On the spectral equations and the statistical energy spectrum of atmospheric motions in the frequency domain[1]

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ABSTRACT

The kinetic energy spectral equations in the frequency domain was derived for the large-scale atmospheric motions. Some of the more important implications revealed by the spectral equations are: (1) the shape of the spectral density function $\phi_{ii}(\nu)$, is controlled by the out-of-phase relationships among various meteorological variables; (2) the rotation of the earth has an explicit influence on $\phi_{ii}(\nu)$ as well as on the average level of the atmospheric kinetic energy, and (3) an atmosphere that is efficient in spreading its kinetic energy among its various frequency components, so that there is no particular tendency for spectral peaks to form, should have a red-noise spectrum. A detailed discussion of these as well as of other implications is given in the text.

1. Introduction

The ultimate source of energy for atmospheric motions is the radiative energy of the sun. The kinetic energy of the atmosphere, once created, is constantly depleted by molecular dissipation and by the work which the atmosphere performs on the earth's surfaces. Since atmospheric motions never cease (at least in the time span of recorded history), the supply of energy is, on the average, equal to the exit of energy. Between the two ends of the energy cycle, however, there is a complicated chain of transformations among different forms of energy, energy exchange among different scales of motion, and energy transportation from one location to another. One of the major tasks of meteorologists has been to find out how energy transformation, exchange and transportation are carried out in the atmosphere.

The exchange of kinetic energy between different scales of fluid motion was first considered by Reynolds. In connection with his study of turbulence, he derived two kinetic energy equations from the equations of motion; one expressed the time rate of change of the kinetic energy of the mean motion, and one the time rate of change of the mean kinetic energy of the eddy motion (Reynolds, 1895). This was perhaps the first instance that the formulation of the energy equations for more than one scale of motion was considered necessary and beneficial. Reynolds' formulation was adopted, and at times modified and extended to include other forms of energy, by Ertel (1943), Calder (1949), Miller (1950), Van Mieghem (1952) and others, for the study of atmospheric energies.

The so-called eddy motions in the atmosphere, however, are themselves a composite of many scales of motion, ranging from minute fluctuations recorded by sensitive micrometeorological instruments, to the very large-scale planetary waves observed in the upper atmosphere. The interactions among various scales of eddy motions are often physically significant and interesting. It would therefore be unsatisfactory to speak only of the transformation and flow of energies between the mean motion on the one hand and all the eddy motions on the other. The need for a closer examination of the

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interactions among all scales of motion has been generally realized by meteorologists in recent years. Spectral equations of the atmospheric motions in the domain of wave number was presented by Saltzman (1957) and Dutton (1963), in the frequency domain by Chiu (1961, 1968) and in a mixed wave number and frequency domain by Kao (1968).

A close scrutiny of these studies on atmospheric spectral equations reveal that there are many interesting and significant properties and implications of these equations that have not been clearly brought out and emphasized on. We shall try to remedy this in this paper.

The purposes of this paper are:

1. To formulate the spectral equations of atmospheric motions from statistical point of view.
2. To compare the spectral equations derived by various investigators, and to point out their similarities and differences.
3. To discuss the meanings and implications of the derived spectral equations, and, out of these discussions, draw some statements concerning the possible characteristics of atmospheric energy spectrum.

2. Atmospheric motion as a random process

Although atmospheric motion changes incessantly with time, past records indicate that the general level of atmospheric kinetic energy has remained about the same for a long time. This is viewed as a consequence of the constancy of the factors which control the statistical features of atmospheric motions, such as the solar constant, the rate of earth rotation, the distribution and physical characteristics of land and sea, etc. In this sense atmospheric motion may be regarded as a quasi-stationary random process.

For a random process it is necessary to talk in terms of an ensemble. Since in reality, we only have a limited time record of meteorological phenomena which occurred on a single planet —the earth—we mentally make up the ensemble by conceiving that meteorological phenomena are being produced, not only by our own planet, but also by infinitely many other planet-earths, each of which is identical to our earth in every physical aspect, in age and in its relationship with its own solar system. However, at any one particular moment in the astronomical time of all these identical systems, the meteorological phenomena occurring at any one planet-earth, being but one out of many possible outcomes, may be quite different from those occurring at other planet-earths; and, together, they form an ensemble. As will be seen later, every meteorological parameter may be considered to consist of a climatological mean, a periodic component and a deviation from them. The deviation is considered to be stationary. The periodic component, which has a fixed phase angle, and the climatological mean, which may have a very slow trend, are not stationary. However, within a time span of decades and centuries, the controlling factors are very nearly constant, and the trend may be ignored. Because of the presence of the periodic component, we therefore elected to call the atmospheric motion a quasi-stationary random process.

3. The spectral equations of kinetic energy in the frequency domain

In the conventional x, y, p and t coordinate system employed by meteorologists, the equations of large-scale atmospheric motions may be written as:

$$\frac{\partial u^*}{\partial t}+u^*\frac{\partial u^*}{\partial x}+v^*\frac{\partial u^*}{\partial y}+\omega^*\frac{\partial u^*}{\partial p}-fv^*=-g\frac{\partial z^*}{\partial x}+F_1^* \tag{1}$$

$$\frac{\partial v^*}{\partial t}+u^*\frac{\partial v^*}{\partial x}+v^*\frac{\partial v^*}{\partial y}+\omega^*\frac{\partial v^*}{\partial p}+fu^*=-g\frac{\partial z^*}{\partial y}+F_2^* \tag{2}$$

where ω^* is the individual rate of change of pressure, z^* the height of the isobaric surface, and F_1^* and F_2^* the x and y components of the frictional forces respectively. The remaining symbols have their conventional meanings. The superscript, *, is used to denote the instantaneous value of the quantity.

Equations (1) and (2) are valid for any space-time point (x, y, p, t). When Eqs. (1) and (2) are multiplied by u^* and v^* respectively, for that same space-time point, and the products added, we obtain the well-known kinetic energy equation:

$$\frac{\partial K^*}{\partial t} + u^* \frac{\partial K^*}{\partial x} + v^* \frac{\partial K^*}{\partial y} + \omega^* \frac{\partial K^*}{\partial p} = -g \left[u^* \frac{\partial z^*}{\partial x} + v^* \frac{\partial z^*}{\partial y} \right] + u^* F_1^* + v^* F_2^* \qquad (3)$$

where $K^* = 1/2(u^{*2} + v^{*2})$ is the total (horizontal) kinetic energy of the air per unit mass.

