On the variation of ocean circulation produced by bottom topography

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(Manuscript received June 30, 1970; revised version November 10, 1970)

ABSTRACT

The three dimensional circulation produced by a given wind stress in a homogeneous ocean with an arbitrary bottom topography is studied. The solution is illustrated by the use of various bottom shapes and a particular wind stress distribution. It is shown that in general a current deviates southwards in a region of decreasing depth and northwards in a region of increasing depth. However, if the wind stress has a perturbation out of phase with the bottom corrugations this result is considerably modified. Moreover, when there is a deepening of the ocean towards the north there may exist a "critical line" at which $\partial/\partial y(f/H)$ is zero (f is the Coriolis parameter, H is the depth and y is the northward co-ordinate). The oceanic circulation is then divided into two parts: if the wind stress curl is positive, then for latitudes north of the "critical line" where $\partial/\partial y(f/H)$ is positive the Sverdrup-topographic interior flow is basically northward and there is an eastern boundary current, while for latitudes south of the critical line, where $\partial/\partial y(f/H)$ is negative the interior flow is basically southward and there is a western boundary current.

1. Introduction

Observations show that ocean currents are significantly influenced by bottom topography. The deflection of currents in the Atlantic as they pass over the mid-Atlantic ridge and the deflection of the Antarctic circumpolar current as it passes over various meridional ridges are well known. Observations by Barrett (1965) show that part of the western boundary undercurrent in the North Atlantic is diverted away from the coast by a submarine ridge extending south eastwards from Cape Fear.

Various theories have been developed to correlate the behaviour of ocean currents with the underlying bottom topography. Warren (1963) has shown numerically that the path of the Gulf Stream is closely related to the bottom topography. Hsueh (1967) has discussed the modification of a uniform current as it flowed into a region of oscillatory wind stress and oscillatory bottom contours. He correlated the strength of the deflections of the current with the difference in phase between the wind stress and the bottom topography. Clarke & Fofonoff (1969) used a transport model for a barotropic ocean to consider the flow over depth discontinuities for constant wind stress curl.

In this paper the barotropic flow over any continuous bottom topography is discussed for an arbitrary wind stress. A linear, three dimensional model on the β-plane is used and each component of the velocity field is calculated. The model is very simplified in the sense that baroclinic effects are excluded, eddy viscosities are assumed constant and only steady flows are considered. Consequently, caution must be exercised when comparing the solutions obtained for flow over continental slopes and submarine ridges with actual ocean currents.

The analytical solutions which are found for general wind stress and bottom topography are illustrated by introducing a particular wind stress. Bottom topography is chosen to represent the folds in the western Pacific, the continental slopes and a zonal slope. In the last case a critical layer may be present in which the normal interior solution breaks down, and the non-linear and frictional terms become important. This is discussed in section 9.

The method derived below may also be applied to the almost isothermal bottom layer of a stratified ocean. In this case the circulation is driven by the upwelling into the thermocline layer instead of by the Ekman layer suction produced by the wind stress curl.

2. Formulation

The model considered is a homogeneous ocean contained between the surface $z = 0$ and the bottom $z = -1 + h(x, y)$ and bounded by west and east coasts at $x = 0$ and $x = 1$, respectively. x, y, z are dimensionless cartesian co-ordinates with x eastwards, y northwards and z vertically upwards, and $\mathbf{u} = (u, v, w)$ are the corresponding velocity components. The linear steady equations of motion are:

$$-fv = -P_x + E_H(u_{xx} + u_{yy}) + Eu_{zz} \tag{1a}$$

$$fu = -P_y + E_H(v_{xx} + v_{yy}) + Ev_{zz} \tag{1b}$$

$$0 = -\delta^{-2}P_z + E_H(w_{xx} + w_{yy}) + Ew_{zz} \tag{1c}$$

$$u_x + v_y + w_z = 0 \tag{1d}$$

where $f = 1 + \beta y$ is the familiar Coriolis parameter on the β-plane, P is the reduced pressure and the small parameters are given by

$$E_H = \frac{\nu_H}{f_0 L^2},\; E = \frac{\nu_V}{f_0 L^2},\; \delta = D/L$$

ν_H, ν_V are the horizontal and vertical coefficients of eddy viscosity, D, L are typical depth and width of the ocean respectively, and f_0 is the Earth's angular velocity. The derivation of these equations and the method of introducing the dimensionless variables are given in Johnson (1968).

