Optical measurement of salt fingers

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ABSTRACT

The optical distortion of a ruled vertical grating, when viewed through a layer of salt fingers, is described and discussed by a simple random walk model for a ray path. Intensity fluctuations are computed and the magnitude of the effect appears to be large enough to measure in the thin transition region separating the step-like structure of the main thermocline (Tait & Howe, 1968).

I. Introduction

The convective phenomenon called "salt fingers" occurs when a layer of hot, salty and light water is placed above a layer of cold fresh and dense water. In a recent paper (Stern & Turner, 1969) the effect was also observed and measured using concentrated salt and sugar solutions at constant temperature. We also observed an interesting optical effect, which will be reported in this paper together with a calculation which indicates the feasibility of related photometric measurements in the main thermocline.

II. Descriptive

The adjoining figure, prepared for the author by Mr. Ian Fletcher of U.R.I., shows a rectangular tank 27 cm wide, 26 cm tall and 10 cm thick. Two overlapping grids have been mounted on the back side, one with vertical lines, one with horizontal lines, and in the middle of the tank one sees the crossed overlap region. Twenty-four hours before the time of this photograph the tank was half filled with a salt solution of specific gravity 1.12 and then a 1.10 sugar solution was carefully pured on top. When the pouring is completed one observes a thin (millimeters) horizontal interface containing quasi-laminar salt fingers, as described by Stern & Turner (*loc. cit.*). At this time the interface is optically translucent, compared to the transparent convecting layers on either side. The potential energy decreases slowly with time as the interface (and salt fingers) spreads in the vertical dimension. This interface occupies the middle third of the tank at the time of the photograph, and the salt fingers are neatly detected by the refractive distortion of the grid lines on the left side of the picture. Note that the horizontal grid lines in the right-center of the picture are undistorted, even though there are vertical salt fingers in this region too. All grid lines immediately above and below the salt finger layer are undisturbed, although some micro-structure is visible at the top and bottom of the tank. If one waits another twelve hours and takes another photograph all grid lines appear undistorted. Furthermore if the observer holds his eye steady in front of the tank no manifestation of the salt fingers is apparent. However if one steps back several feet and moves his eye parallel to the field of view then a weak "scintilliation" effect (variations in light intensity) is observed. It is believed that this is due to the small variations of light intensity on a fixed region of the retina—these variations being due to the collective focusing of many weak salt finger lenses. This effect is weaker for heat and salt solutions, because of the relatively small density variations. However it has also been observed in this case by increasing the optical path. The effect also decreases when the grid spacing is increased.

III. Refraction of light rays by salt fingers

A field of salt fingers has a great deal of short range order since each sinking finger is surrounded by (say) four rising fingers of lower density. However it is apparent that the refraction of a light ray which passes through a large number of fingers must be treated by a statisti-

