Studies of a coalescent world geodetic system[1]

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ABSTRACT

A study has been made of the gravity field of the earth using a combined solution from free air anomalies and satellite data. Spherical harmonics up to order 8.8 have been computed in a joint solution. Satellite data from Doppler tracking (NWL) and optical observations (SAO) were added in a combined solution with full consideration of the complete covariance matrix.—This study was based on the most complete set of gravity data so far presented (approximately 500,000).

The simplest type of a world geodetic system includes only geometrical observations. Such systems have a limited geodetic application because no dynamic properties can be derived from purely geometrical observations. A geodetic system that is going to be used for ballistic applications has to incorporate a solution of the *gravimetric boundary value problem* consistent with the geometrical representation. Present geodetic systems already include a number of dynamic parameters presented in the form of so-called geoids. These geoids are establishing valuable reference systems mainly in a conservative application, i.e., when geometrical measurements are reduced to an analytical reference surface (normally the international ellipsoid). However, it has to be noted that these geoids are not adequate for computations of dynamical parameters. Any computation of ballistic data based on these geoids can be quite misleading. The vertical deflections computed from a geoid can give errors in the dynamical parameters corresponding to several hundred meters in distance. The following study outlines a method to link all available geodetic observations in a "coalescent world geodetic system".

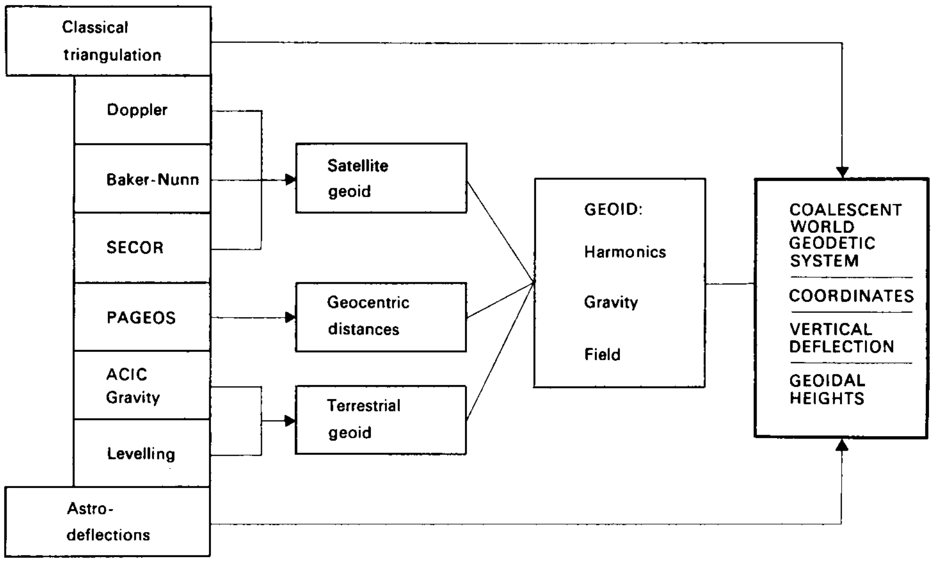
The study outlined here is making use of the following classes of data: classical triangulation, satellite observations, gravity measurements, and astro-geodetic observations.

The final goal is a world geodetic system without any *significant* inconsistencies between the dynamical and geometrical parameters. Preliminary solutions can be obtained at any wanted time intervals. The "final solution" in a coalescent world geodetic system will not be obtained without considerable improvement of several data classes. Our analysis is based on "a geoid" which has specific properties suitable for a world geodetic system.

The complete solution is broken down in several steps. The first step will give a satellite geoid from all available satellite observations. The second step gives a

[1] This study has been completed 1967 at the Research Institute for Geodetic Sciences, Alexandria Va, U.S.A. and was presented to the IAG meeting in Lucern 1967.

Coalescent world geodetic system



terrestrial geoid from gravity measurements. The third step includes PAGEOS and astrogeodetic data (if available). The fourth step will give the space geoid. All these geoid computations will also include solutions for station coordinates common to two or more systems and the solutions will be made with full consideration of given covariances.

When the harmonics (and coordinates) of the first step have been obtained we find the final solution for remaining geometrical data by back substitutions in the partial normal equations of the total system. Using this technique we can fulfil the solution on available computers and still obtain the same result as if all data had been processed simultaneously. The present study is the first step of this approach.

1. Definition of the geoid[1]

Given: Discrete gravity data at the physical surface of the earth.

Wanted: An equipotential surface coinciding with mean sea-level when all external mass is moved inside the reference surface (ellipsoid or sphere) in such a way that the gravity field at the physical surface and in space is maintained.

Solution: It is obvious that there is no unique solution of our problem when gravity is given only in discrete points. Therefore our solution has to include a prediction of gravity in the unsurveyed areas. (In the classical problem gravity was given all over the physical surface.) This prediction of gravity is a part of the solution in the general case.

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[1] For details cf. Bjerhammar 1963, 1964.

The disturbance potential is defined in the classical way.

$$T = W - U \tag{1}$$

where

W = potential of the earth

U = potential of the reference body (ellipsoid or sphere)

T = disturbance potential

After differentiation we have

$$\frac{\partial T}{\partial n} = \frac{\partial W}{\partial n} - \frac{\partial U}{\partial n} = -(g - \gamma) \tag{2}$$

where

g = gravity at a surface point (P_j) of the earth

γ = theoretical gravity at the point (P_j)

$g - \gamma$ = gravity disturbance

n = normal of the reference ellipsoid through the point (P_j)

In equation (2) we only know g because the actual height above the reference body is unknown.

A "theoretical height" can be computed from the geopotential difference between a point, P_0, and an arbitrary point at the reference surface. (Method of Rune.)

$$W_j = W_0 + \frac{\partial W_0}{\partial n} z + \frac{\partial^2 W_0}{2 \cdot \partial n^2} z^2 + \dots \tag{3}$$

with the approximate solution

$$z = \frac{W_j - W_0}{\partial U_0 / \partial n} = \frac{W_0 - W_j}{\gamma_0} \tag{3a}$$

where

z = theoretical height

W_0 = potential at mean sea level

U_0 = potential on the reference surface (theoretical)

γ_0 = theoretical gravity at the reference surface

In most applications the geopotential differences will not be available and the theoretical heights have to be replaced by the orthometric heights or any useful height information.