Since the averages of the meteorological parameters, such as u^*, v^*, z^*, etc. at one particular location, over a long period to time may not be zero (the precise definition of time average will be given later), we shall consider the instantaneous value of each parameter to be the sum of a climatological mean, a periodic component (or the sum of several periodic components such as the annual and diurnal variations), and a deviation from the mean and the periodic component. That is

$$\left. \begin{array}{l} u^*(x, y, p, t) = \bar{u}(x, y, p) + u_p(x, y, p, t) + \\ u(x, y, p, t) \\ \text{and similar expressions for } v^*, \omega^*, z^*, \\ F_1^*, \text{ and } F_2^*. \end{array} \right\} \qquad (4)$$

The means, designated by an overbar, are considered to be independent of time (or approximately so). The periodic component, u_p, and the deviation (which may be called the fluctuating or eddying component), u, are function of both space and time. In addition, the means obtained from any one time record of the ensemble are considered to have the same values as that from any other time record of the ensemble. The deviations are considered stationary.

Later, we shall introduce covariance functions and their Fourier transforms between various pairs of meteorological parameters. Calculations of covariance functions are greatly facilitated if the means of the parameters concerned are removed. The existence of the Fourier transforms of the covariance functions also requires that the mean and periodic part of the parameters be removed. We therefore have resolved each parameter according to the manner given above in the anticipation of this requirement.

When Eq. (1), assigned to a space-time point (x, y, p, t), is multiplied by the velocity deviation $u(x, y, p, t')$ for the same geometrical position but for a different time t', we obtain:

$$u(t') \frac{\partial u^*(t)}{\partial t} + u(t') \left[u^*(t) \frac{\partial u^*(t)}{\partial x} + v^*(t) \frac{\partial u^*(t)}{\partial y} + \omega^* \frac{\partial u^*(t)}{\partial p} \right] - f u(t') v^*(t) = -g u(t') \frac{\partial z^*(t)}{\partial x} + u(t') F_1^*(t) \qquad (5)$$

(the common arguments in x, y, and p are not written out).

Since t' and t are selected independently of each other, they are considered as two independent variables. We shall focus our attention on an arbitrary geometrical point x, y, and p. We therefore only have to consider the dependency of each term of Eq. (5) on the two independent variables t' and t. We shall introduce a primitive transformation of variables such that:

$$\left\{ \begin{array}{l} \xi = \xi(t', t) = t' \\ \tau = \tau(t', t) = t - t' \end{array} \right. \qquad (6)$$

The inverse transformation is:

$$\left. \begin{array}{l} t = t(\xi, \tau) = t' + \tau = \xi + \tau \\ t' = t'(\xi, \tau) = \xi \end{array} \right\} \qquad (7)$$

Any function $F(t', t)$ may, under the above transformation, be expressed as:

$$F(t', t) = F[t'(\xi, \tau), t(\xi, \tau)] = G(\xi, \tau) = G[\xi(t', t), \tau(t', t)] = F(t', t) \qquad (8)$$

then

$$\left. \frac{\partial F}{\partial t} \right|_{t' = \text{const.}} = \frac{\partial G}{\partial \xi} \frac{\partial \xi}{\partial t} + \frac{\partial G}{\partial \tau} \frac{\partial \tau}{\partial t} = \left. \frac{\partial G}{\partial \tau} \right|_{\xi = \text{const.}} \qquad (9)$$

Introducing this transformation and redesignating ξ by t' after the transformation has been applied we may rewrite Eq. (5) as:

$$\frac{\partial}{\partial \tau} \left[u(t') u^*(t' + \tau) \right] + u' \left[u^* \frac{\partial u^*}{\partial x} + v^* \frac{\partial u^*}{\partial y} + \omega^* \frac{\partial u^*}{\partial p} \right] - f u' v^* = -g u' \frac{\partial z^*}{\partial x} + u' F_1^* \qquad (10)$$

where, for the sake of simplicity, the arguments in t' and $t = t' + \tau$ are written out only for the first term in the equation. In other terms $u(t')$ is designated by u', $u(t' + \tau)$ by u, and so forth. We shall often make use of these brief notations.

Combining Eq. (10) and a similar one obtained from Eq. (2), we have

$$\frac{\partial}{\partial \tau}[u' u^* + v' v^*] + u' \left[u^* \frac{\partial u^*}{\partial x} + v^* \frac{\partial u^*}{\partial y} + w^* \frac{\partial u^*}{\partial p}\right]$$
$$+ v' \left[u^* \frac{\partial v^*}{\partial x} + v^* \frac{\partial v^*}{\partial y} + \omega^* \frac{\partial v^*}{\partial p}\right] + f[u^* v' - u' v^*]$$
$$= -g \left[u' \frac{\partial z^*}{\partial x} + v' \frac{\partial z^*}{\partial y}\right] + u' F_1^* + v' F_2^* \qquad (11)$$

Designating the operation of ensemble average by $< >$ and applying this operation to Eq. (11), we have:

$$\frac{\partial}{\partial \tau}[\langle u' u\rangle + \langle v' v\rangle] + \left\langle u' \left[u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \omega \frac{\partial u}{\partial p}\right]\right\rangle$$
$$+ \left\langle v' \left[u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial p}\right]\right\rangle$$
$$+ \left\langle u' \frac{\partial u}{\partial x}\right\rangle (\bar{u} + u_p) + \left\langle u' \frac{\partial u}{\partial y}\right\rangle (\bar{v} + v_p) + \left\langle u' \frac{\partial u}{\partial p}\right\rangle$$
$$(\bar{\omega} + \omega_p)$$
$$+ \langle u' u\rangle \frac{\partial(\bar{u} + u_p)}{\partial x} + \langle u' v\rangle \frac{\partial(\bar{u} + u_p)}{\partial y} + \langle u' w\rangle$$
$$\frac{\partial(\bar{u} + u_p)}{\partial p}$$
$$+ \left\langle v' \frac{\partial v}{\partial x}\right\rangle (\bar{u} + u_p) + \left\langle v' \frac{\partial v}{\partial y}\right\rangle (\bar{v} + v_p) + \left\langle v' \frac{\partial v}{\partial p}\right\rangle$$
$$(\bar{\omega} + \omega_p)$$
$$+ \langle v' u\rangle \frac{\partial(\bar{v} + v_p)}{\partial x} + \langle v' v\rangle \frac{\partial(\bar{v} + v_p)}{\partial y} + \langle v' \omega\rangle \frac{\partial(\bar{v} + v_p)}{\partial p}$$
$$+ f[\langle uv'\rangle - \langle u'v\rangle] = -g\left[\left\langle u' \frac{\partial z}{\partial x}\right\rangle + \left\langle v' \frac{\partial z}{\partial y}\right\rangle\right] +$$
$$\langle u'F_1\rangle + \langle v'F_2\rangle \qquad (12)$$

We obtained the above equation by virtue of the fact that the means and the periodic components are constants when the operation of ensemble average is taken, and that the ensemble averages of the deviations are zero.