The boundary conditions at the surface are:

$$u_z = E^{-1/2}\tau^x,\; v_z = E^{-1/2}\tau^y,\quad w = 0,\quad \text{at}\quad z = 0 \tag{2}$$

where $\tau = (\tau^x, \tau^y, 0)$ is the applied wind stress. The conditions at the east coast $x = 1$, the west coast $x = 0$, and the bottom $z = -1 + h(x, y)$ are:

$$u = v = w = 0 \text{ at } x = 0,\ 1;\ z = -1 + h(x, y)$$

The north and south boundaries are unspecified and hence remain free.

The analysis will proceed by the method of matched asymptotic expansions, each expansion being valid only in the region under consideration. The ocean will be divided into an interior region away from the influence of boundaries, surface and bottom Ekman layers, and coastal boundary layers.

3. The interior

In this region the frictional terms of equation (1) are negligible and expansions of the form

$$\mathbf{u}_I = E^{1/2}\mathbf{u}_1 + E\mathbf{u}_2 + \dots,\; P = E^{1/2}P_1 + EP_2 + \dots$$

substituted in (1) yield the interior solutions

$$u_I = E^{1/2}u_1(x, y) + 0(E) \tag{3a}$$

$$v_I = E^{1/2}v_1(x, y) + 0(E) \tag{3b}$$

$$w_I = E^{1/2}(\beta/f)z\,v_1(x, y) + E^{1/2}w^*(x, y) + 0(E) \tag{3c}$$

where $w^*(x, y)$ is an arbitrary function of integration.

4. The surface Ekman layer

The solution in this layer is well known and the only condition required here is the matching condition on the interior region. This is the Ekman layer suction condition

$$w_I(x, y, 0) = E^{1/2}\mathbf{k}\cdot\text{curl}\,(\boldsymbol{\tau}/f) \tag{4}$$

where $\mathbf{k} = (0, 0, 1)$. Applying this condition to (3c) gives

$$w^*(x, y) = \mathbf{k}\cdot\text{curl}\,(\boldsymbol{\tau}/f) \tag{5}$$

In the case where there is an intermediate thermocline layer between the bottom layer and the Ekman layer, then w^* in the bottom layer will be the upwelling velocity into the thermocline layer.

5. The bottom Ekman layer

In this layer an oblique stretched co-ordinate system (x, y, ζ) is introduced where $E^{1/2}\zeta = z + 1 - h(x, y)$. It is shown in the appendix that provided

$$h_x, h_y, h_{xx} + h_{yy} << E^{1/2}/E_H$$

and $(h_x^2 + h_y^2)^{1/2} << E_H^{-1/2}$, $(\delta E_H)^{-1/2}$, which are not very strong restrictions on the gradients of $h(x, y)$, then the vertical velocity in this Ekman layer is, from (A 9)

$$w_B = E^{1/2}(h_x u_1 + h_y v_1) + \text{exponential terms} \tag{6}$$

The exponential terms tend to zero as $\zeta \to \infty$. This implies that the component of velocity normal to the bottom is zero at the edge of the Ekman layer. That is, there is no $0(E^{\frac{1}{2}})$ suction into the bottom Ekman layer, which therefore carries no transport to this order.

6. The completion of the interior solution

Equation (3c) gives the vertical velocity in the interior region with $w^*(x, y)$ a known function determined by (5). The remaining function $v_1(x, y)$ may be found by matching (3c) with (6) as follows:

$$\lim_{z\to -1+h(x,y)} w_I = \lim_{\zeta\to\infty} w_B$$

Hence

$$\frac{\beta}{f}(h-1)v_1 + w^* = h_x u_1 + h_y v_1 \quad (7)$$

The continuity equation (1d) gives

$$u_{1x} + v_{1y} + w_{1z} = 0$$

and using the derivative of (3c) this may be written

$$\frac{\partial}{\partial x}(f u_1) + \frac{\partial}{\partial y}(f v_1) = 0 \quad (8)$$

which implies the existence of a function ψ_1 such that

$$f u_1 = \frac{\partial \psi_1}{\partial y}, \; f v_1 = -\frac{\partial \psi_1}{\partial x} \quad (9)$$

Substituting (9) into (7) gives

$$h_x \psi_{1y} + \left\{\frac{\beta}{f}(h-1) - h_y\right\}\psi_{1x} = f w^* \quad (10)$$

a first order linear, partial differential equation for ψ_1 which may be solved by the method of characteristics.