The theoretical gravity at the physical surface can now be computed (higher order terms are omitted)

$$\gamma = \gamma_0 + \frac{\partial \gamma}{\partial n}\left(z + \frac{T}{\gamma_0}\right) = \gamma_z + \frac{T}{\gamma_0} \cdot \frac{\partial \gamma}{\partial n} \quad (4)$$

Thus we have

$$\frac{\partial T}{\partial n} = -(g - \gamma) = -\left(g - \gamma_z - \frac{T}{\gamma_0}\frac{\partial \gamma}{\partial n}\right) \quad (5)$$

or

$$\frac{\partial T}{\partial n} - \frac{T}{\gamma_0}\frac{\partial \gamma}{\partial n} = -(g - \gamma_z) = -\Delta g \quad \text{(Brun's formula)} \quad (6)$$

where Δg = gravity anomaly in the classical definition

Molodensky (1948, 1962) used this boundary condition for a solution of the gravimetric boundary value problem of the physical surface. The method is elegant but very tedious for a non-analytical surface because all formulas are related to the external irregular surface. Here we shall first make use of a reduction of the gravity anomalies to the reference surface and then use highly simplified formulas for the sphere.

2. Gravity reduction

The boundary condition (1:6) is approximated for a sphere thus

$$\Delta g = -\frac{\partial T}{\partial r_0} - \frac{2T}{r_0} \quad (1)$$

where r_0 = radius of the sphere.

The potential in space is then

$$T_j = \frac{1}{4\pi r_j}\iint_S \Delta g^* \sum_{n=2}^{\infty} \frac{2n+1}{n-1}\left(\frac{r_0}{r_j}\right)^n P_n(\cos\omega)\, dS \quad (2)$$

(Generalization of Stokes' formula)

or

$$T_j = \frac{1}{4\pi r_j}\iint_S \Delta g^* \left(\frac{2}{r} - 3r + 1 - 5t\cos\omega - 3t\cos\omega \ln\phi\right) dS \quad (3)$$

$$2\phi = 1 - t\cos\omega + r$$

(Bjerhammar, 1959, 1960; Hirvonen, 1960; Molodensky, 1960) where

Δg^* = gravity anomaly at the reference surface

r_0 = radius of the reference surface

r_j = distance between the origin and the point P_j at the physical surface or in space

T = disturbance potential

ω = geocentric angle between the moving point and the fixed point

$P_n(\cos\omega)$ = Legendre polynomial

S = reference surface

$t = r_0/r_j$

$r = r_{ij}/r_j$

r_{ij} = distance between the moving point P_i on the reference surface and the actual point P_j at the physical surface or in space.

From (1) and (2) we obtain after omitting the two terms of lowest order

$$\Delta g_j = \frac{h_j}{2\pi}\iint_S \frac{\Delta g^*}{r_{ij}^3}\,dS \tag{4}$$

where $$h_j = (r_j^2 - r_0^2)/2r_j$$

Equation (4) is an integral equation where Δg_j has to be known all over the physical surface. In geodesy we know the gravity anomaly only in discrete points which means that any solution will include an estimation of gravity in the unsurveyed areas.

In solving equation (4) we start with considering Δg εs an approximate solution for Δg^*.

Then we have the condition

$$\Delta g_j = \frac{h_j}{2\pi}\iint_S \frac{\Delta g}{r_{ij}^3}\,dS \tag{5}$$

The condition will normally not be fulfilled and we use the residuals of Δg_j to correct our first estimate of Δg^*. Then we have

$$\Delta g^* = 2\Delta g_j - \frac{h_j}{2\pi}\iint_S \frac{\Delta g}{r_{ij}^3}\,dS \tag{6}$$

For small values of h_j we have the approximation

$$\Delta g^* = \Delta g_j - \frac{h_j}{2\pi}\iint_S \frac{\Delta g - \Delta g_j}{r_{ij}^3}\,dS \tag{7}$$

This expression has been used as the first approximative solution (Bjerhammar, 1963, p. 72; Moritz, 1964).

The value of Δg^* according to equation (6) or (7) can be used as an improved estimate of our unknown gravity anomalies at the reference surface. However, this technique sometimes gives slow convergence and in test models and we found it convenient to use another method.

Here we express Δg^* by a polynomial in h

$$\Delta g^* = \Delta g + c_1 h + c_2 h^2 + \dots + c_m h^m \tag{8}$$

where

h = height above the reference surface

c_i = unknown constants

Equation (4) and (8) now define a system of n linear equations for n discrete gravity values

$$a_{11} c_1 + a_{12} c_2 + \dots + a_{1m} c_m = \Delta G_1$$

$$a_{21} c_1 + a_{22} c_2 + \dots + a_{2m} c_m = \Delta G_2$$

$$- \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad -$$

$$a_{n1} c_1 + a_{n2} c_2 + \dots + a_{nm} c_m = \Delta G_n$$

where

$$\Delta G_j = \Delta g_j - \frac{h_j}{2\pi} \iint_S \frac{\Delta g}{r_{ij}^3} dS \tag{9}$$

For $n > m$ we obtain a unique solution according to method of least squares. This is true also when the rank of the A-matrix is less than m. Then the sum of the squares of the residuals and the c-values is minimized.

For $n < m$ (and rank $= n$) we have a unique solution satisfying all measured gravity values when minimizing the sum of the squares of the c-values. (All solutions include prediction of gravity.)

The *geoid* according to our new definition is obtained from equation (3) for $t = 1$. Geoidal heights $N = T/\gamma_0$.

The *disturbance* potential in space is obtained from equation (2).