In addition to the assumption of stationarity, we further assume that the deviations are ergodic. With this assumption, we may replace the ensemble average of the deviations, or of their combinations, by a time average defined as

$$\overline{(\quad)} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} (\quad)\, dt' \qquad (13)$$

Replacing the ensemble averages in Eq. (12) by the time averages, we have:

$$\frac{\partial}{\partial \tau} \overline{[u' u + v' v]} + \overline{u' \left[u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \omega \frac{\partial u}{\partial p}\right]}$$
$$+ \overline{v' \left[u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \omega \frac{\partial v}{\partial p}\right]} + \overline{u' \frac{\partial u}{\partial x}} (\bar{u} + u_p)$$
$$+ \overline{u' \frac{\partial u}{\partial y}} (\bar{v} + v_p) + \overline{u' \frac{\partial u}{\partial p}} (\bar{\omega} + \omega_p) + \overline{u' u} \frac{\partial(\bar{u} + u_p)}{\partial x}$$
$$+ \overline{u' v} \frac{\partial(\bar{u} + u_p)}{\partial y} + \overline{u' \omega} \frac{\partial(\bar{u} + u_p)}{\partial p} + \overline{v' \frac{\partial v}{\partial x}} (\bar{u} + u_p)$$
$$+ \overline{v' \frac{\partial v}{\partial y}} (\bar{v} + v_p) + \overline{v' \frac{\partial v}{\partial p}} (\bar{\omega} + \omega_p) + \overline{v' u} \frac{\partial(\bar{u} + u_p)}{\partial x}$$
$$+ \overline{v' v} \frac{\partial(\bar{v} + v_p)}{\partial y} + \overline{v' \omega} \frac{\partial(\bar{v} + v_p)}{\partial p} + \overline{f[u v' - u' v]}$$
$$= -g \overline{\left[u' \frac{\partial z}{\partial x} + v' \frac{\partial z}{\partial y}\right]} + \overline{u' F_1} + \overline{v' F_1} \qquad (14)$$

We shall define a covariance function $R_{ik}(\tau)$ as:

$$R_{ik}(\tau) = \overline{u_i(t') u_k(t' + \tau)}$$
$$= \overline{[u_i^*(t') - (\bar{u}_i + u_{pi})(t')][u_k^*(t' + \tau) - (\bar{u}_k + u_{pk})(t' + \tau)]} \qquad (15)$$

where (for this study) i and k assume the values of 1 and 2 only. For example $u_1 = u$ and $u_2 = v$. Summation convention of repeated index also applies with the same limitation. From Eq. (15), it is observed that

$$R_{ik}(\tau) = R_{ki}(-\tau) \qquad (16)$$

$$R_{ii}(0) = \overline{u_i^2(t')} = \overline{u_i^2(t)} = \overline{u^2 + v^2} \qquad (17)$$

Similarly we may introduce the following notations:

$$R_{u'\omega}(\tau) = \overline{u(t')\, \omega(t' + \tau)} \qquad (18)$$

$$R_{u'(\partial z/\partial x)}(\tau) = \overline{u(t') \frac{\partial x(t' + \tau)}{\partial x}} \qquad (19)$$

$$R_{\mathbf{V}\cdot\nabla z}(\tau)=\overline{\mathbf{V}(t')\cdot\nabla z(t'+\tau)}$$

$$=R_{u'(\partial z/\partial x)}(\tau)+R_{v'(\partial z/\partial y)}(\tau) \qquad (20)$$

$$\mathbf{R}_{u'\nabla_3 u}(\tau)=\mathbf{e}_1\overline{u(t')\frac{\partial u(t'+\tau)}{\partial x}}+\mathbf{e}_2\overline{u(t')\frac{\partial u(t'+\tau)}{\partial y}}$$

$$+\mathbf{e}_3\overline{u(t')\frac{\partial u(t'+\tau)}{\partial p}} \qquad (21)$$

$$T_{u'u(\partial u/\partial x)}(\tau)=\overline{u(t')\,u(t'+\tau)\frac{\partial u(t'+\tau)}{\partial x}} \qquad (22)$$

where $\mathbf{v}$ is the horizontal velocity vector, ∇ the horizontal del-operator, ∇_3 the three-dimensional del-operator and $\mathbf{e}_1$ $\mathbf{e}_2$ and $\mathbf{e}_3$ the unit vectors in x, y, and p coordinates system. $\mathbf{R}_{u'\nabla_3 u}(\tau)$ is the covariance vector for u' and $\nabla_3 u$. We find that:

$$R_{\mathbf{V}'\cdot\nabla z}(\tau)=R_{\nabla z'\cdot\mathbf{V}}(-\tau) \qquad (23)$$

$$R_{\mathbf{V}'\cdot\mathbf{F}}(\tau)=R_{\mathbf{F}'\cdot\mathbf{V}}(-\tau) \qquad (24)$$

$$\mathbf{R}_{u'\nabla_3 u}(\tau)=\mathbf{R}_{(\nabla_3 u')u}(-\tau) \qquad (25)$$

$$T_{\mathbf{V}'\cdot[(\mathbf{V}_3\cdot\nabla_3)\mathbf{V}]}(\tau)=T_{[(\mathbf{V}'_3\cdot\nabla_3)\mathbf{V}']\cdot\mathbf{V}}(-\tau) \qquad (26)$$

where $\mathbf{F}$ is the horizontal frictional force vector, and

$$(\mathbf{V}_3\cdot\nabla_3)=u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+\omega\frac{\partial}{\partial p} \qquad (27)$$

With these notations Eq. (14) may be written as:

$$\frac{\partial R_{ii}(\tau)}{\partial\tau}+T_{\mathbf{V}'\cdot[(\mathbf{V}_3\cdot\nabla_3)\mathbf{V}]}(\tau)+(\bar{\mathbf{V}}_3+\mathbf{V}_{p3})\cdot[\mathbf{R}_{u'\nabla_3 u}(\tau)$$

$$+\mathbf{R}_{v'\nabla_3 v}(\tau)]+\mathbf{R}_{u'\mathbf{V}_3}(\tau)\cdot\nabla_3(\bar{u}+u_p)$$

$$+\mathbf{R}_{v'\mathbf{V}_3}(\tau)\cdot\nabla_3(\bar{v}+v_p)-f[R_{12}(\tau)-R_{21}(\tau)]$$

$$=-gR_{\mathbf{V}\cdot\nabla z}(\tau)+R_{\mathbf{V}'\cdot\mathbf{F}}(\tau) \qquad (28)$$

where $\bar{\mathbf{v}}_3$, $\mathbf{v}_{p3}$, and $\mathbf{v}_3$ are the three dimensional velocity vectors for the mean, the periodic motion and the fluctuating motion respectively. $[R_{12}(\tau)-R_{21}(\tau)]$ represents the phase relationship between u and v and in general is not equal to zero when τ is not zero. When $\tau=0$, $R_{12}(0)-R_{21}(0)=0$ and an equation, similar to Eq. (3) but for the mean kinetic energy of the fluctuating motion, can be deduced from Eq. (28).

