The effect of horizontal shear flow on geostrophic adjustment in a barotropic fluid

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ABSTRACT

The effect of a basic horizontal shear flow on the linear geostrophic adjustment process in an unbounded barotropic fluid is investigated. It is shown that the basic flow is absolutely stable to axially symmetric transverse disturbances and that energy is not abstracted from the basic flow through the action of Reynolds stresses. A potential vorticity equation is derived and solved numerically in order to determine the relative amount of initial energy which is partitioned to geostrophic kinetic and available potential energy, as a function of the initial current width. It is also shown that, in contradistinction to the adjustment process in an atmosphere with a basic state at rest, a significant portion of geostrophic energy resides in the low wave-number part of the energy spectrum. This latter feature is most noticeable when a large gradient of the basic flow exists.

1. Introduction

There are a number of mechanisms in the atmosphere which may produce a large spectrum of internal gravity waves. For example, internal gravity waves may be generated by the flow of a current over mountainous terrain (Queney, 1947), by fluid impacts at the base of a stably stratified layer by rising convective clouds in the layer below (Townsend, 1966) and by the adjustment of an unbalanced current to a balanced state (Monin & Obukhov, 1958).

Some mesoscale phenomena observed in the stratosphere and mesosphere (Newell *et al.*, 1966) and in the lower ionosphere (Hines, 1963) have been interpreted as manifestations of internal gravity waves. These waves are observed to have periods of the order of an hour, dominant vertical wave lengths of one scale height or less and dominant horizontal wave lengths of a few hundred kilometers or less. The basic zonal wind and temperature distribution in the atmosphere commonly exhibits pronounced spatial variability over a distance of a scale height or less, while sharp horizontal gradients are less common, occurring mainly in the vicinity of frontal zones and jet streams. Gravity waves with vertical and horizontal components of propagation would then be expected to undergo a selective reflection with increasing height (Pitteway and Hines, 1965), but could travel large horizontal distances relatively unaffected by broad horizontal gradients of wind and temperature.

Observations in the troposphere and stratosphere, presented by Sawyer (1960), Weinstein *et al.* (1966) and DeMandel and Scoggins (1967), also reveal the existence of mesoscale oscillations but with periods of several hours, vertical scales of the order of one kilometer and horizontal dimensions of hundreds of kilometers. This type of oscillation has been interpreted as a gravity-inertia wave, a gravity wave strongly affected by the earth's rotation when the frequency of the oscillation σ approaches the inertial frequency $f = 2\Omega \sin \phi$. (Ω is the angular velocity of the earth's rotation, ϕ is latitude and $f \approx 10^{-4}$ sec^{-1} in middle latitudes.) The properties of this type of wave have been discussed by Eckart (1960), where it is shown that wave propagation is confined primarily to horizontal surfaces and when $\sigma \to f$ the vertical flux of wave energy approaches zero. Since the observations, cited above, indicate that these oscillations are confined to thin horizontal layers in the atmosphere, horizontal gradients of the basic flow variables could have an important effect on the propagation of these waves.

In this paper, attention is focused on the problem of horizontal propagation of gravity-inertia waves through horizontal shear flow. Since the purpose here is to illustrate some basic properties of this type of motion, we consider only a simple fluid model which is amenable to analysis. The nonlinear equations governing fluid motion in this model, consisting of two incompressible fluid layers in stable stratification, are presented in nondimensional form in section 2. These equations are then reduced to a linear system, which describes small perturbation motion about a basic time-independent geostrophic horizontal shear flow. The absolute stability of this geostrophic current to symmetric transverse disturbances is established in section 3. The conservation of potential vorticity for this model is derived in section 4 and in section 5 this conservation principle is used in the derivation of the horizontally-averaged energy equation. In section 6.1 solutions of the potential vorticity equation for ageostrophic initial conditions are presented and the partition of energy between the geostrophic and ageostrophic components is discussed. The time-dependent and steady solutions of this model, for a spatially uniform initial condition, are presented in 6.2 in order to show the importance of the low wave-number part of the spectrum in the process of energy partition. Finally, in section 7 we conclude by pointing out the important effects which have been introduced by the inclusion of a basic shear flow in the present model.

2. Basic equations

The fluid model used in this study consists of two incompressible layers of finite depth with densities $\varrho^* > \varrho$, separated by an interface at height h^*. The lower layer is bounded from below by a flat unbounded plane, tangent to the earth at latitude ϕ. The upper layer is assumed inert and the pressure is hydrostatic in both layers. A Cartesian coordinate system is used with x^*, y^*, z^* directed eastward, northward and upward respectively. Gravity g is antiparallel to z^*, the vertical component of the earth's rotation vector is parallel to z^* and the horizontal component of rotation is neglected. The velocity components in the x^*, y^* directions are u^*, v^* respectively and the motion is assumed independent of $x^*(\partial/\partial x^* \equiv 0)$. The equations of motion and mass continuity for this system have been presented elsewhere (e.g., Tepper, 1955). In nondimensional form they are

$$\partial u/\partial t + \lambda v(F\partial u/\partial y - 1) = 0 \tag{2.1}$$

$$\lambda(\partial v/\partial t + \lambda Fv\partial v/\partial y) + u + F^{-1}\partial h/\partial y = 0 \tag{2.2}$$

$$\partial h/\partial t + \lambda F(v\partial h/\partial y + h\partial v/\partial y) = 0. \tag{2.3}$$