3. The terrestrial geoid

The gravity field of the rotating earth can be represented by a series expansion in spherical harmonics.

$$W = \frac{GM}{r_j} \left[\sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{r_0}{r_j}\right)^n \bar{P}_{nm}(\sin\phi) \left(\bar{C}_{nm} \cos m\lambda + \bar{S}_{nm} \sin m\lambda\right) \right] + 0.5\, r_j^2 (\cos^2\phi)\, \omega^2 \tag{1}$$

where W = potential

G = Newton's gravitational constant

M = mass of the earth

r_j = geocentric distance to the fixed point

r_0 = radius of the earth

$\bar{P}_{nm}(\sin\phi)$ = associated Legendrians (fully normalized)

$\bar{C}_{nm}, \bar{S}_{nm}$ = fully normalized harmonic coefficients

ϕ = latitude

λ = longitude

ω = rotational velocity

The series expansion corresponds directly to the pure Newtonian gravitation and the additional part is equivalent with the contribution from the centrifugal force which can be transcribed in the following way:

$$0.5\, r_j^2(\cos^2\phi)\,\omega^2 = \frac{\omega^2 r_j^2}{3}\,[1 - P_{20}(\sin\phi)] \tag{2}$$

If the potential function is determined from satellite observations then the centrifugal force of the earth is eliminated.

In terrestrial geodesy the centrifugal force is eliminated by introducing the disturbance potential (T).

$$T = W - U \tag{3}$$

where

W = actual potential

U = potential of the reference body (normally the international reference ellipsoid)

The disturbance potential (T) is a harmonic function which fulfils the Laplace condition

$$\Delta T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{4}$$

The gravity anomaly (Δg) has to fulfil the boundary value (Brun's formula).

$$\frac{\partial T}{\partial r_j} - \frac{T}{\gamma}\frac{\partial \gamma}{\partial r_j} = -(g-\gamma) = -\Delta g \tag{5}$$

where

γ = theoretical gravity in "normal field" of the reference body.

From the international gravity formula Jefferys (1943) computed

$$\gamma = 979{,}770 + 3446.0\ P_{20}(\sin\phi) + 5.3\ P_{40}(\sin\phi) \tag{6}$$

$\gamma = 978{,}049(1 + 0.0052884 \sin^2\phi - 0.000059 \sin^2 2\phi)$ milligal

$P_{20}(\sin\phi)$, $P_{40}(\sin\phi)$ = zonal harmonics (conventional)

Zonal harmonics of higher order are computed from satellite observations by King-Hele and Kozai. Gravity anomalies for this study can also be computed by the aid of any reference body which includes the centrifugal force of the earth.

For our further calculation we introduce the approximation

$$\frac{1}{\gamma}\frac{\partial \gamma}{\partial r} = -\frac{2}{r_j} \tag{7}$$

Then we have from (1) and (5) (harmonics of order zero and one are omitted)

$$\frac{\partial T}{\partial r_j} = -\frac{GM}{r_j^2} \sum_{n=2}^{\infty} \sum_{m=0}^{n} (n+1) \left(\frac{r_0}{r_j}\right)^n \bar{P}_{nm}(\sin\phi)\,[\bar{C}_{nm}\cos m\lambda + \bar{S}_{nm}\sin m\lambda] \tag{8}$$

and

$$\Delta g = \frac{GM}{r_j^2} \sum_{n=2}^{\infty} \sum_{m=0}^{n} (n-1) \left(\frac{r_0}{r_j}\right)^n \bar{P}_{nm}(\sin\phi)\,[\bar{C}_{nm}\cos m\lambda + \bar{S}_{nm}\sin m\lambda] \tag{9}$$

Here we can solve for $\bar{C}_{nm}$ and $\bar{S}_{nm}$ according to method of least squares.

In a strict application of this formula we have to take the topography in consideration by proper evaluation of $(r_0 : r_j)$.

Equation (9) gives the most convenient direct application of equation (2:4).

4. Spherical harmonics on the ellipsoidal reference surface

The terrestrial free air gravity anomalies are referred to the International Gravity Formula and the International Reference Ellipsoid. Any expansion of the potential field of the earth using spherical harmonics on an ellipsoid is introducing a bias in the final solution. Only Lamé functions (ellipsoidal harmonics) can be used on an ellipsoidal surface of reference without any inconsistencies between the individual harmonics. (For a solution with Lamé functions cf. Bjerhammar, 1962 and Molodensky, 1962.) This projection error is here partly compensated by a primary reduction of gravity anomalies from the physical surface of the earth to a reference *sphere* with radius equal to the mean equatorial radius of the earth.

The gravity reduction is made in accordance with the theory presented above by using free air anomalies and adequate values of r_0/r_j in the expansion of spherical harmonics. The *spherical* harmonics of the terrestrial gravity field is then to be referred to a *sphere* instead of an ellipsoid. The equatorial radius has been chosen because all satellite harmonics are referred to this radius.

In the practical procedure the free air anomalies are first converted to "spherical free air anomalies". These new gravity anomalies are considerably larger than the original ones. At the equator most of the anomalies are in the order of 500 milligals. The reduction to the reference sphere ("Geosphere") is then a part of the solution with spherical harmonics and the time consuming gravity reduction according to equation (2:5) is eliminated. Spherical harmonics computed in this way are compatible with the satellite solutions. The numerical results from a computation exclusively based on terrestrial data using this technique show a remarkable agreement with low order harmonics from satellite data. (Cf. below.)

5. Reduced normal equations

All different satellite solutions of interest are combined in the following way:

Contribution 1

Observation equations

$$\mathbf{A}_1\mathbf{X}_1+\mathbf{B}_1\mathbf{Y}=\mathbf{L}_1 \tag{1}$$

where

$\mathbf{X}$ = station coordinates

$\mathbf{Y}$ = harmonics

Normal equations

$$\begin{aligned}\mathbf{A}_1^*\mathbf{A}_1\mathbf{X}_1+\mathbf{A}_1^*\mathbf{B}_1\mathbf{Y}&=\mathbf{A}_1^*\mathbf{L}_1\\ \mathbf{B}_1^*\mathbf{A}_1\mathbf{X}_1+\mathbf{B}_1^*\mathbf{B}_1\mathbf{Y}&=\mathbf{B}_1^*\mathbf{L}_1\end{aligned} \tag{2}$$

There we obtain the reduced normal equation

$$(\mathbf{B}_1^*\mathbf{B}_1-\mathbf{B}_1^*\mathbf{A}_1^0\mathbf{B}_1)\mathbf{Y}=\mathbf{B}_1^*\mathbf{L}_1-\mathbf{B}_1^*\mathbf{A}_1^0\mathbf{L}_1 \tag{3}$$

when $\mathbf{A}_1^0=\mathbf{A}_1(\mathbf{A}_1^*\mathbf{A}_1)^{-1}\mathbf{A}_1^*$

