# Spectra of thermally stratified turbulent flow with no shear

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ABSTRACT

In this paper a new analytical approximation, concerning the integral $\int_k^\infty \phi_{WT}(p)\,dp$ in the spectral equations (3), (4) is suggested. It is based on the physical idea about natural realization of the degree of interaction between the mean and fluctuating fields and is analytically expressed by the formulae (10, (12) and (13). In the extreme cases of strong and weak interaction the general expression (13) has (11) as its asymptotes.

The system of spectral equations (14), (15) is solved numerically and the result is presented in the figure. It is seen that in case of stable stratification, a general inertial range exists, in the middle of which a buoyancy subrange appears with slopes of the spectral curves as shown in the figure. The results eliminate the so far unresolved contradiction between the theories of Bolgiano and Lumley-Shur and agree well with the experimental data.

In a medium with density fluctuations, where the gravity force acts, the buoyancy effect plays an essential role. In the case of atmospheric air the density fluctuations are caused mainly by temperature fluctuations, so that in practice thermally stratified flows are under investigation.

In a stably stratified medium the vertical motions, created by the turbulence (which has dynamic origin), are accompanied by loss of energy for work against the buoyancy force. In this way a part of the kinetic energy of the turbulent vertical motions and therefore a part of the total turbulent energy transforms into potential energy of the density stratification. In other words, the stability suppresses the turbulence.

Inversely, in unstably stratified medium the potential energy of the density stratification releases and transforms into kinetic energy of the turbulent motion, i.e. the instability stimulates the turbulence.

In the same time, if an energy source exists supporting the initial stratification, inflow or draining of temperature energy into or from the temperature fluctuation field occurs respectively.

The briefly described qualitative picture has been more or less successfully reflected in both the Bolgiano's and Lumley-Shur's hypotheses for the spectral energy exchange between the velocity and temperature fields in case of stably stratified turbulent flow with no shear (no production of kinetic energy in the buoyancy subrange).

Bolgiano (1959) postulates that the spectral characteristics of the turbulence depend only on the following dimensional parameters: the rate of total dissipation of temperature inhomogeneity $N$ due to molecular diffusivity, buoyancy parameter $\beta = g/T_0$ and wave number $k$. The other dimensional parameter $\varepsilon$ (the rate of total dissipation of turbulent energy by molecular viscosity) drops from the list of determining parameters, since in case of strong stability as a result of the intensive turbulent energy drainage the value of $\varepsilon$ is negligible as compared to the energy flux in the buoyancy subrange. For the turbulent kinetic and temperature energy spectra dimensional arguments give respectively

$$\phi(k) \sim N^{2/5}\beta^{4/5}k^{-11/5}, \quad \phi_T(k) \sim N^{4/5}\beta^{-2/5}k^{-7/5} \qquad (1)$$

The experimental work by Shur (1962) provokes the creation of a new theory for the buoyancy subrange, developed by Lumley (1964, 1965). Based on the extended Kovasznay's hypothesis of local dependence and on the assumption that the heat flux is proportional to the mean temperature gradient Lumley obtains

$$\phi(k) \sim k^{-3}, \quad \phi_T(k) \sim k^{-1} \tag{2}$$

There is an obvious contradiction between these two theories because of the different physical backgrounds they are based on. It can not be solved by the experiment for the present, so the solution has to be found theoretically. This question has been widely discussed at the international colloquiums in Moscow (1965) and Stockholm (1969).

For the first time Bolgiano's spectra have been obtained as solutions of the spectral equations by Monin (1962). Later Gissina (1966), applying the Tchen's ideas about strong and weak interaction to the temperature field, obtains the same spectra in case of weak interaction in the velocity field and strong interaction in the temperature field. Monin (1965) and Phillips (1965) make comparative analyses of these two hypotheses. Monin shows that the results of both theories come from dimensional reasonings with different determining parameters. It is shown in Phillips' paper that (1) and (2) are consequences from one and the same system of equations, i.e. both theories are compatible. A closer relation between the two hypotheses was established by Lin (1969). He was the first to introduce the concept of intermediate interaction and suggested a model, in which the degree of interaction changes with respect to an external numerical parameter. Applying this model to the temperature field and solving the spectral equations, in case of weak interaction in the velocity field and stable conditions Lin obtained a power law for the turbulent energy spectrum. The exponent is a function of the above mentioned external parameter. It becomes equal to $-11/5$ (i.e. Bolgiano's spectrum) in case of strong interaction in the temperature field and to $-3$ (i.e. Lumley-Shur's spectrum) in case of weak interaction. In this way Lin (1969) obtained the results of both theories as asymptotic parametric cases of a general solution of the spectral equations.