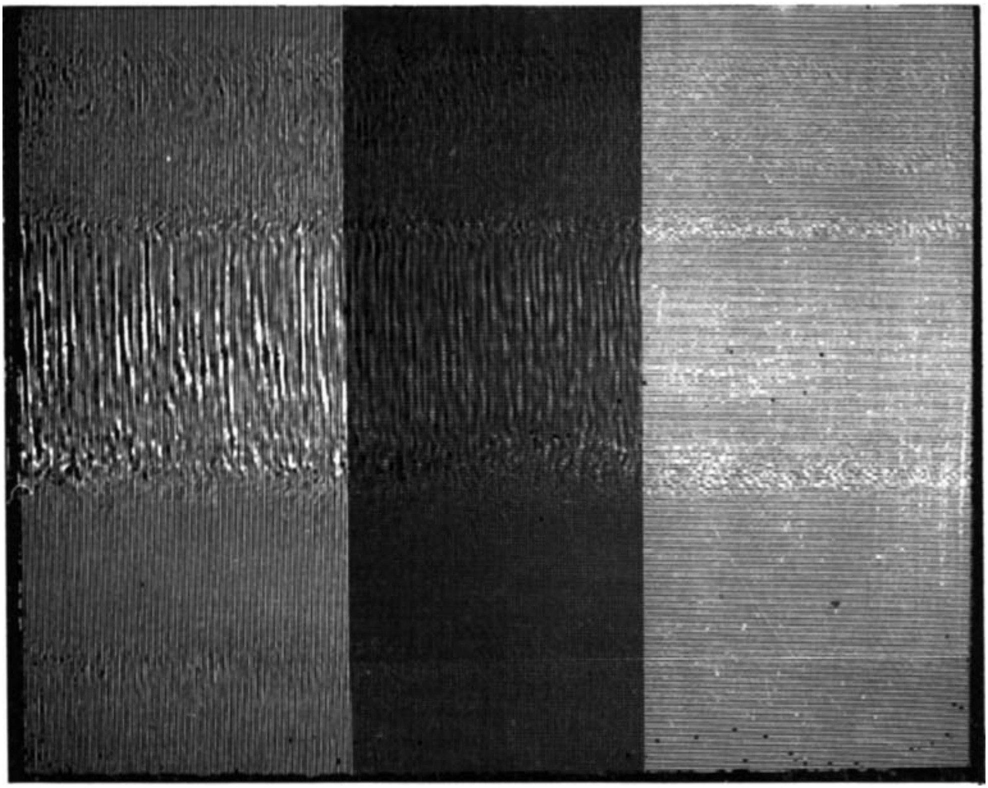

Fig. 1. The refractive distortion of ruled grids by salt fingers. The two overlapping grids are mounted on the back side of the rectangular tank. The main region of salt fingering is in a horizontal layer occupying the mid-third of the tank. Note that there is no distortion of the horizontal grid lines on the right, even though there are vertical salt fingers in this region. Some micro-structure is also visible at the top and bottom of the photograph, which was taken twenty-four hours after the start of the run.

cal theory. This is because the fluid motion (especially the relatively strong convection on either side of the salt finger layer) precludes a correlation of the gradient of index of refraction (∇z) over distances much larger than a finger width. In addition the refraction also depends on the orientation of the beam relative to ∇z, and this too will be a matter of chance. Therefore we anticipate that a ray which passes through N salt fingers will have an r.m.s. deviation angle which is $N^{1/2}$ times the expected angular deviation due to one salt finger. Actually we need to know the angular deviation of the end-point of the ray from the initial direction, and although this is of the same order as the afore-mentioned angle it is instructive to reproduce the simple derivation given below.

Consider a light ray which traverses a small distance l in a uniform medium of relatively low index of refraction z. The ray then encounters the interface of an adjacent region whose index of refraction is $z + \Delta z$. (Think of the latter as a down-going salt finger of square planform.) The ray is then bent to the right or the left of its original path by an angle $\Delta\phi$, depending on the chance orientation of interface and ray direction. The ray then traverses another distance l whereupon it encounters a second interface (which corresponds to the boundary of the first salt finger) and is again deflected to its right or left by the angle $\Delta\phi$. The sense of the latter is also a matter of chance since the second interface encountered by the ray may be either parallel or perpendicular to the first interface. And so on for N interfaces (salt fingers). The path of the light ray is a random walk but one which is different from that taken by a "drunkard". Our "perambulator" is quite conscious of the direction and length (l) of the last step. He is only prevented from traveling in a straight path by navigational uncertainties, which make the next step deviate $\pm\Delta\phi$ (with equal probability) from the previous step. We want to know the r.m.s. transverse displacement (y_N) from the original direction after N steps. We also need to identify the parameters in the statistical model with the physical phenomenon. l^{-1} is proportional to and of the same order of magnitude as the average number of salt fingers per unit length; Δz is proportional to $l|\Delta z|$ where $|\Delta z|$ is the r.m.s. gradient of index of refraction of the convecting fluid; $\Delta\phi < 1$ is proportional to Δz according to the r.m.s. average of Snells law. More detailed calculations may be found in the large mathematical literature on statistical optics (Frisch, 1968).

