Finite amplitude disturbances in a rotating fluid

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ABSTRACT

A liquid is rotating with angular velocity vector along the z-axis. A uniform gravity field acts along the negative z-axis and the liquid is stratified with arbitrary density gradient. A steady disturbance exists with all properties independent of one of the horizontal coordinates, say, y. It is then shown that the primitive equations can be integrated once to yield a second-order vorticity equation. This equation is generally non-linear and complicated but greatly simplifies when the fluid is homogeneous. In one special and interesting case the problem becomes identical to two-dimensional flow of a certain stratified fluid in a uniform gravity field.

1. Introguction

In certain cases the problem of solving the equations of motion and continuity of a fluid can be reduced to that of solving a second-order differential equation. The best-known case is that of irrotational flow; then the second-order equation is Laplace's equation for the velocity potential. Other cases require the assumption of a steady state and of two dimensional or axially symmetric motion. Examples include the two-dimensional flow of a stratified liquid (Long, 1953) and stratified gas (Long, 1956; Yih, 1960; Long, 1966); the axially symmetric flow of a rotating liquid (Long, 1953), and the flow of a liquid on a rotating spherical shell (Ertel, 1943).

In the present paper an analogous result is obtained for steady flow of a homogeneous rotating liquid when the disturbance is independent of one of the Cartesian coordinates. If we neglect the centrifugal force arising from the basic rotation, we may obtain a similar result when the fluid is stratified.

2. Integration of the primitive equations

We consider a liquid rotating about the z-axis in a uniform gravity field along the negative z-axis. The equations of motion in rotating coordinates contain a Coriolis force and a centrifugal force arising from the basic rotation. If the fluid is homogeneous in density the latter may be ignored because it has a potential and is eliminated by taking the curl of the equations of motion. If the density of the fluid varies, we will still neglect the centrifugal force by assuming it is small or by assuming, as on the earth, that a component of gravity cancels the horizontal component and that the vertical component is a small part of apparent gravity. In the geophysical case, we also neglect variations in the Coriolis parameter, f. Finally we assume the existence of steady disturbances whose properties are independent of the y-coordinate. The equations are

$$\varrho\left(\frac{du}{dt} - fv\right) = -p_x \tag{1}$$

$$\varrho\left(\frac{d}{dt} + fu\right) = 0 \tag{2}$$

$$\varrho\frac{dw}{dt} = -p_z - \varrho g \tag{3}$$

$$u = -\psi_z, \; w = \psi_x \tag{4}$$

$$\frac{d\varrho}{dt} = 0 \tag{5}$$

where ψ is the streamfunction of the motion in the xz-plane and

$$\frac{d}{dt} = u\frac{\partial}{\partial x} + w\frac{\partial}{\partial z} \tag{6}$$

Eq. (5) may be integrated to show that the density is a function of the streamfunction. Eq. (2) may also be integrated to yield

$$v + fx = H(\psi) \qquad (7)$$

where H is arbitrary. If we now cross-differentiate to eliminate the pressure, we get the equation

$$\varrho \frac{d}{dt}(\nabla^2\psi) - \varrho f H' \frac{dx}{dt} + \varrho' \frac{d}{dt}\left(\frac{u^2}{2} + \frac{w^2}{2} + gz\right) - f\frac{dx}{dt}\varrho'(H - fx) = 0$$

or

$$\frac{d}{dt}\left[\varrho\nabla^2\psi - \varrho f H' x + \varrho'\left(\frac{u^2+w^2}{2} + gz - xfH + \frac{f^2x^2}{2}\right)\right] = 0 \qquad (8)$$

where the prime denotes differentiation with respect to ψ. Equation (8) may be integrated to yield

$$\nabla^2\psi - fxH' + \frac{\varrho'}{\varrho}\left[\frac{(\nabla\psi)^2}{2} + gz - xfH + \frac{f^2x^2}{2}\right] = K(\psi) \qquad (9)$$

where $K(\psi)$ is arbitrary. Notice that when $f = 0$ this reduces to the vorticity equation for two-dimensional non-rotating flow of a stratified liquid (Long, 1953). In the non-rotating case one of the basic solutions of the vorticity equation is horizontal flow ($v = w = 0$). When there is rotation, however, a flow along the x-axis requires a pressure gradient along the y-axis. We have assumed this is zero.

Linear models in Eq. (9) may be obtained by defining a function

$$F = \int \varrho^{\frac{1}{2}} d\psi$$

Then (9) takes the form

$$\nabla^2 F - \varrho^{\frac{1}{2}} x H' f + \frac{\varrho'}{\varrho^{\frac{1}{2}}}\left(gz - xfH + \frac{f^2x^2}{2}\right) = \varrho^{\frac{1}{2}} K(\psi)$$