In the present paper, for modelling the spectral heat flux, we shall use the idea of natural realization of the degree of interaction between the mean and turbulent motions over different wave numbers, developed by Syrakov (1970, 1971).

The equations describing the spectral balances of turbulent kinetic and temperature energy in the flow under consideration after neglecting the dissipative effects (at high wave numbers $k$) are of the type:

$$\varepsilon = \int_k^\infty F(p)\,dp + \beta \int_k^\infty \phi_{WT}(p)\,dp \tag{3}$$

$$N_* = \int_k^\infty F_T(p)\,dp - 2\frac{d\bar{T}}{dz}\int_k^\infty \phi_{WT}(p)\,dp \tag{4}$$

where $N_* = 2N$, $d\bar{T}/dz$ is the mean temperature gradient and $\phi_{WT}(k)$ is the vertical heat flux spectrum.

The first terms on the right hand side of (3) and (4) describe the inertial transfers of turbulent kinetic and temperature energy over the hierarchy of eddies and for their modelling we shall use Heisenberg's eddy-viscosity approximation:

$$\int_k^\infty F(p)\,dp = \nu_T(k)\int_0^k 2p^2\phi(p)\,dp = \nu_T(k)\,\omega_T^2(k) \tag{5}$$

$$\int_k^\infty F_T(p)\,dp = \nu_T^T(k)\int_0^k 2p^2\phi_T(p)\,dp = \nu_T^T(k)\,\gamma_T^2(k) \tag{6}$$

Here $\nu_T(k)$ and $\nu_T^T(k)$ are spectral turbulent exchange coefficients for momentum and heat respectively, $\omega_T(k)$ and $\gamma_T(k)$ are the mean squared vorticity and the turbulent temperature gradient. Further we assume:

$$\nu_T^T(k) = f\nu_T(k) \tag{7}$$

where $f$ is a constant. It is easy to determine that $f = 2\alpha/\alpha_T \approx 2.5$. $\alpha$ and $\alpha_T$ are the universal numerical constants in the Kolmogorov's and Obukov's "$-5/3$ laws" for the inertial and convective subranges

$$\phi(k) = \alpha\varepsilon^{2/3}k^{-5/3}, \quad \phi_T(k) = \alpha_T N\varepsilon^{-1/3}k^{-5/3} \tag{8}$$

In modelling the third terms in (3) and (4) the semiempirical relation is used

$$\int_0^\infty \phi_{WT}(k)\,dk = \overline{wT'} = -K_T\frac{d\bar{T}}{dz} \tag{9}$$

By analogy with (9) and in agreement with the general idea for the diffusion nature of the transfer mechanism both in the temperature and velocity fields, the following expression may be written:

$$\int_k^\infty \phi_{wT}(p)\,dp = -\nu_T^T(k)\,\Gamma \qquad (10)$$

where $\Gamma$, still undermined, is temperature gradient. To determine this gradient Gisina (1966) extended Tchen's (1953) original conception about weak and strong interactions as follows:

$$\Gamma = \begin{cases} \gamma & \text{at} \quad \gamma < \gamma_T(k) \\ \gamma_T(k) & \text{at} \quad \gamma \gg \gamma_T(k) \end{cases} \qquad (11)$$

where $\gamma = |d\bar{T}/dz|$. Further she interprets these cases in the following way: in case of weak interaction the characteristic scale of change of the mean temperature field (it may be identified with $\gamma^{-1} = |dz/d\bar{T}|$) is much bigger than the same one of the turbulent temperature field ($\gamma_T^{-1}(k)$ and the heat transfer will be realized by the mean temperature gradient. Inversely, in case of strong interaction the characteristic scale of change of the mean temperature field will be so small, that the big eddies from the equilibrium range will perceive the mean gradient as a turbulent fluctuation, due to eddies of certain scale. For them the mean temperature gradient does not exist and the heat flux will be realized by the temperature fluctuation gradient. There exists a case of intermediate interaction when both scales are of one and the same order.