The dependent variables have been nondimensionalized by

$$u^* = Uu, \quad v^* = Vv,$$
$$h^*H^{-1} = h = 1 + fULc^{-2}\hat{h} = 1 + F\hat{h}, \tag{2.4}$$

where U, V, denote characteristic amplitudes of u^*, v^* respectively, $\hat{h}$ is the non-dimensional deviation of the interface from the constant mean value H, f is the constant Coriolis parameter, the characteristic horizontal scale of the motion is $L = f^{-1}(g^*H)^{\frac{1}{2}}$ (the *radius of deformation*, Rossby 1938) and $c^2 = g^*H$ ($g^* \equiv \varrho^{*-1}(\varrho^* - \varrho)g$ is "reduced" gravity). The independent variables have been nondimensionalized by

$$y^* = f^{-1}(g^*H)^{\frac{1}{2}}y, \quad t^* = f^{-1}t \tag{2.5}$$

and derivatives with respect to space and time are of order unity. The nondimensional parameters are

$$\lambda = VU^{-1},$$
$$F = U(g^*H)^{-\frac{1}{2}} \sim O(1), \text{ the Froude number.} \tag{2.6}$$

Values characteristic of middle latitudes are: $f \sim 10^{-4}$ sec^{-1}, $H \sim 10$ km (H represents the mean tropopause height), $\varrho^{*-1}(\varrho^* - \varrho)g \sim 10^{-1}$ msec^{-2}, which corresponds to a jump in the potential temperature of about 3K across the tropopause and $L \equiv f^{-1}(g^*H)^{\frac{1}{2}} \sim 300$ km.

Equations (2.1), (2.2) and (2.3) may be reduced to a linear system by assuming that $\lambda = VU^{-1} \sim 10^{-1}$ and expanding all dependent variables in a power series in λ. The coefficients of each power of λ satisfy (2.1), (2.2) and (2.3). We shall retain only the zero- and first-order systems of linear equations, which are

$$\partial u_0/\partial t = 0, \quad \partial h_0/\partial t = 0, \quad u_0 + \partial\hat{h}_0/\partial y = 0 \tag{2.7}$$

and

$$\partial u_1/\partial t + v_1(FdU_0/dy - 1) = 0 \quad (2.8)$$

$$\partial v_1/\partial t + u_1 + \partial h_1/\partial y = 0 \quad (2.9)$$

$$\partial h_1/\partial t + \partial(h_0 v_1)/\partial y = 0, \quad (2.10)$$

where $h_0 \equiv 1 + F\hat{h}_0$ and $\hat{h}_1 \equiv h_1$. The basic state is one of time-independent geostrophic balance and the first-order variables represent deviations from it.

If a basic state of rest ($u_0 = 0$, $h_0 = 1$) had been assumed, then (2.8), (2.9) and (2.10) would represent the nondimensional form of the equations solved by Cahn (1945) in his study of the geostrophic adjustment process. The flow in Cahn's problem is always stable since no external energy source is available for the unbounded growth of the wave disturbance. In the present problem the energy of the basic geostrophic current is available as a possible source and the stability of this flow must be considered.

3. Stability of geostrophic motion

If all the variables but v_1 are eliminated from (2.8), (2.9) and (2.10) the following equation with y-dependent coefficients is derived

$$\partial^2 v_1/\partial t^2 + (1 - Fdu_0/dy)v_1 - \partial^2 h_0 v_1/\partial y^2 = 0. \quad (3.1)$$

By setting $h_0 v_1 \equiv q$, (3.1) becomes

$$\{h_0^{-1}[\partial^2/\partial t^2 + (1 - Fdu_0/dy)] - \partial^2/\partial y^2\}q = 0. \quad (3.2)$$

As Drazin and Howard (1966) have pointed out, the continuous as well as the discrete spectrum of time-dependent motions must be considered in connection with the intitial-value problem. However it appears that the modes leading to instability are associated with the discrete spectrum alone. Since we are concerned with the stability problem here, we shall assume that each component of q satisfies

$$q(y, t) = Q(y)e^{i\sigma t}, \quad (3.3)$$

where $Q(y) = h_0(y)V_1(y)$ and $\sigma = \sigma_r + i\sigma_i$. Instability of the flow to this mode corresponds to $\sigma_i \neq 0$; otherwise the flow is stable $\sigma_r \neq 0$ or neutrally stable $\sigma_r = 0$.