After passing through the first interface the ray has an angle $\alpha_1\,\Delta\phi$ relative to the initial direction, where $\alpha_n = \pm 1$ with equal probability. After passing through N interfaces the ray has an angle $\Delta\phi\sum_1^N\alpha_m$, relative to the initial angle. The statistics of that angle constitute a normal Gaussian process, but we want to know the transverse displacement of the end point of the ray and that requires a somewhat more involved calculation.

Since $\Delta\phi\sum_1^n\alpha_m$ is the angle of the ray between the n and $n+1$ interfaces, the transverse displacement of the ray in this interval is $l\sin(\phi\sum_1^n\alpha_m)$. The total transverse displacement after N interfaces is then:

$$y_N = \sum_{n=1}^{N} l \sin\left(\Delta\phi \sum_{m=1}^{n}\alpha_m\right) \qquad (1)$$

Subsequent calculations show that in the main thermocline $\Delta\phi$ is small of order 10^{-6} and $N\Delta\phi < 1$ (for practical reasons), so that the approximation

$$y_N = \sum_{n=1}^{N} l\Delta\phi \sum_{m=1}^{n}\alpha_m \qquad (2)$$

is quite satisfactory. Upon squaring eq. (2) and using the abbreviation

$$E_n = \sum_1^n \alpha_m \qquad (3)$$

one gets

$$\left(\frac{y_N}{l\Delta\phi}\right)^2 = (E_1 + E_2 + \cdots E_N)^2$$

$$= (E_1^2 + E_2^2 + \cdots E_N^2) + 2\sum_{n=2}^{N} E_n(E_1 + E_2 + \cdots E_{n-1}) \qquad (4)$$

The basic statistical assumption is:

$$\langle\alpha_n\alpha_m\rangle = 0 \quad \text{if } n \neq m$$
$$(\langle\alpha_n^2\rangle = 1) \qquad (5)$$

where brackets $\langle\ \rangle$ denote an (ensemble) average over different realizations of the ray path. If $n \geqslant m$ the correlation of E_n and E_m obtained from (3) is

$$\langle E_n E_m\rangle = \langle(\alpha_1 + \alpha_2 + \ldots \alpha_n)\,(\alpha_1 + \alpha_2 + \ldots \alpha_m)\rangle$$
$$= \langle\alpha_1^2\rangle + \langle\alpha_2^2\rangle + \ldots \langle\alpha_m^2\rangle$$
$$\langle E_n E_m\rangle = m \quad \text{for } m \leqslant n \qquad (6)$$

Therefore when we take the average of [4] we get

$$\left\langle\left(\frac{y_N}{l\Delta\phi}\right)^2\right\rangle = (1 + 2 + \cdots N) + 2\sum_{n=2}^{N}(1 + 2 + \cdots (n-1)) \qquad (7)$$

The right-hand side of the preceeding equation is evaluated from the progression summation formula:

$$\sum_1^N n = \frac{N^2 + N}{2}$$

$$\sum_1^N n^2 = \frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}$$

and (7) then becomes

$$\left\langle\left(\frac{y_N}{l\Delta\phi}\right)^2\right\rangle = \frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}$$

$$= \frac{N^3}{3} \quad \text{for } N \gg 1 \qquad (8)$$

Let $x = lN$ be the horizontal distance between N consecutive fingers, and substitute in (8). We then find that the angular deflection of the *end point* of the ray:

$$\left(\frac{\langle y_N^2\rangle}{x^2}\right)^{\frac{1}{2}} = \Delta\phi\left(\frac{N}{3}\right)^{\frac{1}{2}} \qquad (9)$$

is smaller by a factor of $\sqrt{3}$ than the r.m.s. angle of the ray itself. Of course, if the array of salt fingers is highly ordered then there will be certain orientations for which the deflection will be much larger than that given by (9).