But by fixing *a priori* the degree of interaction (strong, weak, intermediate), as is done by Tchen (1953), Gissina (1966, 1969), Lin (1969) and others, actually means that the ratio between the gradients of the mean and fluctuation temperature fields is given in advance. However, only one of these quantities ($\gamma$) can be considered as constant; the other one ($\gamma_T(k)$) varies with respect to the wave number $k$. The ratio between them will vary with respect to $k$ too and will be different in different spectral domains. Actually at low wave numbers $\gamma_T(k)$ is small and values of $k$ will exist, at which the inequality $\gamma_T(k) < \gamma$ is fulfilled. At these eddies strong interaction occurs. The increase of $k$ weakens this inequality and at eddies for which $\gamma_T(k) \sim \gamma$ intermediate interaction will take place. The wave number $k_T^i$, defined by $\gamma_T(k_T^i) = \gamma$, will be called "intermediate interaction centre" for the temperature. Further increase of $k$ leads to $\gamma_T(k) \gg \gamma$, i.e. to the weak interaction subrange. At higher $k$ any interaction disappears, i.e. the inertial-convective subrange begins. According to these arguments such a function $\Gamma(k)$ must be constructed, in which the change of $k$ will result in realizing consecutively the above mentioned cases. Following Syrakov (1970, 1971) we assume that the characteristic scale of change of the total (mean plus turbulent) temperature field $\Gamma^{-1}(k)$ is a sum of the corresponding scales of the component fields:

$$\Gamma^{-1}(k) = \gamma^{-1} + \gamma_T^{-1}(k) \qquad (12)$$

i.e.

$$\Gamma = \frac{\gamma\gamma_T(k)}{\gamma + \gamma_T(k)} \qquad (13)$$

With the use of this expression for $\Gamma$ the eqs. (3) and (4) become:

$$\varepsilon = \nu_T(k)\left[\omega_T^2(k) \pm f\beta\frac{\gamma\gamma_T(k)}{\gamma + \gamma_T(k)}\right] \qquad (14)$$

$$N_* = f\nu_T(k)\left[\gamma_T^2(k) + 2\frac{\gamma^2\gamma_T(k)}{\gamma + \gamma_T(k)}\right] \qquad (15)$$

where the sign "—" refers to stable ($d\bar{T}/dz > 0$) and the sign "+" to unstable ($d\bar{T}/dz < 0$) stratification.

For the sake of convenience and for universality of the results we pass to the following dimensionless variables:

$$x = k/k_d, \quad k_d = \alpha^{-3/4}(\varepsilon N_*^{-3}\gamma^6)^{1/4}$$

$$\psi(x) = \phi(k)/\phi_d, \quad \phi_d = \alpha^{9/4}(\varepsilon N_*^5\gamma^{-10})^{1/4}$$

$$\psi_T(x) = \phi_T(k)/\phi_T^d, \quad \phi_T^d = \alpha^{9/4}(\varepsilon^{-3}N_*^9\gamma^{-10})^{1/4}$$

$$\tilde{\nu}_T(x) = \nu_T(k)/\nu_T^d, \quad \nu_T^d = N_*\gamma^{-2} \qquad (16)$$

In the new variables the spectra (8) become:

$$\psi(x) = x^{-5/3}, \quad \psi_T(x) = f^{-1}x^{-5/3} \qquad (17)$$

and they are convenient asymptotes of the solutions to be obtained at $x \to \infty$.

Let

$$Z(x) = \left[\int_0^x 2q^2\psi(q)\,dq\right]^{1/2} \left(\text{i.e. } \psi(x) = \frac{Z}{x^2}\frac{dZ}{dx}\right) \quad (18)$$

$$Y(x) = \left[\int_0^x 2q^2\psi_T(q)\,dq\right]^{1/2} \left(\text{i.e. } \psi_T(x) = \frac{Y}{x^2}\frac{dY}{dx}\right) \quad (19)$$

Then the eqs. (14) and (15) become

$$1 = \tilde{\nu}_T(x)\left(Z^2 - fm\frac{Y}{1+Y}\right) \quad (20)$$

$$1 = f\tilde{\nu}_T(x)\left(Y^2 + 2\frac{Y}{1+Y}\right) \quad (21)$$

where

$$m = \beta N/\varepsilon \frac{dT}{dz} \quad (22)$$

is a nondimensional parameter. In case of stable stratification $m > 0$ (because $dT/dz > 0$) and it increases with the stability. The conditions for existence of a buoyancy subrange, obtained by Lumley (1965) and later by Gissina (1966) are equivalent to $m \gg 1$. In case of unstable stratification $m < 0$ and the restriction $m > -1$ is in force.