The characteristics of (10) are given by

$$\frac{dy}{dx} = \frac{h_x}{\frac{\beta}{f}(h-1) - h_y}$$

which on integretion becomes,

$$\frac{h(x\ y) - 1}{f} = \text{constant.} \quad (11)$$

Let $\eta = \dfrac{f}{h(x, y) - 1}$ so that (10) reduces to

$$\frac{\partial \psi_1}{\partial x} = \frac{f^2 w^*}{\beta(h-1) - f h_y}$$

which implies that

$$\psi_1(x, \eta) = \int \frac{f^2 w^*}{\beta(h-1) - f h_y} dx' + \int C(\eta)\, d\eta \quad (12)$$

where $\int C(\eta) d\eta$ is an arbitrary function of η and the first integrand is regarded as a function of x' and η. Equations (9) then give the horizontal velocity components

$$u_1(x, y) = \frac{\beta(h-1) - f h_y}{f(h-1)^2}\left\{\int \frac{\partial}{\partial \eta}\left(\frac{f^2 w^*}{\beta(h-1) - f h_y}\right) dx' + C(\eta)\right\} \quad (13)$$

$$v_1(x, y) = \frac{h_x}{(h-1)^2}\left\{\int \frac{\partial}{\partial \eta}\left(\frac{f^2 w^*}{\beta(h-1) - f h_y}\right) dx' + C(\eta)\right\} - \frac{f w^*}{\beta(h-1) - f h_y} \quad (14)$$

$C(\eta)$ may be found from a zero transport condition at the east coast ($x = 1$). As the bottom Ekman layer transport is only $0(E)$, the $0(E^{\frac{1}{2}})$ surface Ekman layer transport towards the east coast must descend in an eastern boundary layer and then enter the interior region. This condition may be written

$$[1 - h(1, y)]\, u_1(1, y) = -\frac{\tau^y(1, y)}{f} \quad (15)$$

Equations (13) and (14) give the horizontal velocity components for any bottom topography $h = h(x, y)$ and for any wind stress curl $w^*(x, y)$. They are singular at points where $h(x, y) = 1$, and hence the solution will not be valid near where the bottom surface rises to meet the top surface. This will occur at the west and east coasts, but in these regions coastal boundary layers will be important and the interior solution will not be applicable. In the examples which follow we shall ensure that $h(x, y) \neq 1$ for any x, y. The western boundary layer that is required to bring the east-west flow to zero at the coast and to satisfy continuity by providing an intense north-south return flow will be of the form discussed in Johnson (1968).

7. Solutions for a zonal wind stress when $h = h(x)$

The velocity components given by (13) and (14) will now be simplified for a zonal wind stress $\boldsymbol{\tau} = \boldsymbol{\tau}(y)$ and a bottom topography of the form $h = h(x)$. The wind stress is chosen as

$$\tau^x = -\cos \pi y, \tau^y = 0$$

to represent trade winds and westerlies. τ^y is so chosen to avoid downwelling at the coasts. There is no mathematical difficulty involved in chosing τ^y to be non-zero. The horizontal velocity components then reduce to

$$u_I(x, y) = E^{1/2} \frac{\pi^2}{\beta(h-1)^2} \int_x^1 (h(x') - 1) \cos\left\{\frac{\pi}{\beta}[(h(x') - 1)\eta - 1]\right\} dx' + 0(E) \quad (16)$$

$$v_I = E^{1/2} \frac{\pi^2 f h_x}{\beta^2 (h-1)^3} \int_x^1 (h(x') - 1) \cos\left\{\frac{\pi}{\beta}[(h(x') - 1)\eta - 1]\right\} dx' + \frac{E^{1/2}}{\beta(h-1)} \left(\pi \sin \pi y + \frac{\beta}{f} \cos \pi y\right) + 0(E) \quad (17)$$

where $\eta = f/[h(x) - 1]$ is regarded as a constant in the integration.

Then (3c) gives the vertical component of velocity as

$$w_I(x, y, z) = \frac{\beta z}{f} v_I(x, y) - \frac{E^{1/2}}{f} \left(\pi \sin \pi y + \frac{\beta}{f} \cos \pi y\right)$$

Consider the particular bottom topography

$$h(x) = A e^{-kx} \cos l\pi x \quad (18)$$

When $l = 0$, this represents a bottom profile that rises up towards the surface as the western boundary at $x = 0$ is approached. When $l \neq 0$, (18) may be used to represent a topography that is typical of the Pacific in which meridional folds occur on the western side of the ocean.

The circulation obtained for (i) $l = 0$, $A = 0.8$, $k = 10$ is shown in Fig. 1, and for (ii) $l = 10$, $A = 0.8$, $k = 10$ in Fig. 2. It can be seen in both cases that in the eastern part of the ocean where the bottom is essentially flat, the solution is the same as that given in Johnson (1968). However, for small x the circulation is considerably modified by the bottom topography. Further comparison with Johnson (1968) shows that in the Northern Hemisphere currents are deflected to the right when the gradient of the bottom is increasing, and to the left when the gradient is decreasing (having regard for the sign of the gradient). Consequently in a flow over a single ridge the total deflection will be zero and the maximum displacement will occur at the crest of the ridge.