Contribution 2

Observation equations

$$\mathbf{A}_2\mathbf{X}_2+\mathbf{B}_2\mathbf{Y}=\mathbf{L}_2 \tag{4}$$

Normal equations

$$\begin{aligned}\mathbf{A}_2^*\mathbf{A}_2\mathbf{X}_2+\mathbf{A}_2^*\mathbf{B}_2\mathbf{Y}&=\mathbf{A}_2^*\mathbf{L}_2\\ \mathbf{B}_2^*\mathbf{A}_2\mathbf{X}_2+\mathbf{B}_2^*\mathbf{B}_2\mathbf{Y}&=\mathbf{B}_2^*\mathbf{L}_2\end{aligned} \tag{5}$$

Reduced normal equations

$$(\mathbf{B}_2^*\mathbf{B}_2-\mathbf{B}_2^*\mathbf{A}_2^0\mathbf{B}_2)\mathbf{Y}=\mathbf{B}_2^*\mathbf{L}_2-\mathbf{B}_2^*\mathbf{A}_2^0\mathbf{L}_2 \tag{6}$$

Contribution 3

Observation equations

$$\mathbf{A}_3\mathbf{X}_3+\mathbf{B}_3\mathbf{Y}=\mathbf{L}_3 \tag{7}$$

Normal equations

$$\begin{aligned}\mathbf{A}_3^*\mathbf{A}_3\mathbf{X}_3+\mathbf{A}_3^*\mathbf{B}_3\mathbf{Y}&=\mathbf{A}_3^*\mathbf{L}_3\\ \mathbf{B}_3^*\mathbf{A}_3\mathbf{X}_3+\mathbf{B}_3^*\mathbf{B}_3\mathbf{Y}&=\mathbf{B}_3^*\mathbf{L}_3\end{aligned} \tag{8}$$

Reduced normal equations

$$(\mathbf{B}_3^*\mathbf{B}_3-\mathbf{B}_3^*\mathbf{A}_3^0\mathbf{B}_3)\mathbf{Y}=\mathbf{B}_3^*\mathbf{L}_3-\mathbf{B}_3^*\mathbf{A}_3^0\mathbf{L}_3 \tag{9}$$

Any number of contributions can be added in this way.

Joint satellite solution

Observation equations

$$
\begin{aligned}
\mathbf{A}_1\mathbf{X}_1 + \quad\quad\quad\quad\quad\quad & \mathbf{B}_1\mathbf{Y} = \mathbf{L}_1 \\
\mathbf{A}_2\mathbf{X}_2 + \quad\quad\quad & \mathbf{B}_2\mathbf{Y} = \mathbf{L}_2 \\
\mathbf{A}_3\mathbf{X}_3 + & \mathbf{B}_3\mathbf{Y} = \mathbf{L}_3
\end{aligned}
\tag{10}
$$

Normal equations

$$
\begin{aligned}
\mathbf{A}_1^*\mathbf{A}_1\mathbf{X}_1 + \quad\quad\quad\quad\quad\quad & \mathbf{A}_1^*\mathbf{B}_1\mathbf{Y} = \mathbf{A}_1^*\mathbf{L}_1 \\
\mathbf{A}_2^*\mathbf{A}_2\mathbf{X}_2 + \quad\quad\quad & \mathbf{A}_2^*\mathbf{B}_2\mathbf{Y} = \mathbf{A}_2^*\mathbf{L}_2 \\
\mathbf{A}_3^*\mathbf{A}_3\mathbf{X}_3 + & \mathbf{A}_3^*\mathbf{B}_3\mathbf{Y} = \mathbf{A}_3^*\mathbf{L}_3
\end{aligned}
\tag{11}
$$

$$\mathbf{B}_1^*\mathbf{A}_1\mathbf{X}_1 + \mathbf{B}_2^*\mathbf{A}_2\mathbf{X}_2 + \mathbf{B}_3^*\mathbf{A}_3\mathbf{X}_3 + (\mathbf{B}_1^*\mathbf{B}_1 + \mathbf{B}_2^*\mathbf{B}_2 + \mathbf{B}_3^*\mathbf{B}_3)\mathbf{Y} = \mathbf{B}_1^*\mathbf{L}_1 + \mathbf{B}_2^*\mathbf{L}_2 + \mathbf{B}_3^*\mathbf{L}_3$$

Partial solutions

$$
\begin{aligned}
\mathbf{X}_1 &= (\mathbf{A}_1^*\mathbf{A}_1)^{-1}\mathbf{A}_1^*\mathbf{L}_1 - (\mathbf{A}_1^*\mathbf{A}_1)^{-1}\mathbf{A}_1^*\mathbf{B}_1\mathbf{Y} \\
\mathbf{X}_2 &= (\mathbf{A}_2^*\mathbf{A}_2)^{-1}\mathbf{A}_2^*\mathbf{L}_2 = (\mathbf{A}_2^*\mathbf{A}_2)^{-1}\mathbf{A}_2^*\mathbf{B}_2\mathbf{Y} \\
\mathbf{X}_3 &= (\mathbf{A}_3^*\mathbf{A}_3)^{-1}\mathbf{A}_3^*\mathbf{L}_3 - (\mathbf{A}_3^*\mathbf{A}_3)^{-1}\mathbf{A}_3^*\mathbf{B}_3\mathbf{Y}
\end{aligned}
\tag{12}
$$

Thus we have the final solution for the $\mathbf{Y}$ matrix

$$
\begin{aligned}
&[\mathbf{B}_1^*\mathbf{B}_1 + \mathbf{B}_2^*\mathbf{B}_2 + \mathbf{B}_3^*\mathbf{B}_3 - \mathbf{B}_1^*\mathbf{A}_1^0\mathbf{B}_1 - \mathbf{B}_2^*\mathbf{A}_2^0\mathbf{B}_2 - \mathbf{B}_3^*\mathbf{A}_3^0\mathbf{B}_3]\mathbf{Y} = \\
&\quad = \mathbf{B}_1^*\mathbf{L}_1 + \mathbf{B}_2^*\mathbf{L}_2 + \mathbf{B}_3^*\mathbf{L}_3 - \mathbf{B}_1^*\mathbf{A}_1^0\mathbf{L}_1 - \mathbf{B}_2^*\mathbf{A}_2^0\mathbf{L}_2 - \mathbf{B}_3^*\mathbf{A}_3^0\mathbf{L}_3
\end{aligned}
\tag{13}
$$

Equation (13) is however, identical with the sum of all reduced normal equations for $\mathbf{Y}$.