IV. Fluctuations of light intensity

Since the intensity is the number of rays per unit area it is not difficult to convert the foregoing into intensity statistics. Accordingly we consider a collimated vertical slit of light of width $2t$ located in the "object" plane. Let η measure distance across this slit, with $\eta = 0$ at the center. The intensity of light, $I_0(\eta) = I^*$, is uniform from $\eta = -t$ to $\eta = +t$, and $I_0(\eta) = 0$ outside the slit. Thus we say that there are $I_0(\eta)d\eta$ light rays within the interval $d\eta$, all of which enter the fluid normal to the plane of the slit. The rays from this infinitesimal interval suffer identical refractions by the salt fingers through which they pass. They are received in the "image" plane, located at a distance x from the object plane. Here the transverse coordinate of the afore-mentioned ray lies in some interval $\xi = \eta + y$ to $\eta + y + d\eta$ where y is the random transverse deflection of the ray. The distance $\xi = \eta + y$ is the ray coordinate measured from the projection of the slit axis on the image plane. Let $P(y)dy$ denote the probability that a single ray will be transversely displaced through an interval between y and $y + dy$, where

$$\int_{-\infty}^{+\infty} P(y)\,dy = 1$$

$$\int_{-\infty}^{+\infty} y^2 P(y)\,dy = \langle y_N^2 \rangle \qquad (10)$$

Thus $(d\eta P(y))\ (I_0(\eta)d\eta)$ is the average number of rays between $\eta + y$ and $\eta + y + d\eta$ contributed by the differential element $d\eta$ on the object plane. Dividing by $d\eta$ to get the intensity and then summing all such contributions gives the following value for the average intensity at a fixed point ξ in the image plane

$$I(\xi)\rangle = \int_{-\infty}^{+\infty} I_0(\eta)\,P(\xi - \eta)\,d\eta$$

$$= I^* \int_{-t}^{+t} P(\xi - \eta)\,d\eta \qquad (11)$$

The first non-trivial moment of (11) is obtained by multiplying it by ξ^2 and integrating:

$$\frac{1}{I^*}\int_{-\infty}^{+\infty} \xi^2 \langle I(\xi)\rangle\,d\xi = \int_{-t}^{+t} d\eta \int_{-\infty}^{+\infty} (x+\eta)^2\,P(x)\,dx$$

$$= \int_{-t}^{+t} d\eta [\langle y_N^2\rangle + \eta^2]$$

or
$$\frac{\langle y_N^2\rangle}{t^2} = \frac{1}{2t^3}\int_{-\infty}^{+\infty} \xi^2 \frac{\langle I\rangle}{I^*}\,d\xi - \frac{1}{3}$$

By reintroducing the undisturbed intensity function $I_0(\eta)$ the preceeding equation may be written as:

$$\left\langle \frac{y_N^2}{t^2} \right\rangle = \frac{1}{2t^3}\int_{-\infty}^{+\infty} d\xi\, \xi^2 \left\{ \frac{\langle I\rangle - I_0(\xi)}{I^*} \right\} \qquad (12)$$

Now substitute (9) into (12) and replace $\Delta\phi$ by

$$\Delta\phi = \frac{l\,|\nabla z|}{z}$$

and N by

$$N = \frac{x}{l}$$

to get

$$\frac{1}{3}\,\frac{|\nabla z|^2}{z^2}\,\frac{lx^3}{t^2} = \frac{1}{2t^3}\left\langle \int_{-\infty}^{+\infty} d\xi\, \xi^2 \left\{ \frac{I(\xi) - I_0(\xi)}{I^*} \right\} \right\rangle \qquad (13)$$

All quantities in this equation are well-defined except the "mean free optical path" (l). Eq. (13) may then be regarded as a definition of l and the content resides in the correlation of l with the dimension of salt fingers as determined by other means.