Introduction of (3.3) reduces (3.2) to

$$d^2Q/dy^2 + h_0^{-1}[\sigma^2 - (1 - Fdu_0/dy)]Q = 0. \quad (3.4)$$

Suppose no deviations from the geostrophic height field are permitted, i.e., $\partial h_1/\partial t = -\partial h_0 v_1/\partial y = 0$. Then d^2Q/dy^2, in (3.4), vanishes and, if $Q(y)$ is not identically zero,

$$\sigma^2 = 1 - Fdu_0/dy. \quad (3.5)$$

From the above definition of stability, (3.5) yields the classical criteria for the centrifugal stability of geostrophic motion (e.g., Van Mieghem, 1951)

$$1 - Fdu_0/dy \gtreqless 0, \quad (3.6)$$

corresponding to stability, neutral stability and instability respectively.[1]

We shall now consider the effect of non-vanishing height deviations $h_1(y, t)$ upon the stability of geostrophic flow. We multiply (3.4) by Q^*, the complex conjugate of Q, and integrate over the domain $-\infty \leqslant y \leqslant \infty$. The expression obtained for σ^2 is

$$\sigma^2 = \frac{\int_{-\infty}^{\infty} h_0^{-1}(1 - Fdu_0/dy)|Q|^2 dy + \int_{-\infty}^{\infty} |dQ/dy|^2 dy}{\int_{-\infty}^{\infty} h_0^{-1}|Q|^2 dy}, \quad (3.7)$$

since Q and its derivatives vanish at $y = \pm\infty$, because v_1 and its derivatives vanish there. If we let $Q = h_0 V_1$, then

$$\int_{-\infty}^{\infty} \left|\frac{dQ}{dy}\right|^2 dy = \int_{-\infty}^{\infty} \left(h_0^2 \left|\frac{dV_1}{dy}\right|^2 + Fh_0 \frac{du_0}{dy} |V_1|^2\right) dy, \quad (3.8)$$

upon integration by parts and use of (2.7). Substitution of (3.8) into (3.7), and separation of the real and imaginary parts of σ^2, yields

$$\sigma_i = 0$$

$$\sigma_r^2 = \frac{\int_{-\infty}^{\infty} h_0 |V_1|^2 dy + \int_{-\infty}^{\infty} h_0^2 |dV_1/dy|^2 dy}{\int_{-\infty}^{\infty} h_0 |V_1|^2 dy} > 0. \quad (3.9)$$

[1] The terms "inertial stability" or "inertial instability" have been used in referring to the criteria in (3.6) (e.g., Van Mieghem (1951)). In order to avoid confusion we shall not use this nomenclature, since "intertial instability" has also been used in reference to the instability of two-dimensional parallel shear flow (e.g., Drazin and Howard, 1966).

This result (3.9) means that geostrophic motion is stable to transverse disturbances ($v_1 \neq 0$) which are constrained to move in the $y-z$ plane.

In the derivation of (3.6) it has been tacitly assumed that fluid displacements conserve their momentum but the pressure field remains unchanged throughout the fluid, since $\partial h_1/\partial t = 0$. If the net restoring force on the displaced fluid due to the pressure gradient is less than the centrifugal force, which it conserves upon displacement, the fluid will be accelerated away from its initial position. If this takes place the fluid is said to be unstable to transverse displacements. The criteria for this type of instability have been expressed in terms of the sign of the absolute vorticity, in (3.6). If pressure deviations are permitted ($\partial h_1/\partial t \neq 0$), pressure changes occur in response to fluid displacements and no mechanism exists to produce the type of instability mentioned above. We note stability is still maintained when the velocity divergence vanishes ($\partial v_1/\partial y = 0$). In this case (2.10) reduces to

$$\partial h_1/\partial t + v_1 dh_0/dy = 0. \tag{3.10}$$

Fluid displacements still produce pressure changes but the manner in which these changes take place is constrained by not allowing velocity divergence in the fluid.

4. Conservation of potential vorticity

A model, in which the basic state was one of rest ($u_0 = 0$, $h_0 = 1$), was studied by Obukhov (1949) in application to the adjustment of an initial unbalanced current to geostrophic balance. Obukhov showed that the linear equations of his model possess a time-independent invariant, the potential vorticity, which he expressed in both one and two space dimensions. This invariant is a condition which allows the eventual steady geostrophic state to be determined uniquely from the initial state of the motion field without solving the intermediate initial-value problem. Papers by Cahn (1945), Washington (1964) and Blumen (1967*b*) have been concerned with the solution of the time-dependent problem. Their results show that the time it takes to reach an approximate steady geostrophic state increases with the ratio of the characteristic scale of the initial motion to the radius of deformation. Since this latter aspect of the adjustment problem has been considered elsewhere, we shall not consider it here.