Thus we have proved that the sum of all reduced normal equations gives the same solution as—all observation equations used in a simultaneous solution with all unknowns included.

6. Formation of final normal equations

Given:

Contribution 1

$$(\mathbf{B}_1^*\mathbf{B}_1 - \mathbf{B}_1^*\mathbf{A}_1^0\mathbf{B}_1)\mathbf{Y} = \mathbf{B}_1^*\mathbf{L}_1 - \mathbf{B}_1^*\mathbf{A}_1^0\mathbf{L}_1 = \bar{\mathbf{L}}_1 \tag{1}$$

$$\mathbf{T}_1^*\mathbf{T}_1 = \mathbf{B}_1^*\mathbf{B}_1 - \mathbf{B}_1^*\mathbf{A}_1^0\mathbf{B}_1 \tag{2}$$

Contribution 2

$$(\mathbf{B}_2^*\mathbf{B}_2 - \mathbf{B}_2^*\mathbf{A}_2^0\mathbf{B}_2)\mathbf{Y} = \mathbf{B}_2^*\mathbf{L}_2 - \mathbf{B}_2^*\mathbf{A}_2^0\mathbf{L}_2 = \bar{\mathbf{L}}_2 \tag{3}$$

$$\mathbf{T}_2^*\mathbf{T}_2 = \mathbf{B}_2^*\mathbf{B}_2 - \mathbf{B}_2^*\mathbf{A}_2^0\mathbf{B}_2 \tag{4}$$

Contribution 3

$$(\mathbf{B}_3^* \mathbf{B}_3 - \mathbf{B}_3^* \mathbf{A}_3^0 \mathbf{B}_3)\,\mathbf{Y} = \mathbf{B}_3^* \mathbf{L}_3 - \mathbf{B}_3^* \mathbf{A}_3^0 \mathbf{L}_3 = \bar{\mathbf{L}}_3 \quad (5)$$

$$\mathbf{T}_3^* \mathbf{T}_3 = \mathbf{B}_3^* \mathbf{B}_3 - \mathbf{B}_3^* \mathbf{A}_3^0 \mathbf{B}_3 \quad (6)$$

The joint solution of these three systems is given by the rectangular system of "observation equations".

$$\begin{bmatrix} \mathbf{T}_1 \\ \mathbf{T}_2 \\ \mathbf{T}_3 \end{bmatrix} \mathbf{Y} = \begin{bmatrix} \mathbf{T}_1^{*-1} \bar{\mathbf{L}}_1 \\ \mathbf{T}_2^{*-1} \bar{\mathbf{L}}_2 \\ \mathbf{T}_3^{*-1} \bar{\mathbf{L}}_3 \end{bmatrix} \quad (7)$$

with the normal equations

$$[\mathbf{T}_1^* \mathbf{T}_1 + \mathbf{T}_2^* \mathbf{T}_2 + \mathbf{T}_3^* \mathbf{T}_3]\,\mathbf{Y} = \mathbf{T}_1^* \mathbf{T}_1^{*-1} \bar{\mathbf{L}}_1 + \mathbf{T}_2^* \mathbf{T}_2^{*-1} \bar{\mathbf{L}}_2 + \mathbf{T}_3^* \mathbf{T}_3^{*-1} \bar{\mathbf{L}}_3 \quad (8)$$

$$[\mathbf{T}_1^* \mathbf{T}_1 + \mathbf{T}_2^* \mathbf{T}_2 + \mathbf{T}_3^* \mathbf{T}_3]\,\mathbf{Y} = \bar{\mathbf{L}}_1 + \bar{\mathbf{L}}_2 + \bar{\mathbf{L}}_3 \quad (9)$$

Here equation (9) is identical with equation (5:13) after the interchange of variables.

In our solution we have to consider that the $\mathbf{Y}$-matrix will probably be given in different sequence for the individual contributions.

Then we have

$$\begin{aligned} \mathbf{T}_1 \mathbf{Y}_1 &= \mathbf{T}_1^{*-1} \bar{\mathbf{L}}_1 \\ \mathbf{T}_2 \mathbf{Y}_2 &= \mathbf{T}_2^{*-1} \bar{\mathbf{L}}_2 \\ \mathbf{T}_3 \mathbf{Y}_3 &= \mathbf{T}_3^{*-1} \bar{\mathbf{L}}_3 \end{aligned} \quad (10)$$

where

$$\begin{aligned} \mathbf{Y}_1 &= \boldsymbol{\phi}_1 \mathbf{Y} \\ \mathbf{Y}_2 &= \boldsymbol{\phi}_2 \mathbf{Y} \\ \mathbf{Y}_3 &= \boldsymbol{\phi}_3 \mathbf{Y} \end{aligned} \quad (11)$$

Then we have

$$\begin{aligned} \mathbf{T}_1 \boldsymbol{\phi}_1 \mathbf{Y} &= \mathbf{T}_1^{*-1} \bar{\mathbf{L}}_1 \\ \mathbf{T}_2 \boldsymbol{\phi}_2 \mathbf{Y} &= \mathbf{T}_2^{*-1} \bar{\mathbf{L}}_2 \\ \mathbf{T}_3 \boldsymbol{\phi}_3 \mathbf{Y} &= \mathbf{T}_3^{*-1} \bar{\mathbf{L}}_3 \end{aligned} \quad (12)$$

($\boldsymbol{\phi} = \mathbf{I}$ is recommended)

After this transformation all systems refer to a unique sequence of the unknowns.