The following transformations are introduced:

$$R_{ik}(\tau)=\int_{-\infty}^{\infty}\phi_{ik}(\nu)\,e^{i\nu\tau}\,d\nu \qquad (29)$$

$$\phi_{ik}(\nu)=\frac{1}{2\pi}\int_{-\infty}^{\infty}R_{ik}(\tau)\,e^{-i\nu\tau}d\tau \qquad (30)$$

where ν is the angular frequency. The i in the exponential stands for $\sqrt{-1}$. The existence of the above Fourier transformation pair requires that $R_{ik}(\tau)$ satisfies certain conditions, such as the condition that $R_{ik}(\tau)$ should have piecewise continuity and that the integral of the absolute value of $R_{ik}(\tau)$ with respect to all values of τ should be finite. Since we are dealing with a stationary random process, and that we have removed the means and the periodic components of the parameters concerned on the definition of R_{ik}, these conditions are likely to be met by $R_{ik}(\tau)$. In any case we could calculate $R_{ik}(\tau)$ only for τ up to a finite value. This means that in practice it would be necessary to approximate $R_{ik}(\tau)$ by a truncated $R_{ik}(\tau)$ which is equal to $R_{ik}(\tau)$ for $|\tau|$ less than a certain finite value and is zero otherwise. The truncated $R_{ik}(\tau)$ clearly permits the above transformations. The effect of such a truncated $R_{ik}(\tau)$ on $\phi_{ik}(\nu)$ is well-known.

From Eqs. (29) and (30), we have

$$\frac{\partial R_{ik}(\tau)}{\partial\tau}=\int_{-\infty}^{\infty}i\nu\phi_{ik}(\nu)\,e^{i\nu\tau}d\nu \qquad (31)$$

$$i\nu\phi_{ik}(\nu)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\partial R_{ik}(\tau)}{\partial\tau}e^{-i\nu\tau}d\tau \qquad (32)$$

Here, in order that these equations may be valid, we have assumed that the integral on the right hand side of Eq. (31) converges uniformly to $\partial R_{ik}(\tau)/\partial\tau$ for all τ. Our justification for this is that in the present study we are interested only in the spectrum of the large-scale atmospheric motions. Therefore, $\phi(\nu)$ not only is integrated to a finite value, but also itself becomes identically zero after a certain large value of ν. This effectively reduces the integral in Eq. (31) to a definite integral, and thus uniformly convergent. It also most likely renders $\partial R_{ik}(\tau)/\partial\tau$ absolutely integrable and the existence of its Fourier transform, Eq. (32).

From Eqs. (17) and (29) it follows that:

$$R_{ii}(0) = \overline{u^2 + v^2} = 2\int_0^\infty \phi_{ii}(\nu)d\nu \tag{33}$$

Therefore $\phi_{ii}(\nu)d\nu$ is the spectral contribution to the mean kinetic energy of the fluctuating motion $1/2(u^2+v^2)$, from the frequency interval ν to $\nu + d\nu$. Similarly, we introduce the following transformation pairs:

$$R_{\mathbf{V}'\cdot\nabla z}(\tau) = \int_{-\infty}^\infty \phi_{\mathbf{V}'\cdot\nabla z}(\nu)\, e^{i\nu\tau} d\nu \tag{34}$$

$$\phi_{\mathbf{V}'\cdot\nabla z}(\nu) = \frac{1}{2\pi}\int_{-\infty}^\infty R_{\mathbf{V}'\cdot\nabla z}(\tau)\, e^{-i\nu\tau} d\tau \tag{35}$$

$$T_{\mathbf{V}'\cdot[(\mathbf{V}_3\cdot\nabla)\mathbf{V}]}(\tau) = \int_{-\infty}^\infty \phi_{\mathbf{V}'\cdot[(\mathbf{V}_3\cdot\nabla_3)\mathbf{V}]}(\nu)\, e^{i\nu\tau} d\nu \tag{36}$$

$$\phi_{\mathbf{V}'\cdot[(\mathbf{V}_3\cdot\nabla_3)\mathbf{V}]}(\nu) = \frac{1}{2\pi}\int_{-\infty}^\infty T_{\mathbf{V}'\cdot[(\mathbf{V}_3\cdot\nabla)\mathbf{V}]}(\tau)\, e^{-i\nu\tau} d\tau \tag{37}$$

and others like them for $\mathbf{R}_{u'\Delta_3 u}(\tau)$, $R_{u'v_3}(\tau)$, etc.

When Eq. (28) is multiplied by $(1/2\pi)\, e^{-i\nu\tau}$ and integrated with respect to τ, we obtain, in view of the above transformation pairs, the following spectral equation:

$$\begin{aligned} i\nu\phi_{ii}(\nu) = &-\phi_{\mathbf{V}'\cdot[(\mathbf{V}_3\cdot\nabla_3)\mathbf{V}]}(\nu) - (\overline{\mathbf{V}}_3 + \mathbf{V}_{p3})\cdot[\boldsymbol{\phi}_{u'\nabla_3 u}(\nu) \\ &+ \boldsymbol{\phi}_{v'\nabla 3v}(\nu)] - \boldsymbol{\phi}_{u'\mathbf{V}_3}(\nu)\cdot\nabla_3(\bar{u}+u_p) \\ &- \boldsymbol{\phi}_{v'\mathbf{V}_3}(\nu)\cdot\nabla_3(\bar{v}+v_p) + f[\phi_{12}(\nu) - \phi_{21}(\nu)] \\ &- g\phi_{\mathbf{V}'\cdot\Delta z}(\nu) + \phi_{\mathbf{V}'\cdot\mathbf{F}}(\nu) \end{aligned} \tag{38}$$

Equation (38) is a complex equation and may be separated into a real part and an imaginary part. To do this we note that:

$$\begin{aligned} \phi_{12}(\nu) - \phi_{21}(\nu) &= \frac{1}{2\pi}\int_{-\infty}^\infty [R_{12}(\tau) - R_{21}(\tau)]\, e^{-i\nu\tau} d\tau \\ &= -i2\mathrm{Q}_{u'v}(\nu) \end{aligned} \tag{39}$$

where

$$\mathrm{Q}_{u'v}(\nu) = \frac{1}{2\pi}\int_0^\infty [R_{12}(\tau) - R_{12}(-\tau)]\sin \nu\tau\, d\tau \tag{40}$$

$\mathrm{Q}_{u'v}(\nu)$ is the quadrature spectrum for u and v, and

$$\begin{aligned} \phi_{\mathbf{V}'\cdot\nabla z}(\nu) &= \frac{1}{2\pi}\int_{-\infty}^\infty R_{\mathbf{V}'\cdot\nabla z}(\tau)\, e^{-i\nu\tau} d\tau \\ &= CO_{\mathbf{V}'\cdot\nabla z}(\nu) - i\mathrm{Q}_{\mathbf{V}'\cdot\nabla z}(\nu) \end{aligned} \tag{41}$$

where

$$CO_{\mathbf{V}'\cdot\nabla z}(\nu) = \frac{1}{\pi}\int_0^\infty \frac{1}{2}[R_{\mathbf{V}'\cdot\nabla z}(\tau) + R_{\mathbf{V}'\cdot\nabla z}(\tau)]\cos \nu\tau d\tau \tag{42}$$

and

$$Q_{\mathbf{V}'\cdot\nabla z}(\nu) = \frac{1}{\pi}\int_0^\infty \frac{1}{2}[R_{\mathbf{V}'\cdot\nabla z}(\tau) - R_{\mathbf{V}'\cdot\nabla z}(\tau)]\sin \nu\tau d\tau \tag{43}$$