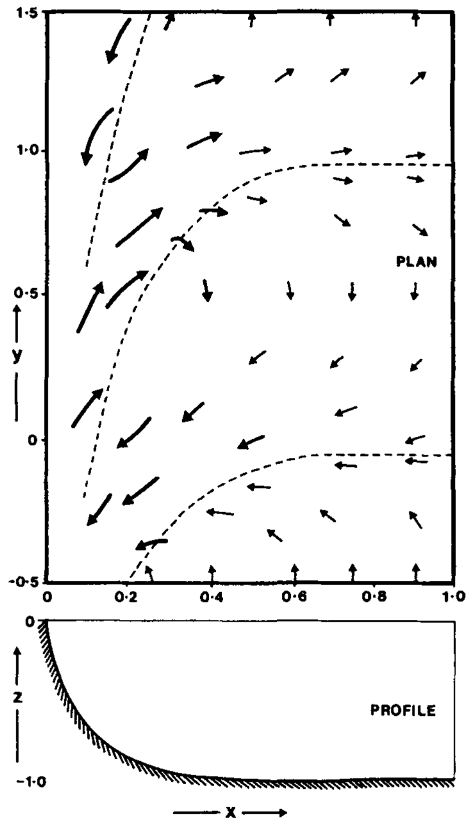

Fig. 1. Flow pattern produced for the wind stress $\tau^x = -\cos \pi y$, $\tau^y = 0$ and for the bottom topography $h(x) = 0.8 \exp(-10x)$. The north-south component of velocity is zero on the broken lines which also indicate the deflection of the flow due to the underlying bottom topography.

Calculations of the flow patterns were made with the topographies of the form

$$h(x) = A e^{-k(x-\alpha)^2}$$

corresponding to a single ridge at $x = \alpha$, and

FIG. 2

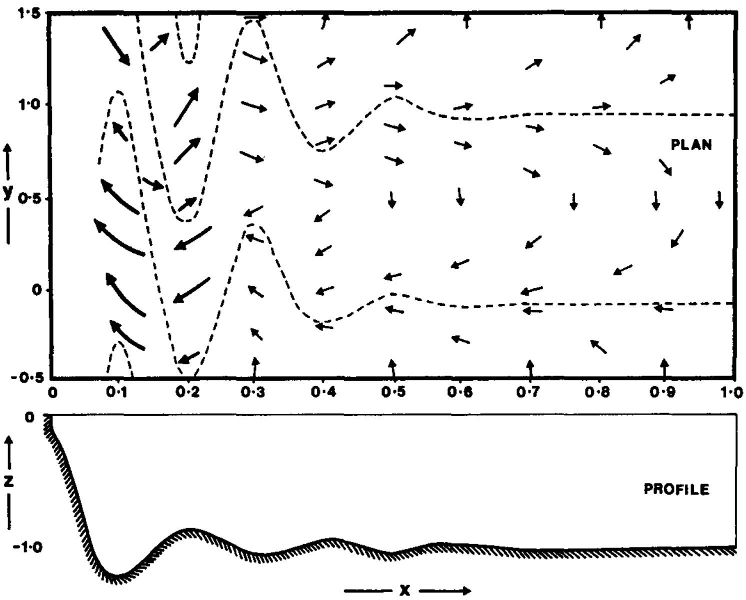

Fig. 2. Flow pattern produced for the wind stress $\tau^x = -\cos \pi y$, $\tau^y = 0$ and for the bottom topography $h(x) = 0.8 \exp(-10x) \cos(10\pi x)$.

the anticipated deflections were observed. This behaviour is illustrated diagramatically in Fig. 3.

8. The influence of an oscillatory wind stress

To provide a comparison with the results of Hsueh (1967), who discussed the deviations of a uniform current on entering a region of changing wind stress and bottom topography, a solution is found for

$$\tau^x = 0, \; \tau^y = \begin{cases} A(1 + T\cos 9\pi x) & 0 \leq x \leq \frac{4}{9} \\ A(1+T) & \frac{4}{9} \leq x \leq 1 \end{cases}$$

$$h(x) = \begin{cases} H \sin 9\pi x & 0 \leq x \leq \frac{1}{2} \\ H & \frac{1}{2} \leq x \leq 1 \end{cases}$$