Special problems

1.0. $\mathbf{Y}_1$, $\mathbf{Y}_2$ *and* $\mathbf{Y}_3$ *include different numbers of unknowns*

Then we have

$$\bar{\mathbf{Y}} = \mathbf{Y}_1 \cup \mathbf{Y}_2 \cup \mathbf{Y}_3$$

Here the $\bar{\mathbf{Y}}$ set is the union of the set $\mathbf{Y}_1 \mathbf{Y}_2$ and $\mathbf{Y}_3$.

In our solution the union $\bar{\mathbf{Y}}$ is used all over. (There might be empty sets in the **T**-matrices.)

2.0. *Preassigned values are used in some of the contributions*

This problem is related with the preceding one to some extent. If preassigned values are used exclusively in only one of the contributions this should remain fixed and no immediate correction should be included. If different preassigned values are used for the same unknowns, then corrections should be applied so that all corrections are consistent. This can be done by adding new unknowns which represent the errors in the preassigned values. The simplest approach is to use the same preassigned values all over.

3.0. *Additional data*

Any additional data from terrestrial gravity, optical satellite triangulater, astro-geodetic levelling, etc., can be added using the technique described above.

7. Normalizations of potentials for the geoid

The potential of a body can be expressed in the following way

$$V = G\int_M \frac{1}{r_{ij}}\,dm \tag{1}$$

when

r_{ij} = distance between the mass element and the actual point

dm = mass element

G = gravitational constant (Newton's)

By using geocentric coordinates we obtain

$$r_{ij}^2 = r^2 + \varrho^2 - 2r\varrho\cos\omega \tag{2}$$

where

ϱ = distance from the center of the earth to the mass element

r = distance from the center of the earth to the actual point

ω = geocentric angle between the actual point and the mass element

Furthermore, we obtain

$$\frac{1}{r_{ij}} = \frac{1}{r}(1+k)^{-\frac{1}{2}} \tag{3}$$

where

$$k = \left(\frac{\varrho}{r}\right)^2 - 2\left(\frac{\varrho}{r}\right)\cos\omega \tag{4}$$

After binomial expansion we obtain

$$\frac{1}{r_{ij}} = \frac{1}{r}\left(1 - \frac{1}{2}k + \frac{3}{8}k^2 - \frac{5}{16}k^3 + \frac{35}{128}k^4 \dots\right) \tag{5}$$

These series can be replaced by

$$\frac{1}{r_{ij}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{\varrho}{r}\right)^n P_n(\cos\omega) \tag{6}$$

where

$$P_n(\cos\omega) = \frac{1}{2^n n!}\frac{\partial^n}{\partial(\cos\omega)^n}(\cos^2\omega - 1)^n \tag{7}$$

Here we have

$$P_0(\cos\omega) = 1 \tag{8}$$

$$P_1(\cos\omega) = \cos\omega \tag{9}$$

$$P_2(\cos\omega) = \tfrac{1}{2}(3\cos^2\omega - 1) \tag{10}$$

$$P_3(\cos\omega) = \tfrac{1}{2}(5\cos^3\omega - 3\cos\omega) \tag{11}$$

$$P_4(\cos\omega) = \tfrac{1}{8}(35\cos^4\omega - 30\cos^2\omega + 3) \quad \text{etc.} \tag{12}$$

For the potential we get

$$V = G\int_M \frac{1}{r_{ij}}\,dm = \frac{G}{r}\int_M \sum_{n=0}^{\infty}\left(\frac{\varrho}{r}\right)^n P_n(\cos\omega)\,dm \tag{13}$$

or

$$V = V_0 + V_1 + V_2 + V_3 + \dots = \sum_{n=0}^{\infty} V_n \tag{14}$$

Where

$$V_0 = \frac{G}{r}\int dm \tag{15}$$

$$V_1 = \frac{G}{r}\int t\cos\omega\,dm \tag{16}$$

$$V_2 = \frac{G}{r}\int \frac{t^2}{2}(3\cos^2\omega - 1)\,dm \tag{17}$$

— — — — — — —

$$\left(t = \frac{\varrho}{r}\right) \tag{18}$$

Now we are relating the series expression to geographical coordinates and then we have by the aid of so called *surface spherical* harmonics (Y_n).

$$V = \sum_{n=0}^{\infty} t^{n+1} Y_n \quad (\varrho = \text{equatorial semi-diameter of the earth}) \tag{19}$$

$$V_n = t^{n+1} Y_n \tag{20}$$

and

$$Y_n = \sum_{m=0}^{n} (a_{nm} \cos m\lambda + b_{nm} \sin m\lambda) P_{nm}(\cos \psi) \tag{21}$$

$$\psi = 90^\circ - \phi$$

$$P_{nm}(\cos \psi) = \frac{d^m P_n(\cos \psi)}{d(\cos \psi)^m} \sin^m \psi$$

$$P_{00}(\cos \psi) = 1 = P_0(\cos \psi)$$

$$P_{10}(\cos \psi) = \cos \psi = P_1(\cos \psi)$$

$$P_{11}(\cos \psi) = \sin \psi$$

$$P_{20}(\cos \psi) = \tfrac{1}{2}(3 \cos^2 \psi - 1) = P_2(\cos \psi)$$

$$P_{21}(\cos \psi) = 3 \sin \psi \cos \psi$$

$$P_{22}(\cos \psi) = 3 \sin^2 \psi \quad \text{etc.}$$

Sometimes so-called *solid spherical harmonics* are used.

$$V = \sum_{n=0}^{\infty} r_0^{n+1} Y_n^* \tag{22}$$

$$Y_n^* = Y_n : r^{n+1} \tag{23}$$

where r_0 = equatorial semidiameters of the earth.

In satellite geodesy often dimensionless coefficients are used in the following way:

$$V = \frac{GM}{r} \left[1 + \sum_{n=2}^{\infty} \sum_{m=0}^{n} t^n (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda) P_{nm}(\cos \psi) \right]$$

where

$$t = \frac{r_0}{r} \tag{24}$$

Some authors use the following notations

$$J_n = -C_{n0}$$

$$J_{nm} = -C_{nm}$$

$$K_{nm} = -S_{nm}$$

Sometimes the following series expression is used

$$V = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{1}{r^{n+1}} (A_{nm} \cos m\lambda + B_{nm} \sin m\lambda)\, P_{nm}(\cos \psi) \tag{25}$$

where the coefficients include the primary constant (GM/r).