In this section we shall show that potential vorticity is conserved in the present model, in which the basic state is one of steady geostrophic balance. Equation (2.8) may be rewritten as

$$\partial u_1/\partial t - v_1 h_0 \Omega_0 = 0, \tag{4.1}$$

where

$$\Omega_0 \equiv h_0^{-1}(1 - F du_0/dy) \tag{4.2}$$

is the potential vorticity of the basic state. If (4.1) is differentiated with respect to y and (2.10) is introduced, we obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial u_1}{\partial y} + \Omega_0 h_1\right) - v_1 h_0 \frac{d\Omega_0}{dy} = 0. \tag{4.3}$$

Multiplication of (4.3) by Ω_0 and use of (4.1) yields the equation for the conservation of first-order potential vorticity

$$\frac{\partial}{\partial t}\left(\Omega_0 \frac{\partial u_1}{\partial y} - \frac{d\Omega_0}{dy} u_1 + \Omega_0^2 h_1\right) = 0. \tag{4.4}$$

If the flow becomes steady it must also become geostrophic

$$u_{1g} + \partial h_{1g}/\partial y = 0. \tag{4.5}$$

The nonsteady ageostrophic part of the flow will be denoted by a prime. Then, upon integration of (4.4), we obtain

$$\Omega_0 \frac{\partial u_1'}{\partial y} - \frac{d\Omega_0}{dy} u_1' + \Omega_0^2 h_1' = 0 \tag{4.6}$$

and

$$\Omega_0 \frac{du_{1g}}{dy} - \frac{d\Omega_0}{dy} u_{1g} + \Omega_0^2 h_{1g} = \Omega_i, \tag{4.7}$$

where

$$\Omega_i \equiv \left[\Omega_0 \frac{\partial u_i}{\partial y} - \frac{d\Omega_0}{dy} u_i + \Omega_0^2 h_i\right] \tag{4.8}$$

denotes the initial-value ($t = 0$) of the first-order potential vorticity. The steady first-order geostrophic field is determined from the solution of the following linear differential equation with variable coefficients

$$\left[\Omega_0 \frac{d^2}{dy^2} - \frac{d\Omega_0}{dy}\frac{d}{dy} - \Omega_0^2\right] h_{1g} = -\Omega_i, \tag{4.9}$$

which has been obtained by using (4.5) to eliminate u_{1g} in (4.7). If $\Omega_i \neq 0$, then a certain portion of the initial energy will go into the steady first-order geostrophic field and the remainder will be dispersed throughout the fluid by horizontally propagating gravity waves. The gravity-wave field may be determined from the solution of the wave equation (3.1) and specified initial conditions. Obukhov's (1949) one-dimensional form of the potential vorticity equation is obtained by setting $\Omega_0 = 1$ in (4.9).

In order to determine some properties of the linear system (2.7–2.10) the following initial conditions will be considered

$$\left.\begin{aligned} u_i &= (u_g)_i \\ h_i &= (h_g)_i \\ v_i &= v_i' \end{aligned}\right\} t = 0. \qquad (4.10)$$

If (4.8) is evaluated by (4.10), then (4.9) becomes

$$\left(\Omega_0 \frac{d^2}{dy^2} - \frac{d\Omega_0}{dy}\frac{d}{dy} - \Omega_0^2\right)\chi = 0 \qquad (4.11)$$

where $$\chi \equiv h_{1g} - (h_g)_i. \qquad (4.12)$$

We shall consider the case $\Omega_0 \neq 0$ in $-\infty \leqslant y \leqslant \infty$, i.e., $1 - F\,du_0/dy \neq 0$. Then (4.11) may be rewritten as

$$\frac{d}{dy}\left(\Omega_0^{-1}\frac{d\chi}{dy}\right) - \chi = 0. \qquad (4.13)$$

Multiplication of (4.13) by χ^*, the complex conjugate of χ, and integration over y yields

$$\int_{-\infty}^{\infty} \Omega_0^{-1}\left|\frac{d\chi}{dy}\right|^2 dy + \int_{-\infty}^{\infty} |\chi|^2 dy = 0, \qquad (4.14)$$

since χ and its derivatives are assumed to vanish at $y = \pm\infty$. Since $\Omega_0 \neq 0$, (4.14) can only be satisfied if $\chi = 0$. This means that the eventual steady fields of u_1 and h_1 are the same as the initial fields, given by (4.10). This result will be used in the following section where the partition of energy between the steady and nonsteady fields is considered.