With this combination the interior flow for $x > 1/2$ is a uniform drift westwards. The perturbation in wind stress and the variation in bottom topography for $x < 1/2$ are $\pi/2$ out of phase. The different x-ranges are required to keep the wind stress and the bottom variation as well as the gradients of these continuous. Otherwise there will be a discontinuity in the v_1 velocity component. Substitution into (13) and (14) and matching of the solutions at $x = 4/9$ and $x = 1/2$ gives the following distributions, for u_1 and v_1

$$x \geq \tfrac{1}{2} \begin{cases} u_1 = -\dfrac{A(1+T)}{f(1-H)} \\ v_1 = 0 \end{cases}$$

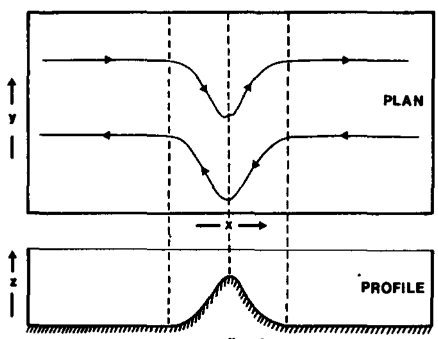

Fig. 3. Diagrammatic representation of the flow over a ridge. The opposite deflection occurs in the flow over a trough.

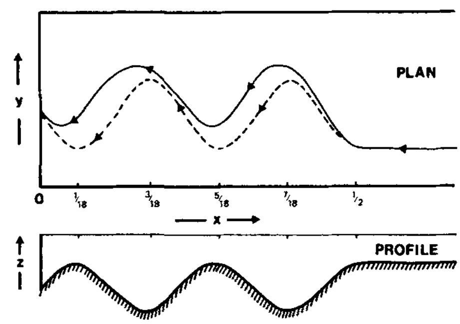

Fig. 4. The deviation caused by a wind stress perturbation that is $\pi/2$ out of phase with the bottom corrugation. The broken streamline indicates the flow with the constant wind stress $A(1+T)$ over the whole ocean.

$$\tfrac{4}{9}\leqslant x\leqslant\tfrac{1}{2}\begin{cases} u_1=\dfrac{-A(1+T)}{f(1-H\sin\pi x)} \\ v_1=\dfrac{9\pi A(HT+T\cos 9\pi x)}{\beta(1-H\sin 9\pi x)^2}\end{cases}\tag{19}$$

$$0\leqslant x\leqslant\tfrac{4}{9}\begin{cases} u_1=\dfrac{-A(1+T\cos 9\pi x)}{f(1-H\sin 9\pi x)} \\ v_1=\dfrac{9\pi A(HT+H\cos 9\pi x-T\sin 9\pi x)}{\beta(1-H\sin 9\pi x)^2}\end{cases}\tag{20}$$

Equation (20) shows how the wind stress and the bottom topography contribute to the deviation of the flow from a purely westward flow. It may be seen that as the magnitude T of the wind stress variation increases, v_1 becomes more out of phase with the gradient of the bottom topography.

In Fig. 4 the results of (19) and (20) are presented for the case $T=1/2$, $H=1/4$. When compared with the results of Hsueh (1967) several differences will be noted. These may be related to the fact that he used a constant Coriolis parameter and did not include the essential driving mechanism of the basic unperturbed wind stress curl.

9. Solutions when h has y-dependence

If h is independent of x, equation (14) may be written as

$$v_1(x,y)=\frac{fw^*}{fh_y-\beta(h-1)}$$

or

$$v_1(x,y)\frac{\partial}{\partial y}\left(\frac{f}{1-h}\right)=\frac{w^*}{(1-h)^2}\tag{21}$$

the second form serving as a comparison with the form given in Welander (1968). At a latitude where

$$\frac{\partial}{\partial y}\left(\frac{f}{1-h}\right)=0$$

or where the equivalent expression $R(y)=\beta(h-1)-fh_y$ vanishes, the interior solutions found are no longer valid. The non-linear and frictional terms will become important in the neighbourhood of such a "critical latitude". Welander (1968) discusses this phenomenon and correctly predicts the existence of a western boundary current and an eastern boundary current on the southern and northern sides of the "critical latitude", respectively. However, Welander also predicts a basically southward flow for both $R<0$ and $R>0$, a result which is not consistent with the circulations deduced in this paper. Inspection of (21) shows that in a region where w^* does not change sign, a change in sign of $R(y)$ means a reversal of the direction of the north-south interior velocity $v_1(x,y)$, as shown in Fig. 5.