If we divide (20) by (21), $\tilde{\nu}_T(x)$ cancels out and we obtain an independent on the wave number relationship between $Y$ and $Z$

$$Y^2 + 2(m+1)\frac{Y}{1+Y} = f^{-1}Z^2 \quad (23)$$

At low $x$ from (20) and (23) the following asymptotic equation it obtained:

$$1 = \tilde{\nu}_T(x)\frac{Z^2}{m+1} \quad (24)$$

At any model for $\nu_T(k)$ it has the solution $\psi(x) = (m+1)^{2/3}x^{-5/3}$.

At high $x$ the equation (20) reduces to

$$1 = \tilde{\nu}_T(x)\,Z^2 \quad (25)$$

It does not depend on the stratification and has Kolmogorov's spectrum $\psi(x) = x^{-5/3}$ as a solution.

The asymptotic solutions just obtained show that a general inertial subrange over the whole spectrum range under consideration exists and the buoyancy effects excite it in a particular subrange. It follows from the solution of the equation (24) that in case of $m > 0$ the long wave " $-5/3$ asymptote" of $\psi(x)$ locates higher than the classical inertial " $-5/3$ curve" ($m = 0$) and therefore assures a steeper slope to the former one in the excited (buoyancy) subrange. This corresponds to a loss of turbulent energy, which is in agreement with the above discussed physical ideas about the mechanism of such a type of turbulence. In case of $m < 0$ the long wave " $-5/3$ asymptote" lowers and the spectrum slope in the buoyancy subrange is less steep, which corresponds to an inflow of turbulent energy in this intermediate spectral subrange. At $m$ close to $-1$ the appearance of humps in the spectral curve may be expected.

The system (20, 23) is solved numerically for the following models of $\nu_T(k)$: the model of Kesic (1969)

$$\nu_T(k) = \sqrt{\frac{2}{3}}\alpha^{-3/2}\frac{k\,\phi(k)}{\omega_T(k)}, \quad \tilde{\nu}_T(x) = \sqrt{\frac{2}{3}}\frac{1}{x}\frac{dZ}{dx} \quad (26)$$

and Kovasznay's model

$$\nu_T(k) = \frac{2}{3}\alpha^{-3/2}\sqrt{\frac{\phi(k)}{k}}, \quad \tilde{\nu}_T(x) = \frac{2}{3}\sqrt{\frac{Z}{x^3}\frac{dZ}{dx}} \quad (27)$$

In the expressions (26) and (27) the relationship (18) has been used.

In Fig. 1 the computed energy and temperature spectra at several different stratifications are represented. The energy spectra confirm previous quantitative conclusions according to the asymptotic solution. In case of stable stratification ($m > 0$) in the transitional spectral range between the two $x^{-5/3}$ asymptotes of $\psi(x)$ spectral subranges with $x^{-n}$ behavior exist. At lower wave numbers $n = 11/5$ and at higher $n = 3$. The arrow in the figure shows the centre of the interval which divide these two subranges. It is defined as wave number $x_T^i$, at which $Y(x_T^i) = 1$.

These results appear as direct consequence of the application of our idea about natural realization of the degree of interaction to the temperature field. In the " $-11/5$ subrange" (Bolgiano's spectrum) strong interaction takes place ($Y < 1$); in the subrange around $x_T^i$ the interaction is of intermediate degree ($Y \sim 1$); at $n = 3$ (Lumley-Shur's spectrum) the interaction is weak ($Y \gg 1$); after that the curves sharply pass into the inertial subrange. With the increase of the stability the slope of the curves in the weak interaction subrange becomes steeper than " $-3$", which is in agreement with the experimental results of Pinus & Scherbakova (1966). They have observed spectra with a slope up to " $-3.5$". When the stability decreases " $-11/5$" and " $-3$ subranges" shrink and at certain low value of $m$ the " $-3$ subrange" can even disappear. The further decrease of $m$ $(m \to 0)$ leads to disappearance of the whole buoyancy subrange, which is in agreement with the results of Lumley and Gissina.