5. Energy equation

An energy equation may be obtained by multiplying (2.8), (2.9) and (2.10) by $h_0 u_1$, $h_0 v_1$ and h_1 respectively. Addition of these equations yields

$$\frac{\partial}{\partial t}\varepsilon + F h_0 v_1 u_1 \frac{du_0}{dy} + \frac{\partial}{\partial y} h_0 v_1 h_1 = 0, \qquad (5.1)$$

where $$\varepsilon \equiv \tfrac{1}{2}(h_0 u_1^2 + h_0 v_1^2 + h_1^2). \qquad (5.2)$$

ε represents the sum of kinetic and available potential energy per unit horizontal area, if we set the constant nondimensional density of the lower layer equal to unity. Integration of (5.1) over y, denoted by a bar, yields

$$\frac{\partial}{\partial t}\bar{\varepsilon} + \overline{F h_0 v_1 u_1 \frac{du_0}{dy}} = 0. \qquad (5.3)$$

The second term in (5.3), *which represents the* total interaction between the Reynolds stress and the horizontal shear of the basic flow, may be rewritten as

$$\overline{F h_0 v_1 u_1 \frac{du_0}{dy}} = F\frac{\partial}{\partial t}\overline{\left[\frac{h_0 u_1^2}{2}\frac{du_0}{dy}\left(1 - F\frac{du_0}{dy}\right)^{-1}\right]} \qquad (5.4)$$

using (2.8) and assuming $1 - F\,du_0/dy \neq 0$. It is now possible, using (4.2) and (5.4), to rewrite (5.3) as

$$\frac{\partial}{\partial t}\tfrac{1}{2}[\overline{\Omega_0^{-1} u_1^2} + \overline{h_0 v_1^2} + \overline{h_1^2}] = 0. \qquad (5.5)$$

Integration of (5.5) yields

$$\tfrac{1}{2}[\overline{\Omega_0^{-1} u_1^2} + \overline{h_0 v_1^2} + \overline{h_1^2}] = \bar{\varepsilon}_i, \qquad (5.6)$$

where ε_i denotes the initial value of the expression in brackets on the left side of (5.6). The first-order variables may be expressed as

$$u_1 = u_{1g} + u_1', \quad v = v_1', \quad h_1 = h_{1g} + h_1'. \qquad (5.7)$$

Then if (5.7) is inserted into (5.6) the cross-product term

$$\mathcal{J} \equiv \overline{\Omega_0^{-1} u_1' u_{1g}} + \overline{h_1' h_{1g}} \qquad (5.8)$$

will arise in addition to the term representing the sum of the geostrophic and ageostrophic energies. With the aid of (4.5) and one integration by parts (5.8) becomes

$$\mathcal{J} \equiv \overline{h_{1g}\left[\frac{\partial}{\partial y}(\Omega_0^{-1} u_1') + h_1'\right]}. \qquad (5.9)$$

We note, however, that (4.6) may be written

$$\frac{\partial}{\partial y}(\Omega_0^{-1}u_1') + h_1' = 0 \qquad (5.10)$$

if $\Omega_0 \neq 0$, so that $\mathcal{J} = 0$.

Further simplification is possible if initial conditions (4.10) are used to evaluate ε_i. Then (5.6) becomes

$$\tfrac{1}{2}[\overline{\Omega_0^{-1}(u_{1g}^2 + u_1'^2)} + \overline{h_0 v_1'^2} + \overline{(h_{1g}^2 + h_1'^2)}] = \tfrac{1}{2}[\overline{\Omega_0^{-1}(u_g)_i^2} + \overline{h_0 v_i'^2} + \overline{(h_g)_i^2}]. \qquad (5.11)$$

This energy equation (5.11) represents an extension of a result obtained by Blumen (1967*a*) who investigated the energy partition of motions superposed on a basic state of rest.

Finally, using the result based on (4.14), equation (5.11) simplifies to

$$\tfrac{1}{2}[\overline{\Omega_0^{-1}u_1'^2} + \overline{h_0 v_1'^2} + \overline{h_1'^2}] = \tfrac{1}{2}\overline{h_0 v_i'^2}, \qquad (5.12)$$

which expresses the partition of the initial ageostrophic energy among the ageostrophic components. It is interesting that, in this model, the ageostrophic components do not abstract energy from the basic current. However the distribution of the potential vorticity in the basic state does determine the relative partitioning of energy among the components.

6. Solutions

6.1. *Potential vorticity equation*

Solutions of the potential vorticity equation (4.9) for ageostrophic initial conditions have been determined by numerical integration in the region $-5.0 \leq y \leq 5.0$. The basic flow variables (2.7) are given by

$$\hat{h}_0 = -\tfrac{1}{4}\tanh 2y$$

$$u_0 = -\partial\hat{h}_0/\partial y = \tfrac{1}{2}\,\mathrm{sech}^2\, 2y \qquad (6.1)$$

and the initial conditions by

$$u_i = e^{-\frac{1}{2}(y/r)^2}$$

$$v_i = 0 \qquad (6.2)$$

$$h_i = 0,$$

where $r = r^*/(f^{-1}(g^*H)^{\frac{1}{2}})$ denotes the nondimensional measure of the initial current width. The basic state variables, together with the potential vorticity

$$\Omega_0 \equiv (1 + F\hat{h}_0)^{-1}(1 - F\,du_0/dy) \qquad (6.3)$$

for $F = 0.5$, are displayed in Fig. 1. The initial velocity field, delineated by dots, appears in Fig. 2 for $r = 0.5$ and $r = 1.5$.