This is linear, of the form,

$$\nabla^2 F - (a_2 + b_2 F)x + (a_3 + 2b_3 F)\left(gz + \frac{f^2x^2}{2}\right) = a_1 + b_1 F$$

if

$$\varrho^{\frac{1}{2}} K(\psi) = a_1 + b_1 F, \quad \frac{\varrho'}{\varrho^{\frac{1}{2}}} = a_3 + 2b_3 F,$$

$$\varrho^{\frac{1}{2}} f H' + \frac{\varrho'}{\varrho^{\frac{1}{2}}} f H = a_2 + b_2 F$$

where a_1, b_1, a_3, b_3, a_2, b_2 are constants. If $b_3 \neq 0$ we get

$$F = \frac{A}{\sqrt{b_3}}\exp(\sqrt{b_3}\,\psi) - \frac{B}{\sqrt{b_3}}\exp(-\sqrt{b_3}\,\psi) - \frac{a_3}{2b_3}$$

$$\varrho^{\frac{1}{2}} = A\exp(\sqrt{b_3}\,\psi) + B\exp(-\sqrt{b_3}\,\psi)$$

$$K\varrho^{\frac{1}{2}} = a_1 + \frac{Ab_1}{\sqrt{b_3}}\exp(\sqrt{b_3}\,\psi) - \frac{Bb_1}{\sqrt{b_3}}\exp(-\sqrt{b_3}\,\psi) - \frac{a_3 b_1}{2b_3}$$

$$H = \frac{1}{\varrho f}\left\{C + \frac{1}{\sqrt{b_3}}\left(a_2 - \frac{b_2 a_3}{2b_3}\right)[A\exp(\sqrt{b_3}\,\psi) - B\exp(-\sqrt{b_3}\,\psi)] + \frac{b_2}{2b_3}[A^2\exp(2\sqrt{b_3}\,\psi) + B^2\exp(-2\sqrt{b_3}\,\psi)]\right\}$$

where A, B, C are constants. If $b_3 = 0$, we get

$$\nabla^2 F - (a_2 + b_2 F)x + a_3\left(gz + \frac{f^2x^2}{2}\right) = a_1 + b_1 F$$

and

$$\varrho^{\frac{1}{2}} = \frac{a_3\psi}{2} + \alpha$$

$$F = \beta + \alpha\psi + a_3\frac{\psi^2}{4}$$

$$\varrho^{\frac{1}{2}} K = a_1 + \beta b_1 + b_1\alpha\psi + \frac{a_3 b_1}{4}\psi^2$$

$$H = \frac{1}{\varrho f}\left[\gamma + (a_2\alpha + b_2\alpha\beta)\psi + \left(\frac{a_2 a_3}{4} + \frac{\beta b_2 a_3}{4} + \frac{b_2\alpha^2}{2}\right)\psi^2 + \frac{\alpha b_2 a_3}{4}\psi^3 + \frac{b_2 a_3^2}{32}\psi^4\right]$$

where α, β, γ are constants.

3. Homogeneous liquid

In the case of a homogeneous fluid, Eqs. (7) and (9) become

$$v + fx = H(\psi) \quad (10)$$

$$\nabla^2\psi - fxH' = K(\psi) \quad (11)$$

We now consider motion which, at $z = +\infty$ or $z = -\infty$ reduces to a simple flow of velocity $w = w_0(x_0)$, $u = 0$, $v = 0$ along the z-axis, where x_0 is the distance of a streamline from the yz-plane in the undisturbed region upstream or downstream. Evidently, $\psi = \psi(x_0)$, and if we go over to x_0 as dependent variable, we get

$$v + fx = fx_0$$

$$w_0 \nabla^2 x_0 + \frac{dw_0}{dx_0}(\nabla x_0)^2 - \frac{f^2 x}{w_0} = \frac{dw_0}{dx_0} - \frac{f^2 x_0}{w_0} \quad (12)$$

where we have assumed that w_0 does not vanish anywhere. If we define

$$\delta = x - x_0 \quad (13)$$

as the deviation of the streamline from the undisturbed distance from the yz-plane we get

$$\nabla^2\delta - \frac{d \ln w_0}{dx_0}\left[(\nabla\delta)^2 - 2\frac{\partial\delta}{\partial x}\right] + \frac{f^2\delta}{w_0^2} = 0 \quad (14)$$

In case the approach velocity w_0 is uniform, we get the linear equation

$$\nabla^2\delta + \frac{f^2}{w_0^2}\delta = 0 \quad (15)$$

This is identical in form to the differential equation for stratified flow in a uniform gravity field along the negative x-axis where the approach velocity w_0 and the density ϱ are related by $w_0^2\varrho$ = constant and when the density is a linear function of x_0 (Long, 1953). This simple stratified flow has been extensively investigated (Long, 1955; Yih, 1958; Debler, 1959; Miles, 1968*a*, 1968*b*; Miles & Huppert, 1969*a*, 1969*b*) and solutions obtained for sources and sinks and flow past obstacles. In the present case this corresponds to a line source or sink along a line parallel to the y-axis and to obstacles whose shape is independent of the y-coordinate moving along the z-axis[1]. Of course in the rotating case, we have the additional velocity component v which may, however, be solved for from the equation

$$v = -f\delta \quad (16)$$

Finally, another phenomenon of stratified fluids (Long, 1965) that has application to the rotating homogeneous case is the solitary wave at rest in a basic flow $w_0(x_0)$. In the present case there is no wave when w_0 is constant because the governing equation (15) is then linear whereas the solitary wave is essentially a nonlinear phenomenon. If w_0 varies, then the wave exists. The governing equations in the two cases are identical if

$$w_0 = \text{constant} \quad (17)$$

$$\varrho = \varrho_0 e^{-bx_0} \quad (18)$$

in the stratified case, and

$$w_0 = w_{00}\, e^{-\frac{bx_0}{2}} \quad (19)$$

in the rotating case, where ϱ_0, w_{00} and b are constants.