$CO_{\mathbf{V}'\cdot\nabla z}$ and $Q_{\mathbf{V}'\cdot\nabla z}$ are the cospectrum and quadrature spectrum respectively for v and ∇z. By similar manipulations we obtain from Eq. (38) the following two equations:

$$\begin{aligned} O = &-CO_{\mathbf{V}'\cdot[(\mathbf{V}_3\cdot\nabla_3)\mathbf{V}]}(\nu) - (\overline{\mathbf{V}}_3 + \mathbf{V}_{p3})\cdot[\mathbf{CO}_{u'\nabla_3 u}(\nu) \\ &+ \mathbf{CO}_{v'\nabla_3 v}(\nu)] - [\mathbf{CO}_{u'\mathbf{V}_3}(\nu)\cdot(\bar{u}+u_p) + \mathbf{CO}_{v'\mathbf{V}_3} \\ &(\nu)\cdot\Delta_3(\bar{v}+v_p)] - gCO_{\mathbf{V}'\cdot\nabla z}(\nu) \\ &+ CO_{\mathbf{V}'\cdot\mathbf{F}}(\nu) \end{aligned} \tag{44}$$

and

$$\begin{aligned} \nu\phi_{ii}(\nu) = &\,\mathrm{Q}_{\mathbf{V}'\cdot[(\mathbf{V}_3\cdot\nabla_3)\mathbf{V}]}(\nu) + (\overline{\mathbf{V}}_3 + \mathbf{V}_{p3})\cdot[\mathbf{Q}_{u'\nabla_3 u}(\nu) \\ &+ \mathbf{Q}_{v'\nabla_3 v}(\nu)] + [\mathbf{Q}_{u'\mathbf{V}_3}(\nu)\cdot\nabla_3(\bar{u}+u_p) \\ &+ \mathbf{Q}_{v'\mathbf{V}_3}(\nu)\cdot\nabla_3(\bar{v}+v_p)] - 2f\mathrm{Q}_{u'v}(\nu) \\ &+ g\mathrm{Q}_{\mathbf{V}'\cdot\nabla z}(\nu) - \mathrm{Q}_{\mathbf{V}'\cdot\mathbf{F}}(\nu) \end{aligned} \tag{45}$$

Equations (44) and (45) are the real and imaginary parts of Eq. (38), respectively.

4. Some general comments about the atmospheric spectral equations

A close inspection of the ways various existing atmospheric spectral equations were derived shows that Saltzman's (1957) and Dutton's (1963) kinetic energy spectral equations are in the same category as that of Eq. (44), and that Kao's (1968) kinetic energy spectral equation is in the same category as that of Eq. (38). If we were to derive the spectral equations in the frequency domain by a method similar to that employed by Kao or Saltzman, we would first truncate u^* and other meteorological variables in the following manner:

$$u_T^*(t) = \begin{cases} u^*(t) & \text{for} \quad -T \leqslant t \leqslant T \\ 0 & \text{for all other } t \end{cases} \tag{46}$$

Then we represent the Fourier transform of u_T^* by

$$U_T^*(\nu) = \int_{-\infty}^{\infty} u_T^*(t)\, e^{-i\nu t} dt \qquad (47)$$

and the reverse transform by

$$u_T^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U_T^*(\nu)\, e^{i\nu t} d\nu \qquad (48)$$

When Eq. (1) is multiplied by $e^{-i\nu t}$ and transformed according to Eq. (47) and others like it, we get an equation which may be symbolically written as

$$i\nu U_T^*(\nu) = \ldots \qquad (49)$$

This equation corresponds to Kao's Eq. (24) and Saltzman's Eq. (40) in their respective papers.

Saltzman obtained his equation for the time rate of change of $|U(n)|^2$ (his Eq. (47)), where n is the wave number, by forming $U(-n)\,[\partial U(n)/\partial t] + U(n)\;[\partial U(-n)/\partial t]$. In our case, this would equivalent to form, via Eq. (49), the sum $U_T^*(-\nu)[i\nu U_T^*(\nu)] + U_T^*(\nu)[-i\nu U(_T^* - \nu)]$. This sum is zero, and so is the sum for $V_T^*(-\nu)[i\nu V_T^*(\nu)] + V_T^*(\nu)[-i\nu V_T^*(-\nu)]$, where $V_T^*(\nu)$ is the Fourier transform of the truncated v_T^*. But this, in essence, is the same way we derived Eq. (44), and the zero term on its left side from Eq. (38).

Therefore Saltzman's kinetic energy spectral equation, like our Eq. (44) is the real part of his complex spectral equation (which, if written down, would be an equation with $U(-n)$ $[\partial U(n)/\partial t] + V(-n)[\partial V(n)/\partial t)$ as its leading term). The same also can be said about Dutton's spectral equation.

At this point, we note that when the spectrum in the wave number domain is interested, as in Saltzman's & Dutton's studies, the real part of the complex spectral equation (or the co-spectral equation) deals with the time rate of change of spectrum. On the other hand, when the spectrum in the frequency domain is interested, as in Kao's and the present studies, th co-spectral equation does not deal with the time rate of change of spectrum, but with the balance among various co-spectra.

If one were to sort out the imaginary part (or the quadrature spectral part) from Saltzman's complex spectral equations, he would take Saltzman's Eq. (40) and perform an operation like $U(-n)[\partial U(n)/\partial t] - U(n)[\partial U(-n)/\partial t]$. This term, however, does not seem to have much physical meaning. In our case, a similar operation applied to Eq. (49) leads to an equation with $U_T^*(-\nu)[i\nu U_T^*(\nu) - U_T^*(\nu)[-i\nu U_T^*(-\nu)] = 2i\nu|U(\nu)|^2$ as its leading term. This, except for a factor of 2, corresponds to the left hand side term of our Eq. (45).

Therefore when the spectrum in the wave number domain is interested, the quadrature spectral equation does not seem to carry much physical meaning. On the other hand, when the spectrum in the frequency domain is interested, the quadrature spectral equation relates the spectrum itself to various factors that influence it and so has great significance.

Another point worth noting is the appearance of Coriolis parameter in the quadrature spectral equations and, with a minor exception, its disappearance in the co-spectral equations, signifying that the earth's rotation has a role to play in the former but in general not in the latter. We shall have more to say about this later.