V. Is it possible to "see" salt fingers in the thermocline?

A calculation of the order of magnitude of the right-hand side of (13) will now be made for a relatively "clean" region of the main thermocline, utilizing existing theoretical information about the expected salt finger structure. At a given depth the variation of index of refraction as a function of salinity S and temperature T may be written in the form:

$$\frac{1}{z}\nabla z = a(\nabla S - b\nabla T$$

$$a = \frac{1}{z}\left(\frac{\partial z}{\partial S}\right)_T$$

$$b = \left(\frac{\partial S}{\partial T}\right)_n$$

where a is the rate of increase of index of refraction with salinity at constant temperature and b is the rate of increase of S with T at constant index of refraction. By squaring and averaging the above equation we obtain the r.m.s. $|\nabla n|$ for use in (13). Denoting the r.m.s. salinity gradient by $|\nabla S|$ we may write the result as:

$$\frac{|\nabla z|^2}{z^2} = a^2|\nabla S|^2\left(1 - \frac{2\,br\sigma}{q} + \frac{\sigma^2 b^2}{q^2}\right) \qquad (15)$$

where r is the correlation coefficient between ∇T and ∇S, σ is a normalizing factor equal to the ratio of the mean vertical temperature gradient divided by the mean vertical salinity gradient and $q = \sigma|\nabla S|/|\nabla T|$. The estimates

$$r \simeq 0.2$$

$$q \simeq 10$$

$$|\nabla S| \simeq 0.01‰\ \mathrm{cm}^{-1}$$

given by Stern (1968) are based on general thermodynamic considerations of the world ocean. The following values of b and a were read from the graphs given by Montgomery (1957):

$$b = \frac{2.5‰}{5°\mathrm{C}}$$

$$a = \frac{0.001}{7‰}$$

and σ can be obtained from classical oceanographical measurements. The value we use is

$$\sigma = \frac{10°\mathrm{C}}{1‰}$$

With these values (15) becomes:

$$\frac{|\nabla n|^2}{n^2} \simeq \left(\frac{10^{-3}}{7}\right)^2 (10^{-2})^2 (1 - 0.2 + \tfrac{1}{4})$$

$$= 2 \times 10^{-12}\ \mathrm{cm}^{-2}$$

An estimate of the salt finger thickness:

$$l \sim 1\ \mathrm{cm}$$

can be obtained from theoretical studies (Stern, 1969). Although this estimate is not too reliable it is hard to believe that the oceanic fingers could be an order of magnitude larger or smaller. If we consider an optical path of

$$x = 10^3\ \mathrm{cm}$$

in the main thermocline and use a grid with

$$t = 10^{-1}\ \mathrm{cm}$$

we then find that the second moment of the mean intensity fluctuation (eq. 13) is 10%. Somewhat larger deviations are expected in a single realization.

If a photometric device were lowered in regions of the thermocline exhibiting regularly spaced "steps" (Tait & Howe, 1968) then the relatively thin transition layer should be associated with larger fluctuations of intensity (13) on the microscale. If such a program is feasible then it might be possible to calibrate (13) in laboratory experiments to obtain an oceanic value of l. Furthermore it may be possible to derive another non-redundant equation for the intensity statistics and these two simultaneous equations will allow a computation of both ∇z and l from the optical measurements. The effect of molecular scattering, absorption and diffraction have not been explicitly accounted for in the foregoing calculation. To some extent they may be compensated for by interpreting $I_0(\xi)$ in eq. (13) as the intensity observed in the absence of microstructure. The same reservations hold for particle scattering and absorption, and measurements should be confined to regions of the *deep* thermocline where such effects are minimal. In any case the modification of a beam of light after it passes through salt fingers is qualitatively different from the modification caused by particles. One should also try to correlate the optical signature with simultaneous measurements of temperature and salinity using an STD. Although there are many practical problems which must arise in the application of the method, we believe that it will be possible to adapt the micro-optical (or related) techniques to the important problem of measuring the dissipation range of oceanic turbulence.

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ОПТИЧЕСКИЕ ИЗМЕРЕНИЯ «СОЛЕНЫХ ПАЛЬЦЕВ»

Оптическая деформация поворачиваемой дифракционной решетки, рассматриваемой через слой жидкости с «солеными пальцами», описывается и обсуждается с точки зрения простой модели случайных блужданий светового луча. Вычисляется флуктуация интенсивности и величина эффекта представляется достаточно большой, чтобы было можно производить измерения в тонкой переходной области, определяющей ступенчатообразную структуру главного термоклина.