)$ are then determined (predicted) by evaluating $\tilde{T}$, i.e. using Bruns formula

$$h(P) = L_h(\tilde{T}(P)) = \sum_{i=1}^{n} a_i L_h(L_i(K(P_i, P)))$$

$$= \sum_{i=1}^{n} a_i L_i K(P_i, P)/\gamma_P. \tag{12}$$

The error of prediction may be estimated by

$$\|L_h\| = L_h L_h K(P, P) - \{L_h L_i K(P_i, P)\}' \{L_i L_j K(P_i, P_j)\}^{-1} \{L_h L_j K(P_j, P)\} \tag{13}$$

Different Hilbert spaces and corresponding different kernels implicate different properties of $\tilde{T}$, i.e. the variations of the first or second derivatives are minimized. At the moment no rules for the selection of the optimal space for collocation can be given, and very different kernels may give nearly the same predictions.

On the contrary, the estimates of the error of prediction can vary much. As shown in Parzen (1959), every covariance function will have a corresponding Hilbert space of which the function will be the reproducing kernel and conversely. Therefore, any reproducing kernel can be interpreted as a covariance function. This suggests the use of a reproducing kernel approximating an empirically determined covariance function. In doing so, computational experiments resulted in realistic estimates of the predicted errors.

Contrary to the model of least squares prediction we will only assume rotational invariance of the covariance function. The corresponding kernel must then have the same property.

The general expression of a rotation invariant reproducing kernel of a Hilbert space, with the underlying set being the space outside a sphere, is

$$K(P, Q) = \sum_{n=0}^{\infty} a_n \left(\frac{R^2}{r_P r_Q}\right)^{n+1} P_n(\cos v) \tag{14}$$

where the "degree-variances" a_n are ≥ 0 and so that the series is absolutely convergent for all P and Q outside the sphere.

Kernels similar to a global covariance function may be represented by letting the "degree-variances" vary as n^{-2} and using a ratio of 0.995 between the radius of the Bjerhammar sphere and the mean radius of the Earth. Local covariance functions are approximated by setting a number of the first coefficients equal to zero and using a ratio closer to 1.

2. Comparison of the two methods

Comparisons of the expressions (4) and (12), (3) and (11), (6) and (10) will make the similarity of the two methods stand out clearly. Furthermore, both require some sort of minimization. Least squares prediction requires the square sum of expression (8) and collocation the norm $\|\tilde{T}\|$ to be minimum. Collocation gives an estimate $\tilde{T}$ which is a harmonic function.

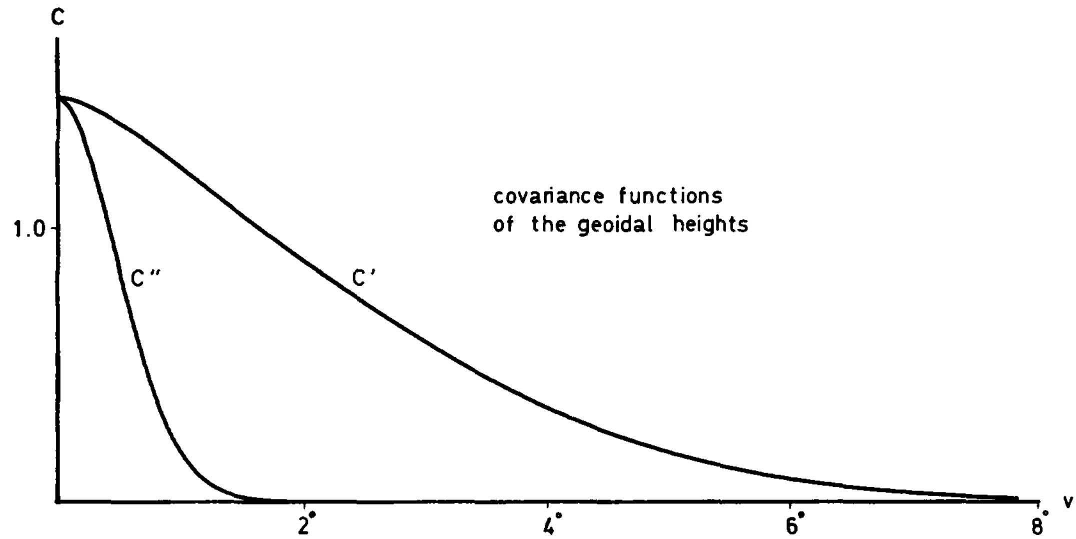

Fig. 2.

This makes it possible to *combine* different measurements for the determination of wanted quantities in a clear and unique way. But it is not an advantage to determine geoid heights based on gravity anomalies *or* astrogeodetic deflections by collocation instead of using Stokes formula or the least squares prediction method respectively. On the other hand, collocation is more suitable than astrogeodetic levelling in cases of local geoid determinations by use of gravity anomalies and astrogeodetic deflections.