4. Linear models for a homogeneous liquid

The most general linear form of Eq. (11) is

$$\nabla^2\psi - (\gamma - \beta\psi)x = c - \alpha^2\psi \quad (20)$$

where γ, β, c, α^2 are constants and

$$\frac{dH}{d\psi} = \frac{\gamma}{f} - \frac{\beta\psi}{f} \quad (21)$$

We again assume that ψ is a function of x_0, so that $u = 0$ in the undisturbed region and $w = w_0(x_0)$. Then in the undisturbed region, if $v = v_0$

$$v_0 = -fx_0 + H(\psi) \quad (22)$$

It follows that v_0 is also a function of x_0.

Eq. (20) now yields a differential equation for ψ as a function of x_0, i.e.,

$$\frac{d^2\psi}{dx_0^2} - (\gamma - \beta\psi)x_0 = c - \alpha^2\psi \quad (23)$$

[1] Bretherton (1967) has considered this problem when the motion of the obstacle is infinitely slow.

Eq. (23) has been solved in connection with an investigation of linear models for stratified fluids (Long, 1958). The result for $\beta \neq 0$ is

$$\psi = \frac{\pi}{\beta\sqrt{3}}(\alpha^2 + \beta x_0)^{\frac{1}{2}} \Bigg\{ J_{\frac{1}{3}}\left[\frac{2(\alpha^2 + \beta x_0)^{\frac{3}{2}}}{3\beta}\right] \times \int (c + \gamma x_0)\,(\alpha^2 + \beta x_0)^{\frac{1}{2}} J_{-\frac{1}{3}}\left[\frac{2(\alpha^2 + \beta x_0)^{\frac{3}{2}}}{3\beta}\right] dx_0 \tag{24}$$

$$- J_{-\frac{1}{3}}\left[\frac{2(\alpha^2 + \beta x_0)^{\frac{3}{2}}}{3\beta}\right] \int (c + \gamma x_0)\,(\alpha^2 + \beta x_0)^{\frac{1}{2}} \times J_{\frac{1}{3}}\left[\frac{2(\alpha^2 + \beta x_0)^{\frac{3}{2}}}{3\beta}\right] dx_0 \Bigg\}$$

This equation may be differentiated with respect to x_0 to yield w_0 as a function of x_0. The function H may be solved for as a function of x_0 from Eq. (21). Then Eq. (22) yields v_0 as a function of x_0 and the model is fully determined.

If $\beta = 0$ we get a much simpler result. Differentiating Eq. (23) we get

$$\frac{d^2 w_0}{dx_0^2} - \gamma = -\alpha^2 w_0 \tag{25}$$

so that

$$w_0 = \frac{\gamma}{\alpha^2} + A \cos \alpha x_0 + B \sin \alpha x_0 \tag{26}$$

where A and B are constants. Eq. (21) yields

$$\frac{dH}{dx_0} = \frac{\gamma^2}{f\alpha^2} + \frac{\gamma A}{f} \cos \alpha x_0 + \frac{\gamma B}{f} \sin \alpha x_0 \tag{27}$$

so that

$$H = D + \frac{\gamma^2}{f\alpha^2} x_0 + \frac{\gamma A}{f\alpha} \sin \alpha x_0 - \frac{\gamma B}{f\alpha} \cos \alpha x_0 \tag{28}$$

where D is a constant. It follows that

$$v_0 = -f x_0 + D + \frac{\gamma^2 x_0}{f\alpha^2} + \frac{\gamma A}{f\alpha} \sin \alpha x_0 - \frac{\gamma B}{f\alpha} \cos \alpha x_0 \tag{29}$$

The model is determined by Eqs. (26) and (29). The linear model of Sec. 3 is a special case.

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ВОЗМУЩЕНИЯ КОНЕЧНОЙ АМПЛИТУДЫ ВО ВРАЩАЮЩЕЙСЯ ЖИДКОСТИ

Жидкость вращается с угловой скоростью, направленной по оси z. Однородное поле силы тяжести действует вдоль отрицательного направления оси z и жидкость стратифицирована с произвольным градиентом плотности. Существует устойчивое возмущение, свойства которого не зависят от одной из горизонтальных координат, скажем, y. Показано, что тогда примитивные уравнения могут быть один раз проинтегрированы, в результате чего получается уравнение второго порядка для вихря скорости. Это уравнение в общем случае нелинейно и сложно, но оно сильно упрощается для однородной жидкости. В одном специальном и интересном случае задача становится идентичной двумерному течению некоторой стратифицированной жидкости в однородном поле силы тяжести.