The transition from a western boundary current to an eastern boundary current when the "critical latitude" is crossed can be explained in terms of the conservation of potential vorticity,

$$\frac{D}{Dt}\left\{\frac{f+\zeta}{1-h}\right\}=0$$

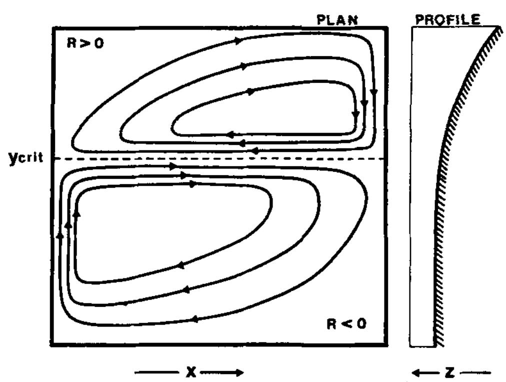

Fig. 5. Diagrammatic representation of the flow produced when the bottom deepens towards the North and causes a "critical layer" to form.

in the vicinity of boundary regions where the local wind stress terms and frictional terms are unimportant (inertial boundary layer) (Stommel, 1958).

Taking for example, the case of w^* negative then south of the "critical latitude" where

$$\frac{\partial}{\partial y}\left(\frac{f}{1-h}\right)$$

is positive ($R<0$) the interior flow is basically southwards and a northward flowing boundary current is required. Since $R<0$, $f/(1-h)$ increases in a northward flowing current and so for conservation of $(f+\zeta)/(1-h)$ this northward flow must give the incoming interior flow clockwise vorticity which is compatible with the distribution of shear for a western boundary only (Lighthill, 1966). North of the "critical latitude" where

$$\frac{\partial}{\partial \eta}\left(\frac{f}{1-h}\right)$$

is negative ($R>0$) the interior flow is basically northwards and a southward flowing boundary current is required. Now since $R>0$, $f/(1-h)$ again increases in a southward flowing current and potential vorticity is conserved if clockwise vorticity is induced into the incoming interior flow which in this case of a southward flowing boundary current is in agreement with the distribution of shear for an eastern boundary only.

An alternative explanation of this transition can be made using similar arguments to those of Pedlosky (1965) for the westward intensification of the oceanic circulation. Pedlosky suggests that the energy put into the ocean by the wind stress is concentrated near the western boundary because large-scale Rossby waves propagate to the west and are reflected as small-scale motions which leave their energy at the western boundary either because they are dissipated by friction or because they are trapped near the western boundary by the westward convection of fluid under the direct action of the wind stress. South of the "critical latitude" $R(y)$ is negative and the β-effect either dominates or is re-inforced by the effect of bottom topography (Veronis, 1966) and a western boundary current should exist. North of the "critical latitude" $R(y)$ is positive and the restoring effect of bottom variations is opposed to and dominates the β-effect. The existence of an eastern boundary current is thus expected with large-scale topographic waves travelling eastward and providing a concentration of energy at the eastern boundary.

The position and thickness of the critical layers, if any exist, may be determined as follows. A critical layer will exist if the continuous function $R(y)$ changes sign. Since $(h-1)$ is always negative, R can only vanish at points where h_y is negative (excluding cases when $h=1$). Hence a critical layer can only exist over topography where $h_y<0$; that is where the depth is increasing towards the North. It is near this critical line that not only the frictional terms become important, but also the non-linear intertial terms. The frictional terms become important in the region where

$$|R(y)| \leqslant E_H^{1/3} f\,|h_{yy}|$$

and the non-linear intertial terms where

$$|R(y)| \leqslant R_0^{1/3} E_H^{1/3} |h_{yy}|^{2/3}$$

where $R_0 = U/f_0 L$ is the Rossby number for the flow. The actual thickness of each region may be calculated for a given $h(y)$. In general the "non-linear region" is narrower than the "frictional region".

When both zonal and meridional variations of bottom topography are allowed, that is when h is a function of x and y, the location of boundary currents on either the eastern or western boundary is again determined by the sign of

$$\frac{\partial}{\partial y}\left(\frac{f}{1-h}\right)$$

However regions of the ocean which have boundary currents at opposite boundaries are no longer separated by a "critical latitude" but rather by a "critical line" or "critical contour" (Welander, 1968). In a quite detailed numerical investigation of a two-dimensional oceanic model Holland (1967) has examined the effect of various simple bottom shapes on the oceanic circulation. It is not surprising that in the sole case in which Holland allowed the ocean depth to increase towards the north the numerical solutions broke down for large values of his parameter characterising the inertial terms. Although the topography is not specifically defined one expects that somewhere $(\partial/\partial y)[f/(1-h)]$ vanishes.