Several authors use "fully normalized associated Legendrians" ($\bar{P}_{nm}(\cos \psi)$) where

$$\bar{P}_{nm}(\cos \psi) = \sqrt{(2n+1)\frac{(n-m)!\,\varkappa}{(n+m)!}}\, P_{nm}(\cos \psi) \tag{26}$$

$$\varkappa = 1 \quad \text{for } m = 0 \quad \varkappa = 2 \quad m \neq 0$$

Then the potential might be given by the function

$$V = \frac{GM}{r}\left[1 + \sum_{n=2}^{m} \sum_{m=0}^{n} t^n (\bar{C}_{nm} \cos m\lambda + \bar{S}_{nm} \sin m\lambda) \bar{P}_{nm} (\cos \psi)\right] \tag{27}$$

In the Space Geoid program we will have

$$\mathbf{T}_{01}\mathbf{Y} = \mathbf{T}_1^{*-1}\bar{\mathbf{L}}_1 \quad \text{(Contribution 1)} \tag{28}$$

$$\mathbf{T}_{02}\mathbf{Y} = \mathbf{T}_2^{*-1}\bar{\mathbf{L}}_2 \quad \text{(Contribution 2)} \tag{29}$$

$$\mathbf{T}_{03}\mathbf{Y} = \mathbf{T}_3^{*-1}\bar{\mathbf{L}}_3 \quad \text{(Contribution 3)} \tag{30}$$

where

$$\mathbf{T}_{01} = \mathbf{T}_1 \boldsymbol{\phi}_1 \tag{31}$$

$$\mathbf{T}_{02} = \mathbf{T}_2 \boldsymbol{\phi}_2 \quad (\boldsymbol{\phi}_i = \text{normalization matrix}) \tag{32}$$

$$\mathbf{T}_{03} = \mathbf{T}_3 \boldsymbol{\phi}_3 \tag{33}$$

From (26) and (27) we obtain

$$\frac{(C, S)_{nm}}{\sqrt{(2n+1)\frac{(n-m)!\,\varkappa}{(n+m)!}}} = (\bar{C}, \bar{S})_{nm} \tag{34}$$

Thus we have for equation (31–33)

$$\phi_i = \sqrt{(2n+1)\frac{(n-m)!\,\varkappa}{(n+m)!}}$$

If normalization is only needed for equation (31) we obtain

$$\mathbf{T}_1 \phi \mathbf{Y}_0 = \mathbf{T}_1^{*-1}\bar{\mathbf{L}}_1 \tag{35}$$

$$\mathbf{T}_2 \mathbf{Y}_0 = \mathbf{T}_2^{*-1} \bar{\mathbf{L}}_2 \qquad (\mathbf{Y}_0 = \text{fully normalized}) \tag{36}$$

$$\mathbf{T}_3 \mathbf{Y}_0 = \mathbf{T}_3^{*-1} \bar{\mathbf{L}}_3 \tag{37}$$

with the final normal equations

$$(\boldsymbol{\phi}^* \mathbf{T}_1^* \mathbf{T}_1 \boldsymbol{\phi} + \mathbf{T}_2^* \mathbf{T}_2 + \mathbf{T}_3^* \mathbf{T}_3) \mathbf{Y}_0 = \boldsymbol{\phi}^* \bar{\mathbf{L}}_1 + \bar{\mathbf{L}}_2 + \bar{\mathbf{L}}_3$$

8. Contribution from singular normal equations

The final normal equations for the space geoid have to be nonsingular in order to give a unique solution according to classical method of least squares. However, the partial "reduced normal equations" might be singular in some cases which means that the covariance matrix doesn't exist in the classical sense. In such cases triangular decomposition according to Cholesky and Gauss is no longer possible. Instead we are using the following technique

Observation equations

$$\mathbf{AX} + \mathbf{BY} = \mathbf{L} \tag{1}$$

where $\mathbf{Y}$ = harmonics

$\mathbf{X}$ = remaining unknowns

Normal equations

$$\mathbf{A^*AX} + \mathbf{A^*BY} = \mathbf{A^*L} \tag{2}$$

$$\mathbf{B^*AX} + \mathbf{B^*BY} = \mathbf{B^*L}$$

We introduce the triangular matrix $\mathbf{T}$ defined by

$$\mathbf{T^*T} = \mathbf{A^*A} \tag{3}$$

$$\mathbf{T} = \begin{bmatrix} t_{11} & t_{12} & t_{13} & \dots \\ & t_{22} & t_{23} & \dots \\ & & t_{33} & \dots \end{bmatrix} \tag{4}$$

$$(\mathbf{T}^*)^{-1}\mathbf{A^*AX} + (\mathbf{T}^*)^{-1}\mathbf{A^*BY} = (\mathbf{T}^*)^{-1}\mathbf{A^*L} \tag{5}$$

or

$$\mathbf{TX} + (\mathbf{T}^*)^{-1}\mathbf{A^*BY} = (\mathbf{T}^*)^{-1}\mathbf{A^*L} \tag{6}$$

After omitting the **TX** matrix we have the remaining "observation equations"

$$(\mathbf{T}^*)^{-1}\mathbf{A^*BY} = (\mathbf{T}^*)^{-1}\mathbf{A^*L} \tag{7}$$

Premultiplication by the transposed coefficient matrix of the left member gives

$$\mathbf{B^*AT}^{-1}(\mathbf{T}^*)^{-1}\mathbf{A^*BY} = \mathbf{B^*AT}^{-1}(\mathbf{T}^*)^{-1}\mathbf{A^*L} \tag{8}$$

or

$$\mathbf{B^*A}(\mathbf{A^*A})^{-1}\mathbf{A^*BY} = \mathbf{B^*A}(\mathbf{A^*A})^{-1}\mathbf{A^*L} \tag{9}$$

With generalized matrix symbols we have

$$\mathbf{B^*A^0BY = B^*A^0L} \qquad (10)$$

By omitting **AX** in the primary observation equation we obtain

$$\mathbf{BY = L} \qquad (11)$$

With the normal equations

$$\mathbf{B^*BY = B^*L} \qquad (12)$$

Subtracting equation (10) from equation (12) gives

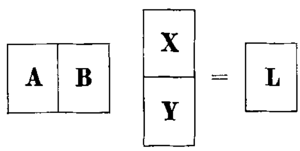
$$\mathbf{(B^*B - B^*A^0B)Y = B^*L - B^*A^0L} \qquad (13)$$

which is the reduced normal equation.