5. Discussion of the spectral equations and the statistical energy spectrum of atmospheric motions in the frequency domain

From Eq. (45), the following deductions may be made:

A. From Eqs. (16) and (30) we find that

$$\int_{-\infty}^{\infty} [\nu\phi_{ii}(\nu)]d\nu = 0 \qquad (50)$$

This is because $\phi_{ii}(\nu)$ is an even and so $\nu\phi_{ii}(\nu)$ is an odd function of ν. However $\phi_{ii}(\nu)$ is in general not zero. Therefore, Eq. (45) links $\phi_{ii}(\nu)$ to the factors that influence it, and makes it possible for us to discuss and to speculate on the characteristics of $\phi_{ii}(\nu)$.

B. The statistical spectral density of the kinetic energy, $\phi_{ii}(\nu)$, of a stationary atmosphere is shaped by quadrature spectral densities only. There are, according to their order of appearance in Eq. (45), the quadrature spectral densities for $\mathbf{v}'\cdot(\mathbf{v}_3\cdot\nabla_3)\mathbf{v}$, for $u'\;\nabla_3 u$, for $v'\nabla_3 v$, for $u'v_3$ for $v'v_3$, for $u'v$, for $\mathbf{v}'\cdot\nabla z$ and for $\mathbf{v}'\cdot\mathbf{F}$ respectively. Since the quadrature spectrum is a measure of the 90 degree out-of-phase relationship between the variables concerned (Panofsky & Brier, 1958), the existence of such an out-of-phase relationship between the various pairs of variables mentioned is essential for molding the shape of $\phi_{ii}(\nu)$.

C. The first term on the right side of Eq. (45) has its origin in the interaction among different frequencies of the motion. This can easily be seen if $u(t')$, $v(t')$, etc., were truncated and represented in the same way as u^* was by Eqs. (46) and (48). Then we have

$$\overline{u(t')\,u(t)\frac{\partial u(t)}{\partial x}} = \frac{1}{8T\pi^2}\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} U(-\nu)\,U(\mu)\,U_x(-\mu+\nu)d\mu\right] e^{i\nu t}d\nu \quad (51)$$

where $U(\nu)$ is the Fourier transform of $u(t')$, $U_x = \partial U/\partial x$, and μ is another notation for the angular frequency. Therefore

$$\overline{u(t')\,u(t)\frac{\partial u(t)}{\partial x}}$$

represents some kind of interaction among different frequencies, and $Q_{\mathbf{V}'[(\mathbf{V}_3\cdot\nabla_3)\mathbf{V}_3]}(\nu)$ is a sine transform of the terms like it. However, for this quadrature spectrum to be nonzero, there must also be an out-of-phase relationship between the fluctuating momentum (u, v) and the advection of fluctuating momentum by the fluctuating motion ($u\partial u/\partial x$, etc.).

Similarly, the second and third terms on the right side of Eq. (45) have their origin in the interaction between the fluctuating motion and the mean and periodic motions. The second term depends on an out-of-phase relationship between the fluctuating momentum and the advection of fluctuating momentum by the mean and periodic motions

$$\left[(\bar{u}+u_p)\frac{\partial u}{\partial x}, \text{etc.}\right];$$

while the third term depends on an out-of-phase relationship between the fluctuating momentum and the advection of momentum of mean and periodic motions by the fluctuating motion

$$\left[u\frac{\partial(\bar{u}+u_p)}{\partial x}, \text{etc.}\right].$$

For all these three quadrature spectra to be nonzero, it is also necessary that there be some velocity shear, either of fluctuating motion or of mean and periodic motions, or of both. Since velocity shear is related to vorticity, one might say that vorticity plays an important role in shaping the spectrum. For example, the third term may be rearranged to yield that

$$\begin{aligned}&[\mathbf{Q}_{u'\mathbf{V}_3}(\nu)\cdot\nabla_3(\bar{u}+u_p)+\mathbf{Q}_{v'\mathbf{V}_3}(\nu)\cdot\nabla_3(\bar{v}+v_p)]\\ &= \frac{1}{2}[\phi_{12}(\nu)-\phi_{21}(\nu)]\left[\frac{\partial(\bar{v}+v_p)}{\partial x}-\frac{\partial(\bar{u}+u_p)}{\partial y}\right]\\ &+\frac{1}{2}[\phi_{v'\omega}(\nu)-\phi_{\omega' v}(\nu)]\left[-\frac{\partial(\bar{v}+v_p)}{\partial p}\right]\\ &+\frac{1}{2}[\phi_{\omega' u}(\nu)-\phi_{u'\omega}(\nu)]\left[\frac{\partial(\bar{u}+u_p)}{\partial p}\right] \quad (52)\end{aligned}$$

That the existence of this term depends on the presence of vorticity in the mean and periodic motions is obvious. It also shows that this term is in nature similar to $2f\,Q_{u'v}(\nu)$.

D. From Eqs. (15) and (29), we have

$$fR_{12}(0)=fR_{21}(0)=f\overline{u(t')\,v(t')} - f\int_{-\infty}^{\infty}\phi_{12}(\nu)d\nu = f\int_{-\infty}^{\infty}\phi_{21}(\nu)d\nu \quad (53)$$

Therefore both $\phi_{12}(\nu)d\nu$ and $\phi_{21}(\nu)d\nu$ are the spectral contribution to the exchange of kinetic energy between u and v velocity components. However from Eqs. (16) and (30), we find that

$$\phi_{12}(\nu) = \phi_{21}(-\nu) \mp \phi_{21}(\nu) \quad (54)$$

Therefore $2f\,Q_{u'v}(\nu)$ is in general not zero. Its association with f means that its contribution to $\nu\phi_{ii}(\nu)$ is brought about by the rotation of the earth. Therefore the rotation of the earth plays an explicit role in shaping the spectral distribution of the kinetic energy. This point was first brought out by Chiu (1961) (although this report contains some errors in other aspects), and later by Dutton (1963) and Kao (1968).

We are used to the notion that the earth's rotation plays no part in the time rate of change of kinetic energy of the atmosphere (disregarding the possible work done by the centrifugal force due to the rotation of the earth, which is conventionally incorporated into the gravity), but we have reason to believe both from our intuition as well as from the results of the dish pan experiment (Fultz et al., 1959) that the rotation of the earth should exert a very strong influence on the types of motion the atmosphere will follow and consequently upon its associated kinetic energy level and distribution. Eq. (45) brings out clearly the influence of earth's rotation on spectrum.