The shape of the turbulent energy spectra, computed in the present paper, shows that in the stratified turbulent flows theory an essential progress is achieved. The contradiction between Bolgiano's and Lumley-Shur's theories seems finally solved. It is convincingly shown that the results of both theories follow not only as two parametric cases of the general solution (Lin, 1969) but that the Bolgiano's and Lumley-Shur's spectra are actually two wave subranges of one spectral cuve. This is one of the most important results of the application to the temperature field of our idea about natural realization of the degree of interaction over the wave numbers in the treated range.

In the temperature spectra, however, no similarity between (1), (2) and the spectra presented in the figure exists. This means that we cannot obtain the shape of the temperature spectra in the buoyancy subrange only by dimensional reasonings. In fact Monin (1965) and Lumley (1965) obtain different expressions ($k^{-3}$ and $k^{-1}$) for the temperature spectrum, though they proceed from correct assumptions. The temperature spectra presented in the figure are more objective because they are obtained as solutions of the spectral equations. The long wave asymptote " $-1/3$" means that the mean temperature gradient feeds the temperature fluctuation field without having influence on the inertial velocity spectrum. At higher wave numbers the buoyancy force action begins. In case of stable stratification the decrease of turbulent energy causes influx of temperature

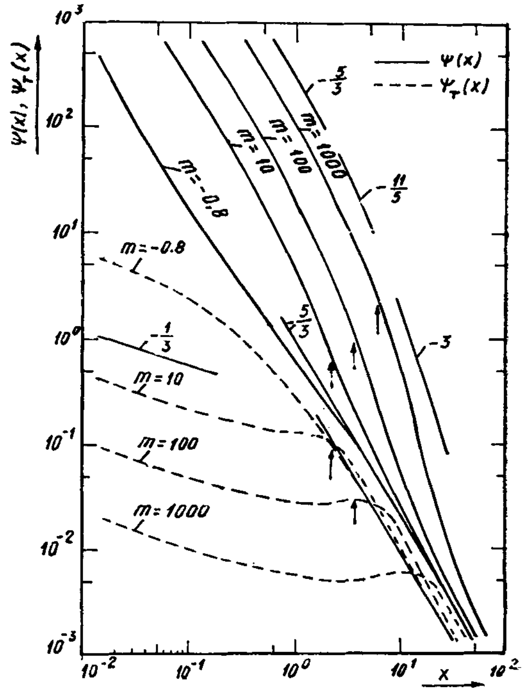


*Fig. 1.* Dimensionless turbulent spectra of velocity $\psi(x)$ and temperature $\psi_T(x)$ at stable $(m > 0)$ and unstable $(m < 0)$ stratifications.

energy and the temperature curves slope becomes less steep and even humps appear. In the case of unstable stratification the temperature spectrum slope is steeper, which corresponds to drainage of turbulent temperature energy and its transformation into kinetic.

In general the buoyancy subrange is located over about two decades of the nondimensional wave numbers $x(10^{-1} \div 10^{1})$ and moves to the right with the increase of $m$. The wave number $x_T^{t}$ moves in an analogical way as a centre of the whole buoyancy subrange.

The authors of this paper see the further development of their idea, concerning the natural realization of the degree of interaction between the mean and turbulent fields over the different wave numbers, in its application to the thermally stratified turbulent shear flow.

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## СПЕКТРЫ ТЕМПЕРАТУРНО СТРАТИФИЦИРОВАННОГО ТУРБУЛЕНТНОГО ПОТОКА БЕЗ СДВИГА СКОРОСТИ

В работе предложен новый подход к вопросу об аналитической аппроксимации интеграла $\int_K^\infty \Phi_{WT}(p)dp$ в спектральных уравнениях баланса (3) и (4). Он выражается формулами (10), (12) и (13) и в его основе лежит физическая идея об естественном осуществлении степени взаимодействия между осредненными и флюктуационными полями. В предельных случаях сильного и слабого взаимодействия общее выражение (13) имеет известные асимптоты (11).

Система спектральных уравнений (14), (15) решалась численно. Результаты вычислений представлены на фигуре. При стабильной стратификации они указывают на существование единого инерционного интервала, в междинном спектральном участке которого появляется интервал плавучести с наклонами спектральной кривой как показаны на фигуре. Устанавливается хорошее согласие теории с экспериментами. Устраняется противоречие между теориями Болжиано и Ламли–Шура.