The first-order geostrophic variables may be obtained from (4.9), using (6.1), (6.2) and (6.3). The geostrophic velocity, $u_{1g} = -\partial h_{1g}/\partial y$, is displayed in Fig. 2 for various values of Ω_0. The case $\Omega_0 = 1$ refers to a resting basic state. Solutions for this latter case were also determined analytically to provide a check on the numerical integration. These geostrophic solutions, for $\Omega_0 = 1$, have been determined from the Green's function for the operator appearing on the left side of (4.9) (Goertzell and Tralli, 1960). They are

$$h_{1g} = \frac{r}{2}\sqrt{\frac{\pi}{2}}\left\{e^{y+(r^2/2)}\,\mathrm{erfc}\left[\frac{r^2+y}{\sqrt{2}r}\right] - e^{-y+(r^2/2)}\,\mathrm{erfc}\left[\frac{r^2-y}{\sqrt{2}r}\right]\right\} \qquad (6.4)$$

and

$$u_{1g} = -\partial h_{1g}/\partial y = u_i(y) - \frac{r}{2}\sqrt{\frac{\pi}{2}}\left\{e^{y+(r^2/2)} \times \mathrm{erfc}\left[\frac{r^2+y}{\sqrt{2}r}\right] + e^{-y+(r^2/2)}\,\mathrm{erfc}\left[\frac{r^2-y}{\sqrt{2}r}\right]\right\}, \qquad (6.5)$$

where $\mathrm{erfc}(z)$ denotes the complementary error function (Gautschi, 1964) and $u_i(y)$ is given (6.2).

The geostrophic velocity fields for $F = 0.5$ and 1.0 in Fig. 2 show some features which do not appear in the case $\Omega_0 = 1$. Most noticeable is the southward shift of the current axis and the relatively large amplitude of the velocity with increasing values of F. The southward shift, first discovered by Rossby (1938), does not appear in the case $\Omega_0 = 1$ because the absolute momentum following fluid parcels is not conserved. The addition of the convective term $Fv_1 du_0/dy$ in (2.8) overcomes this restriction, within the scope of linear theory. Additional features of the motion introduced by the addition of convective terms has been discussed elsewhere (Blumen, 1967*b*).

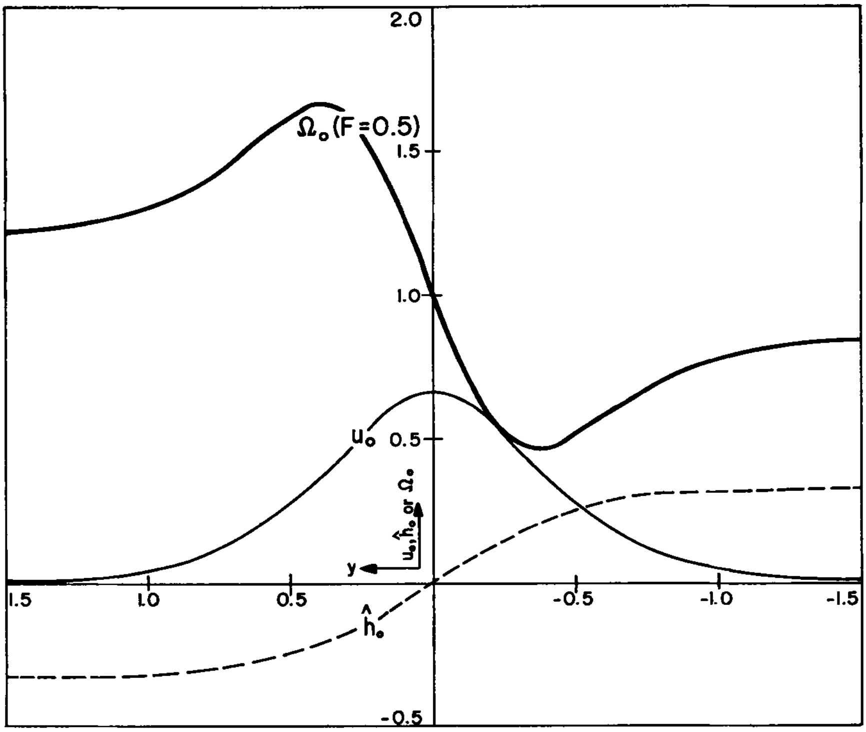


Fig. 1. Distribution of the basic-state geostrophic height field $\hat{h}_0$, zonal velocity u_0 and potential vorticity Ω_0. Ω_0 has been evaluated with the Froude number $F = 0.5$. Positive y is directed northward.