When Eq. (45) is divided by ν, we have:

$$\phi_{tt}(\nu) = \frac{1}{\nu}\{ Q_{\mathbf{V}'\cdot[(\mathbf{V}_3\cdot\nabla_3)\mathbf{V}]}(\nu) + (\bar{\mathbf{V}}_3 + \mathbf{V}_{p3})\cdot[\mathbf{Q}_{u'\nabla_3 u}(\nu) + \mathbf{Q}_{v'\nabla_3 v}(\nu)] + [\mathbf{Q}_{u'\mathbf{V}_3}(\nu)\cdot\nabla_3(\bar{u}+u_p) + \mathbf{Q}_{v'\mathbf{V}_3}\cdot\nabla_3(\bar{v}+v_p)] - 2fQ_{u'v}(\nu) + gQ_{\mathbf{V}'\cdot\nabla z}(\nu) - Q_{\mathbf{V}'\cdot\mathbf{F}}(\nu)\} \tag{55}$$

When this equation is integrated with respect to ν from zero to infinity, we have, in view of Eq. (33):

$$\frac{1}{2}\overline{(u^2+v^2)} = \int_0^\infty \frac{1}{\nu}\{ Q_{\mathbf{V}'\cdot[(\mathbf{V}_3\cdot\nabla_3)\mathbf{V}]}(\nu) + (\bar{\mathbf{V}}_3 + \mathbf{V}_{p3})\cdot[\mathbf{Q}_{u'\nabla_3 u}(\nu) + \mathbf{Q}_{v'\nabla_3 v}(\nu)] + [\mathbf{Q}_{u'\mathbf{V}_3}(\nu)\cdot\nabla_3(\bar{u}+u_p) + \mathbf{Q}_{v'\mathbf{V}_3}(\nu)\cdot\nabla_3(\bar{v}+v_p)] - 2fQ_{u'v}(\nu) + gQ_{\mathbf{V}'\cdot\nabla z}(\nu) - Q_{\mathbf{V}'\cdot\mathbf{F}}(\nu)\}\, d\nu \tag{56}$$

Equation (56) shows the manner in which the various factors, including the rotation of the earth, influence the kinetic energy level of the atmosphere.

Since the earth's rotation plays no role in the time rate of change of kinetic energy, we more or less expect that it also plays no role in the time rate of change of the kinetic energy spectrum. This indeed is so in Saltzman's case.But Dutton's result shows that when the variation in fluid density is taken into consideration, the earth's rotation plays a role even in the time rate of change of the spectrum. [Incidently, since Dutton's spectrum represents a summation over all three space directions, its variation could not be a result of an exchange of spectral energy among different directional components. Therefore, if there has been a change of the spectral density at a certain wave number due to the Coriolis term in his spectral equation, there must be a compensating change (or changes) of spectral density (or densities) at another wave number (or many other wave numbers).] However, for the large-scale quasi-horizontal atmospheric motions the variation in density is small compared to the variation in velocity, making it permissible to disregard the Coriolis term in Dutton's spectral equation (this term drops out when density is considered constant); while for the small-scale atmospheric motions, the Coriolis terms in the equations of motion are usually negligible in the first place. Therefore, in all likelihoods, the effect of the earth's rotation on the time rate of change of spectral energy (in the wave number domain) is negligible.

According to the term $2\,f\,Q_{u'v}(\nu)$ in Eq. (45), the effect of earth rotation on spectrum increases with f (or the rate of earth's rotation), and vanishes when the earth is not rotating. This is what one would expect from intuition. However, the magnitude of this term depends also on $Q_{u'v}(\nu)$. Therefore, one cannot conclude that the effect of earth's rotation on spectrum is necessarily more important in high latitudes than in low latitudes without further looking into $Q_{u'v}(\nu)$. In this connection, it would be interesting to find out: (1) whether there is any tendency for $Q_{u'v}(\nu)$ to arrange itself according to ν (i.e., whether there is any tendency for $Q_{u'v}(\nu)$ to be one sign for a certain range of ν and to be of another sign for another range of ν), and (2) whether there is any systematic change in the pattern of $Q_{u'v}(\nu)$ from high to low latitudes.

E. The last two terms of Eq. (45), $g\,Q_{\mathbf{V}'\cdot\nabla\xi}(\nu)$ and $Q_{\mathbf{V}'\cdot\mathbf{F}}(\nu)$, represent the role of ν frequency component of the pressure gradient force and the ν frequency component of the frictional force, respectively, play in shaping $\phi_{tt}(\nu)$.

F. Equation (55) shows that $\phi_{tt}(\nu)$ would in general decrease with increasing ν, leading to the so-called red-noise spectrum (Gilman, Fuglister & Mitchell, 1963) provided that the bracketed term does not generate a sharp spectral peak or peaks. Since our atmosphere is often in a turbulent state, we could expect the exchange mechanism associated with the turbulent motions acts to redistribute the spectral densities among different frequency scales of the motion, thus any spectral energy input or inputs that the term $gQ_{\mathbf{V}'\cdot\nabla z}(\nu)$ introduces into $\phi_{tt}(\nu)$ spectrum will be smoothed out.[1]

[1] Although one cannot rule out the possibility that the exchange mechanism may occasionally tend to concentrate rather than to spread the spectral energy input, the chances are greater that it will spread, since motions of all scales, directly or indirectly, must eventually derive their kinetic energy from the potential energy. A similar idea concerning the redistribution of the spectral energies in the wave number space was voiced by Kraichnan (1967). The tendency to spread input does not necessarily mean that there is no flow of the spectral energy from the region of low spectral density to the region of high spectral density. The picture visualized here is that if the input of kinetic energy through the conversion from the potential energy is concentrated in certain frequency bands, the ex-

Equation (55) shows that such an atmosphere would have a red-noise spectrum. The available observational evidence (Chiu (1960); Shapiro & Ward (1960); Ward & Shapiro (1961)) seems to indicate that our atmosphere is of this type.

A process that exhibits some persistence possesses a red-noise spectrum (Gilman, Fuglister & Mitchell, 1963; Ward & Shapiro, 1961). Atmospheric motion exhibits some persistence. Because it evolves continuously with time and in so doing is bound to the laws of motion, the motion at one moment must carry with it some memory and influence of the motion at the immediately previous moment. ν in Eq. (55) arises from the terms for the time rate of change of velocity in Eqs. (1) and (2). Therefore $1/\nu$ in Eq. (55) seems to be a subtle mathematical expression of the persistence of the motion and of its red-noise characteristics.