APPENDIX

In the bottom Ekman layer an oblique stretched co-ordinate system (x, y, ζ) is used where $E^{\frac{1}{2}}\zeta = z + 1 - h(x, y)$. Then provided

$$h_x, h_y, h_{xx} + h_{yy} < \frac{E^{1/2}}{E_H}$$

the governing equations become

$$-fv = -P_x + \frac{h_x}{E^{1/2}} P_\zeta + \left\{1 + \frac{E_H}{E}(h_x^2 + h_y^2)\right\} u_{\zeta\zeta} \quad \text{(A 1)}$$

$$fu = -P_y + \frac{h_y}{E^{1/2}} P_\zeta + \left\{1 + \frac{E_H}{E}(h_x^2 + h_y^2)\right\} v_{\zeta\zeta} \quad \text{(A 2)}$$

$$0 = -\delta^{-2} E^{-1/2} P_\zeta + \left\{1 + \frac{E_H}{E}(h_x^2 + h_y^2)\right\} w_{\zeta\zeta} \quad \text{(A 3)}$$

$$u_x - \frac{h_x}{E^{1/2}} u_\zeta + v_y - \frac{h_y}{E^{1/2}} v_\zeta + \frac{1}{E^{1/2}} w_\zeta = 0 \quad \text{(A 4)}$$

With the following expansions

$$\mathbf{u}_B = E^{1/2}\bar{\mathbf{u}}_1 + E\bar{\mathbf{u}}_2 + \ldots,\ P_B = E^{1/2}\bar{P}_1 + E\bar{P}_2 + \ldots$$

where the overbars indicate the solutions in the bottom Ekman layer. Equation (A 4) implies to $0(E)$ that

$$h_x\bar{u}_1 + h_y\bar{v}_1 - \bar{w}_1 = \text{function of } x, y$$

Applying the boundary condition that the velocity is zero at $\zeta = 0$ $(z = 0)$ gives

$$\bar{w}_1 = h_x\bar{u}_1 + h_y\bar{v}_1 \quad \text{(A 6)}$$

From further substitution of (A 5) into (A 1) to (A 4) it may be shown that $\bar{u}_1$ and $\bar{v}_1$ satisfy the Ekman layer equation

$$\left[\{1 + \delta^2(h_x^2 + h_y^2)\}\{1 + \frac{E_H}{E}(h_x^2 + h_y^2)\frac{\partial^5}{\partial\zeta^5} + f^2\frac{\partial}{\partial\zeta}\right](\bar{u}_1, \bar{v}_1) = 0$$

whose bounded solution as $\zeta \to \infty$ is

$$\bar{u}_1 = \bar{u}_{11}(x, y) + \bar{u}_{12}e^{-(1+i)K\zeta} + \bar{u}_{13}e^{-(1-i)K\zeta} \quad \text{(A 7)}$$

$$\bar{v}_1 = \bar{v}_{11}(x, y) + \bar{v}_{12}e^{-(1+i)K\zeta} + \bar{v}_{13}e^{-(1-i)K\zeta} \quad \text{(A 8)}$$

where

$$K = \left(\frac{f}{2}\right)^{\frac{1}{2}}\left\{1 + \frac{E_H}{E}(h_x^2 + {}_y^2)\right\}^{-\frac{1}{4}}\{1 + \delta^2(h_x^2 + h_y^2)\}^{-\frac{1}{4}}$$

Further restrictions on the function $h(x, y)$ must now be made in order that the thickness of the Ekman layer, $E^{\frac{1}{2}}/K$, remains much smaller than unity. As the restrictions will be placed on an upper bound for $(h_x^2 + h_y^2)^{\frac{1}{2}}$ we will consider the case where $(h_x^2 + h_y^2) \gg 1$ coupled with:

(a) $\delta(h_x^2 + h_y^2)^{\frac{1}{2}} \gg 1$. Then $K = 0\{(E/E_H\delta)^{\frac{1}{4}}(h_x^2 + h_y^2)^{-\frac{3}{4}}\}$ and the thickness of the Ekman layer will be much less than unity only if

$$(h_x^2 + h_y^2)^{\frac{1}{2}} < < (\delta E_H)^{-\frac{1}{3}}$$

(b) $\delta(h_x^2 + h_y^2)^{\frac{1}{2}} < < 1$. Then $K = 0\{(E/E_H)^{\frac{1}{4}}(h_x^2 + h_y^2)^{-\frac{1}{4}}\}$ and the corresponding requirement is

$$(h_x^2 + h_y^2)^{\frac{1}{2}} < E_H^{\frac{1}{2}}$$

Hence the Ekman layer solution will only break down for slopes contained within the thickness of the upwelling layers of thickness $E_H^{\frac{1}{2}}$ or $(\delta E_H)^{\frac{1}{3}}$ (Pedlosky, 1968).