In order to more clearly show the significance of including a basic horizontal shear flow we have computed the following quantities, defined by

$$\varepsilon_K \equiv \overline{u_{1g}^2 \Omega_0^{-1}} / \overline{u_i^2 \Omega_0^{-1}} \quad (6.6)$$

and $$\varepsilon_G \equiv [\overline{u_{1g}^2 \Omega_0^{-1}} + \overline{h_{1g}^2}] / \overline{u_i^2 \Omega_0^{-1}}. \quad (6.7)$$

ε_K and ε_G, which are presented as functions of r in Fig. 3, denote the fractions of the total energy $\overline{u_i^2 \Omega_0^{-1}}$ which reside in geostrophic kinetic energy and in total geostrophic energy. The fraction of energy going into the ageostrophic components, determined from (5.6), is

$$\varepsilon_A = 1 - \varepsilon_G \quad (6.8)$$

where $$\varepsilon_A \equiv [\overline{u_1'^2 \Omega_0^{-1}} + \overline{h_0 v_1'^2} + \overline{h_1'^2}] / \overline{u_i^2 \Omega_0^{-1}}. \quad (6.9)$$

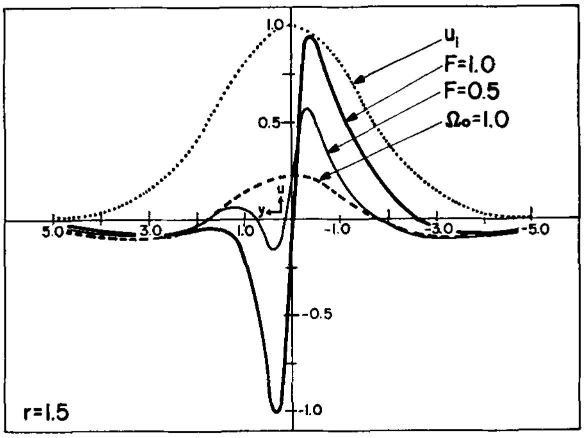
It is noted that the presence of a basic horizontal shear flow inhibits the partition of energy to the ageostrophic components, and that this effect becomes more pronounced with larger shears, i.e., increasing F. A substantial contribution to geostrophic energy is also evident when $r > 1$ and $\Omega_0 \neq 1$. When r is large this contribution must come principally from the low wave number part of the initial velocity spectrum, $k << 1$, since

$$U_i(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u_i(y) e^{-iky} dy = \frac{r}{\sqrt{2\pi}} e^{-(r^2k^2)/2}. \quad (6.10)$$

In contradistinction, Blumen (1967*a*) has shown that with a basic state of rest ($\Omega_0 = 1$) little energy is partitioned to geostrophic motion when k is small. In the limit, $k = 0$, all the initial energy goes into pure inertia motion, since pressure gradients do not develop in response to an initial motion which is uniform over the whole infinite plane.

6.2. *Uniform initial condition*

We now investigate the limiting case ($r = \infty$ or $k = 0$) for the present model, when a basic horizontal shear flow exists. From (3.9), we note



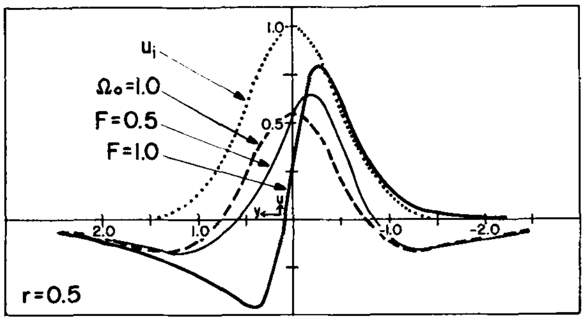


Fig. 2. Distribution of the initial ($t=0$) unbalanced current u_i and steady-state ($t=\infty$) first-order geostrophic velocities u_{1g} for indicated values of the Froude number F and initial current width r. $\Omega_0=1$ corresponds to a basic-state at rest.

that the time-dependent motion oscillates with the inertial frequency, $\sigma=1$, when $\partial v_1/\partial y=0$. The time-dependent solutions for this case may be determined from (2.8), (2.9) and (2.10), which may be expressed as

$$\frac{\partial}{\partial t}(u_1'+h_{1y}'+iv_1')+i(u_1'+h_{1y}'+iv_1')=0, \quad (6.11)$$

where the prime denotes ageostrophic motion. The solution of (6.11), satisfying $u_i=$ constant initially, is

$$v_1' = -u_i \sin t$$
$$u_1'+h_{1y}' = u_i \cos t. \quad (6.12)$$

From (2.10), with $\partial v_1/\partial y=0$, and (6.12) we obtain

$$u_1' = u_i[1-Fu_{0y}]\cos t$$
$$h_1' = u_i F u_0 \cos t. \quad (6.13)$$

The geostrophic part of the solution may be determined from (2.8), which may be expressed as

$$\frac{\partial}{\partial t}\left\{\frac{\partial}{\partial y}\frac{u}{1-Fu_{0y}}\right\}=0, \quad 1-Fu_{0y}\neq 0. \quad (6.14)$$

Integration of (6.14) yields the geostrophic solution

$$u_{1g} = Fu_i u_{0y}$$
$$h_{1g} = -Fu_i u_0, \quad (6.15)$$

where $u_0(\pm\infty)=0$. (It may be verified that this geostrophic solution (6.15) satisfies (4.9).) The geostrophic solutions for $F=1.0$ which are displayed in Fig. 2, clearly show the influence of the low wave-number part of the spectrum. This may be seen by comparing these solutions with $u_g \propto u_{0y}$, where $u_0(y)$ is shown in Fig. 1.