Within the inertial subrange of the turbulent fluid motion, there is no kinetic energy input (from the conversion of the internal and potential energies). The only mechanism operating is the exchange of kinetic energy between scales of motion, which passes the kinetic energy down the scale in the case of a three-dimensional turbulence. As a result, the spectrum of the inertial subrange follows a very simple Kolmogoroff's-5/3 law. On the other hand, the immediate sources of atmospheric kinetic energy are many, and they fuel atmospheric motions from many different scales, such as large-scale differential heating between pole and equator, differential heating between land and sea, latent heat and potential energy associated with cyclones or local thunderstorms (various time scales are associated with each energy source). The manner in which energy is converted to atmospheric motion is therefore much more complicated than that associated with wind tunnel turbulence for which Kolmogoroff's law is most convincingly confirmed. This, however, does not necessarily imply that the spectral law of atmospheric motion must be a complicated one. It could be that such factors as a persistently high degree of turbulence in the atmosphere and interactions on all scales of motion, would maintain a relatively simple statistical spectral distribution of atmospheric motion. From the observed red-noise characteristics of the large-scale atmospheric motion, it appears that the spectrum of atmospheric motion also follows a rather simple law despite numerous kinetic energy inputs within the range of motion that interests us. This probably means that the interaction term also plays a dominant role in shaping the spectrum of the large-scale motion, as it does for the small scale turbulence. Therefore it may be possible that atmospheric spectrum appears to be isotropic, but is really not isotropic in nature. In their study of the mean geostrophic kinetic spectra of the large-scale atmospheric circulation, Horn and Bryson (1963) found that there was a section of the spectra that followed an 8/3 power relationship with respect to (spatial) wave number. They assigned this section to the isotropic range when comparing it with the results of other studies. They also suggested that this section belongs to the range at which the kinetic energy of the atmosphere was produced (from the conversion of the available potential energy). However, the idea of a range of wave length which is both isotropic and energy-producing is itself conflicting. Hence it might have been better to omit its explanation in terms of isotropy.

From Eq. (44), the following statements may be made:

G. When Eq. (44) is integrated over all positive ν, we find, in view of equations like Eq. (42) and its Fourier transform, that:

$$\overline{\mathbf{V}_3^* \cdot \nabla_3 \frac{\overline{\mathbf{V} \cdot \mathbf{V}}}{2}} + \overline{[u\mathbf{V}_3 \cdot \nabla_3(\bar{u}+u_p) + v\mathbf{V}_3 \cdot \nabla_3(\bar{v}+v_p)]} + g\overline{\mathbf{V} \cdot \nabla z} = \overline{\mathbf{V} \cdot \mathbf{F}} \quad (57)$$

That is, at a particular location in a stationary atmosphere, the average rate of advection of eddy kinetic energy through that location by the total velocity vector, $\mathbf{v}_3^*$, plus the average rate of work done by the Reynolds stresses against the gradient of the mean and periodic motions (which represent the exchange of kinetic energy between the eddy motion and the

change mechanism sperads the input to other frequencies. It may also take the kinetic energy from a region of relatively low spectral density and give it to a region of relatively high spectral density, without creating peaks in doing so. The latter transfer, for example, may correspond to a "counter-gradient" flow of energy from the intermediate frequencies to the low frequencies in a red-noise spectrum. This would be the case if the semipermanent highs and lows were found to be maintained by the moving cyclones, and the latter were maintained by the input from the potential energy.

mean and periodic motions, although its physical meaning is not without ambiguity, see Lettau, 1954), and the average rate of work done by the fluctuating pressure gradient force (which represents the conversion from the eddy potential and internal energies into the eddy kinetic energy, see for example, White & Saltzman, 1956) must be equal to the average rate of frictional dissipation of the kinetic energy. Since the frictional dissipation acts to take away the kinetic energy,[1] the sum of the three terms on the lefthand side of Eq. (57) must, on the average, act to increase the kinetic energy. The second term, according to the current belief, is likely to represent a loss of eddy kinetic energy to the kinetic energy of mean and periodic motion (see for example, Starr, 1958).[2] The first term may be of either sign, depending on the location considered. There have, for example, been some indications that in some areas of the stratosphere the third term of Eq. (57) represents a loss rather than a gain of kinetic energy (White & Nolan, 1960; Oort, 1967).[2] In those areas the first term of Eq. (57) must represent a gain of kinetic energy in order to balance the drain due to all other terms. On the other hand, the semi-permanent subtropical high pressure belt, according to the discussion by Starr (1948), should be an area where kinetic energy is produced and transported away. In that area, the loss of kinetic energy by the first, second and fourth terms of Eq. (57) is balanced by the gain of kinetic energy due to the third term. When Eq. (57) is integrated over the whole atmosphere, the first term is zero, as there can be no net transport of kinetic energy out of the whole atmosphere. The total production of kinetic energy by the third term is balanced by the total dissipation by the second and fourth terms in a stationary atmosphere.

In short, important information concerning the energy characteristics of various locations of the atmosphere could be obtained from the calculation of the terms of Eq. (57) at those locations. It should tell us whether they are kinetic energy production or consumption areas, and should thus enhance our understanding of the working and compling of different parts of the atmosphere. However, it is difficult to calculate the terms in Eq. (57) from presently available meteorological observations. Initial effort toward such a calculation, if attempted, is best limited to areas, where observational network is dense and observations accurate and have been carried out for a period long enough to yield stable time means.

In this connection, it also would be of interest to see whether the shape of $\phi_{ii}(\nu)$ for the area of kinetic energy production is substantially different from that of $\phi_{ii}(\nu)$ for the area of kinetic energy consumption.

Since the Coriolis parameter does not appear in Eq. (57), the four terms in the equation should balance regardless of the earth's rotation rate. Indeed, the terms should balance even when the earth is not rotating. The earth's rotation, however, might have a subtle influence on the manner in which these terms maintain their balance, it certainly influences the pattern of the wind field and hence the magnitudes of these terms.

H. The terms in Eq. (44) are, respectively, the cospectral densities (or the spectral contributions, when multiplied by $d\nu$) of the terms in Eq. (57). Consequently a balance like that of Eq. (57) holds true even for the corresponding cospectral densities in a stationary atmosphere. The foregoing remarks on the effect of the earth's rotation apply equally well here.

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[1] $\overline{\mathbf{V}\cdot\mathbf{F}}$ includes, in addition to the molecular dissipation, some work done by molecular and Reynolds stresses which may not be dissipative. But on the average $\overline{\mathbf{V}\cdot\mathbf{F}}$ must be dissipative.

[2] Most of the studies that showed the mean (in space) motion is maintained by eddy motions dealt with motions with different space scales. We are making an assumption here that a similar relationship exists between the mean (in time) motion and eddy motions of different time scales.

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О СПЕКТРАЛЬНЫХ УРАВНЕНИЯХ И СТАТИСТИЧЕСКОМ СПЕКТРЕ ЭНЕРГИИ АТМОСФЕРНЫХ ДВИЖЕНИЙ В ПРОСТРАНСТВЕ ЧАСТОТ

В пространстве частот выводятся спектральные уравнения для кинетической энергии крупномасштабных атмосферных движений. Некоторыми важными следствиями этих уравнений являются: 1) форма функции спектральной плотности $\phi_{ii}(\nu)$ контролируется фазовыми соотношениями между различными метеорологическими переменными; 2) вращение земли непосредственно влияет на $\phi_{ii}(\nu)$, также как и на средний уровень кинетической энергии атмосферы, и 3) атмосфера, эффективная в распределении своей кинетической энергии между различными частотными компонентами, так что нет какой-либо определенной тенденции к образованию спектральных пиков, должна иметь спектр красного шума. В работе производится детальное обсуждение этих следствий, а также ряда других.