Matching (A 7) and (A 8) with the interior solution (3) gives

$$\bar{u}_{11} = u_1 \quad \text{and} \quad \bar{v}_{11} = v_1$$

Then the boundary conditions $\bar{u}_1 = \bar{v}_1 = \bar{w}_1 = 0$ at $\zeta = 0$ and the $0(1)$ terms of (A 1), (A 2) determine the arbitrary functions $\bar{u}_{12}$, $\bar{u}_{13}$, $\bar{v}_{12}$, $\bar{v}_{13}$ in terms of u_1 and v_1. The resulting solution is then

$$\bar{u}_1 = u_1 + e^{-K\zeta}[-u_1\cos K\zeta - v_1\{1 + \delta^2(h_x^2 + h_y^2)\}^{-\frac{1}{2}}\sin K\zeta]$$

$$\bar{v}_1 = v_1 + e^{-K\zeta}[-v_1\cos K\zeta + u_1\{1 + \delta^2(h_x^2 + h_y^2)\}^{\frac{1}{2}}\sin K\zeta]$$

Equation (A 6) then gives the vertical velocity in the lower Ekman layer as

$$w_B = E^{\frac{1}{2}}(h_xu_1 + h_yv_1) + \text{exponential terms} \quad \text{(A 9)}$$

Acknowledgements

The authors would like to express their appreciation to Dr R. K. Smith for many helpful comments. This work was carried out whilst J. A. J. was a visiting lecturer in the Mathematics Department at Monash University and whilst C. B. F. was supported by a CSIRO Postgraduate Studentship.

REFERENCES

Barrett, J. R. 1965. Subsurface currents off Cape Hatteras. *Deep Sea Res. 12*, 173–184.

Clarke, R. A. & Fofonoff, N. P. 1969. Oceanic flow over varying bottom topography. *J. Mar. Res. 27*, 226–240.

Holland, W. R. 1967. On the wind-driven circulation in an ocean with bottom topography. *Tellus 19*, 582–600.

Hsueh, Y. 1969. Bottom topography and variable wind stress. *J. Geophys. Res. 72*, 4101–4107.

Johnson, J. A. 1968. A three dimensional model of the wind driven ocean circulation. *J. Fluid Mech. 34*, 721–734.

Lighthill, M. J. 1966. Dynamics of rotating fluids: a survey. *J. Fluid Mech. 26*, 411–431.

Pedlosky, J. 1965. A note on the western intensification of the oceanic circulation. *J. Mar. Res. 23*, 207–209.

Pedlosky, J. 1968. An overlooked aspect of the wind-driven oceanic circulation. *J. Fluid Mech. 32*, 809–821.

Stommel, H. 1958. The Gulf Stream. University of California Press.

Veronis, G. 1966. Rossby waves with bottom topography. *J. Mar. Res. 24*, 338–349.

Warren, B. A. 1963. Topographic influences on the path of the Gulf Stream. *Tellus 15*, 167–183.

Welander, P. 1968. Wind-driven circulation in one and two layer oceans of variable depth. *Tellus 20*, 1–16.

ОБ ИЗМЕНЧИВОСТИ ОКЕАНИЧЕСКОЙ ЦИРКУЛЯЦИИ, ПОРОЖДАЕМОЙ ТОПОГРАФИЕЙ ДНА

Изучается трёхмерная циркуляция, порождаемая заданным напряжением ветра в однородном океане с произвольной топографией дна. Решение иллюстрируется на примере различных форм дна при заданном напряжении ветра. Показано, что в общем случае течение отклоняется на юг в области убывающей глубины и на север в области возрастающей глубины. Однако, если напряжение ветра имеет возмущение, фаза которого противоположна неоднородности дна, то предыдущий результат существенно видоизменяется. Более того, если океан углубляется к северу, то может существовать «критическая линия», на которой $\partial/\partial y(f/H)$ обращается в нуль (f — параметр Кориолиса, H — глубина и y — координата, направленная к северу). Океаническая циркуляция разделяется тогда на две части: если вихрь напряжения ветра положителен, то на широтах севернее «критической линии», где $\partial/\partial y(f/H)$ положительно, внутреннее топографическое течение Свердрупа направлено на север, и существует восточное пограничное течение, тогда как для широт южнее критической линии, где $\partial/\partial y/f/H)$ отрицательно, внутреннее течение направлено на юг, и существует западное пограничное течение.