Finally, the complete solution, obtained from (6.12), (6.13) and (6.15), is

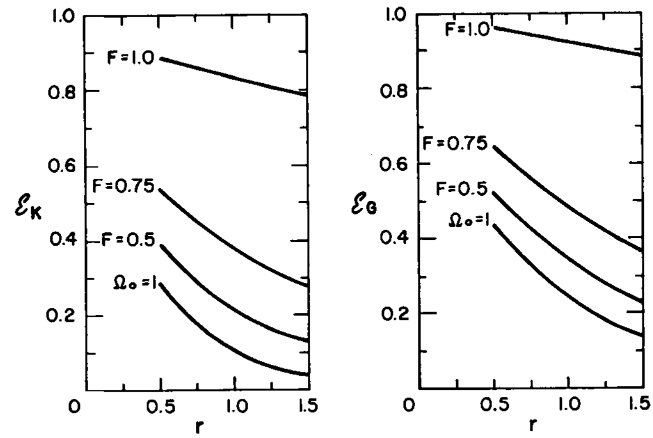


Fig. 3. Fraction of initial energy partitioned to geostrophic kinetic energy ε_K and total geostrophic energy ε_G as functions of the initial current width r for indicated values of the Froude number F. $\Omega_0 = 1$ corresponds to a basic-state at rest.

$$u_1 = u_i\{(1 - Fu_{0y}) \cos t + Fu_{0y}\}$$
$$v_1 = -u_i \sin t$$
$$h_1 = u_i F u_0 (\cos t - 1). \qquad (6.16)$$

This solution (6.16) represents an undamped inertia-motion superposed on a steady geostrophic motion.

The reason that pressure gradients develop in response to a uniform initial motion is related to the presence of a basic shear flow. The spatial variability of the basic state introduces a variable restoring force, per unit mass and unit displacement, which acts on fluid motions. This force is $f(f - u_{0y})H/h_0^*$,[1] expressed in dimensional units. If the basic state is at rest, the force f^2 is constant and the resulting motion is independent of y. When $u_0 = u_0(y)$, pressure gradients must develop in order to balance the variable restoring force and permit the fluid to attain an ultimate steady-state.

The term ε_G (6.7) cannot be determined for the present case, since the total initial energy is infinite. However, from (6.15), we note that the total geostrophic energy is finite and increases like F^2. When the initial motion is y-dependent, so that the initial spectrum is not discrete and the total energy is finite, a nonzero contribution to ε_G comes from the total spectrum, including $k = 0$.

7. Concluding remarks

The principal conclusion that may be drawn from the present investigation is that the process of energy partition between geostrophic and ageostrophic motions is strongly influenced by the presence of a horizontal shear flow. This effect, which is illustrated in Fig. 3, is most noticeable when the shears are large. For example, if $F = 0.5$ the horizontal shear is one-half the Coriolis parameter in a small region near the zonal wind maximum. Horizontal shears of this magnitude are not uncommon in the vicinity of jet streams (Endlich, 1964).

Finally, we note that the absolute stability of the basic geostrophic current has been established by allowing pressure changes to occur in response to fluid displacements. This result suggests that "pure" centrifugal instability should be a rare phenomenon, if it exists at all, in the atmosphere and oceans.

[1] The presence of this term as a variable restoring force is most clearly seen in (3.2), which is the one-dimensional Klein–Gordon equation discussed in Morse and Feshbach (1953, § 2.1).

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ЭФФЕКТ ТЕЧЕНИЯ С ГОРИЗОНТАЛЬНЫМ СДВИГОМ НА ГЕОСТРОФИЧЕСКУЮ АДАПТАЦИЮ В БАРОТРОПНОЙ ЖИДКОСТИ

Исследуется эффект основного течения с горизонтальным сдвигом скорости на процесс линейной геострофической адаптации в неограниченной баротропной жидкости. Показано, что основное течение абсолютно устойчиво по отношению к осесимметричным поперечным возмущениям и что энергия не черпается из основного течения под действием напряжений Рейнольдса. Выводится и решается численно уравнение потенциального вихря, чтобы определить относительное количество начальной энергии, которое подразделяется на геострофическую кинетическую и доступную потенциальную энергию, как функцию ширины начального течения. Показано также, что противоположность процессу адаптации в атмосфере, основным состоянием которой является покой, значительная часть геострофической энергии находится в волнах с малыми номерами. Это обстоятельство наиболее заметно, когда градиент основного течения велик.