Astrogeodetic determination of the geoid by collocation gives a geoid which agrees exactly with the given deflections. But it is necessary to solve equally many equations as there are observations. The least squares prediction method is easier to handle. Both the normal equations and equations (2) will require about half as many equations to be solved as the corresponding collocation problem if $n'' = 2n$. In this case, the deflections will not agree exactly. However, when comparing corresponding geoids it is seen that the two methods give nearly the same results (see section 3). Least squares prediction may therefore be preferred to practical geoid determination if the number n of free coefficients can be chosen much smaller than the number n'' of observations.

3. Numerical examples

For numerical tests, 62 stations with known astrogeodetic deflections of the vertical have been selected, distributed evenly over an area bounded by the meridians 7° and 11° East of Greenwich and the parallels 52° and 54°. As the collocation method described in paragraph 1.2 is the theoretically better founded method the results of a geoid computation using this method together with the above-mentioned data have been chosen for comparisons with the results of special cases of the method 1.1.

The computation according to the collocation method is based on the covariance function

$$K(P, Q) = \sum_{n=16}^{\infty} \frac{a}{n(n-1)} \left(\frac{R^2}{r_P r_Q}\right)^{n+1} P_n(\cos v)$$

$$a/R^2 = 0.645 \text{ mGal}^2$$

for the anomalous potential where $R \cdot 0.998$ is a mean Earth radius. The corresponding covariance function C'_{PQ} of the geoid heights is shown in Fig. 2. Also, the covariance function C''_{PQ} of the geoid heights is shown, which was used for a geoid determination for the Federal Republic of Germany in 1968 (Heitz, 1969). C''_{PQ} is defined by

$$C''_{PQ} = 0.7244 \exp\,(-6.892802 \cdot 10^{-4}\, v^2) + 0.7356 \exp\,(-4.721570 \cdot 10^{-4}\, v^2)$$

where v is the angular distance in minutes of arc.

By means of the method 1.1, three computations were performed based on the same data as used for the collocation method 1.2. The assumptions for these geoid determinations were:

(a) covariance function C', number of redundancies $= 63 = n + 1$
(b) covariance function C'', number of redundancies $= 63$
(c) covariance function C', number of redundancies $= 39$

The relative frequencies of the differences d_a and d_b between the results of the cases (a) and (b) and those of the collocation method are represented by the graphs of Fig. 3. Case (c) yielded similar results except for one point at the southern boundary of the area in which the difference was 60 cm. As the standard error in this somewhat isolated point is 30 cm, this difference is not very important.

Taking into account that the average of the standard error of the centred geoidal heights is about 15 cm, it is seen that there is a close agreement between the results of the compared methods.

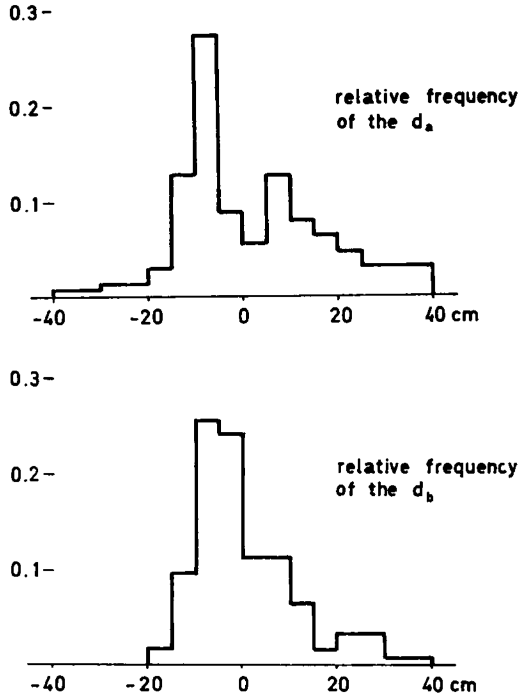

Fig. 3.

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СРАВНЕНИЕ ДВУХ МЕТОДОВ АСТРОГЕОДЕЗИЧЕСКИХ ОПРЕДЕЛЕНИЙ ФОРМЫ ГЕОИДА, ОСНОВАННЫХ НА ПРЕДСКАЗАНИИ С ПОМОЩЬЮ МЕТОДА НАИМЕНЬШИХ КВАДРАТОВ И КОЛЛОКАЦИИ

Описываются и обсуждаются два метода наименьших квадратов для астрогеодезических определений формы геоида в связи с возможностями их практического применения. Первый метод основан на модели предсказания с помощью метода наименьших квадратов, а именно, на линейной автокорреляции высот геоида. Отклонения вертикалей, выведенные из этой модели, дают уравнения относительно данных наблюдений для определе-

ния свободных коэффициентов линейной автокорреляции путем использования метода наименьших квадратов. Во-втором случае используется коллокация, введенная в физическую геодезию Т. Крарупом.

При применении этого метода высоты геоида представляются непосредственно их линейной корреляцией с отклонениями вертикалей в формальном соответствии с предсказанием с помощью метода наименьших квадратов. Необходимые функции корреляции для высот геоида, копоненит отклонений и взаимной корреляции между этими величинами могут быть получены из корреляционной функции или производящего ядра аномалий потенциала с помощью линейных функционалов. Оба метода были использованы для определения части геоида в Федеративной Республике Германии. Сравнение результатов указывает на их хорошее согласие.