In this way we obtain an expression for the partial contribution without forming corresponding inverse which is considered singular.

Block diagrams:

Observations equations:



Normal Equations:

$$\begin{bmatrix} \mathbf{A^*A} & \mathbf{A^*B} \\ \mathbf{B^*A} & \mathbf{B^*B} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix} = \begin{bmatrix} \mathbf{A^*L} \\ \mathbf{B^*L} \end{bmatrix}$$

$$\mathbf{T} \; [\mathbf{(T^*)^{-1}A^*B}] \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix} = [\mathbf{(T^*)^{-1}A^*L}]$$

$$[\mathbf{B^*A(T)^{-1}(T^*)^{-1}A^*B}][\mathbf{Y}] = [\mathbf{B^*A(T)^{-1}(T^*)^{-1}A^*L}]$$

or

$$[\mathbf{B^*A^0B}]\mathbf{Y} = [\mathbf{B^*A^0L}]$$

with the final "reduced normal equations"

$$\mathbf{(B^*B - B^*A^0B)Y = B^*L - B^*A^0L}$$

9. Station coordinates and orbit parameters

Any station coordinates common to two or more systems can be included directly in the **B**-matrix and corresponding final coordinates will be obtained directly from the first step of the adjustment. Then we have in this general case

$$\mathbf{(\Sigma B^*B - \Sigma B^*A^0B)Y = \Sigma B^*L - \Sigma B^*A^0L} \qquad (1)$$

All remaining unknowns, station coordinates as well as orbit parameters can now be obtained from the equation

$$\mathbf{A}_i^* \mathbf{A}_i \mathbf{X}_i = \mathbf{A}_i^* \mathbf{L}_i - \mathbf{A}_i^* \mathbf{B}_i \mathbf{Y} \tag{2}$$

In most applications we only solve for the coordinates and leave the orbit parameter dormant in the reduced normal equations.

10. Weights

The weight problem is highly intricate when completely different measurements are used in a united solution.

The simplest approach seems to be to make use of a normalization which gives each observation equation the weight of a scalar $(=1)$.

Various estimates of the standard deviation can now be formed. Thus we have

Group 1: $$s_1^2 = \frac{\mathbf{L}_1^* \mathbf{L}_1 - \mathbf{L}_1^* \mathbf{A}_1^0 \mathbf{L}_1}{f_1} \quad (f_1 = \text{degrees of freedom of group 1})$$

Group 2: $$s_2^2 = \frac{\mathbf{L}_2^* \mathbf{L}_2 - \mathbf{L}_2^* \mathbf{A}_2^0 \mathbf{L}_2}{f_2} \quad (f_2 = \text{degrees of freedom of group 2})$$

Group 3: $$s_3^2 = \frac{\mathbf{L}_3^* \mathbf{L}_3 - \mathbf{L}_3^* \mathbf{A}_3^0 \mathbf{L}_3}{f_3} \quad (f_3 = \text{degrees of freedom of group 3})$$

$$s_\omega^2 = \frac{f_1 s_1^2 + f_2 s_2^2 + f_3 s_3^2}{f_1 + f_2 + f_3} \quad (s_\omega^2 = \text{"variance within groups"})$$

All groups: $$s^2 = \frac{\mathbf{L}^* \mathbf{L} - \mathbf{L}^* \mathbf{A}^0 \mathbf{L}}{f} \quad (f = \text{degrees of freedom of all groups united})$$

Finally we form the variance ratio

$$F = \frac{s_b^2}{s_\omega^2}$$

where $$s_b^2 = \frac{s^2 f - s_\omega^2 f_\omega}{f - f_1 - f_2 - f_3} \quad (f_\omega = f_1 + f_2 + f_3)$$

Here s_ω^2 is a pooled variance and we use the Bartlett test for verifying if pooling is justified.

The variance ratio F is compared with the theoretical variance ratio in Fisher's distribution.

Primary hypothesis: $\sigma_\omega^2 = \sigma_b^2$

Alternative hypothesis: $\sigma_\omega^2 < \sigma_b^2$

The primary hypothesis is rejected on a 5% risk level.

We can also use the hypothesis that the variance for the whole adjustment is equal to the unity.

Any deviation is tested with the Fisher test. These weight considerations were not fully introduced in this study.

Geoid computations

Primary data. Gravity data from free air anomalies (10° × 10°). Resolvent: Stokes. Reference surface: Ellipsoid.

Free air gravity anomalies (International Formula)

Terrestrial data with zero for unsurveyed areas

Longitude east Greenwich

5°	15°	25°	35°	45°	55°	65°	75°	85°	95°	105°	115°	125°	135°	145°	155°	165°	175°	Lat.
15	−2	14	0	1	8	−15	−10	−3	3	7	7	7	−2	7	7	7	7	+85°
−5	3	13	4	3	6	−3	−5	−2	−3	0	3	2	−10	0	0	0	0	+75°
19	6	−1	2	12	−2	−6	−15	2	−5	0	2	0	0	0	0	0	0	+65°
7	6	9	11	7	10	2	−6	0	−26	−8	5	4	2	−3	5	0	−6	+55°
13	23	31	8	1	−10	−15	−23	−14	0	0	1	7	5	0	−3	−2	−2	+45°
12	5	−12	12	22	8	0	0	11	0	0	−11	11	24	−2	−6	−6	−2	+35°
12	12	1	2	1	−19	6	−4	−16	−5	−4	0	1	0	1	0	−8	−8	+25°
6	3	2	9	0	−9	−5	−27	−43	−12	−10	−3	12	−1	−8	−6	−6	0	+15°
10	7	−10	−1	1	−2	−9	−11	−30	−10	8	16	20	25	20	30	2	0	+ 5°
5	−6	−9	5	−10	0	0	−13	−16	0	9	21	0	0	−11	3	35	−14	− 5°
15	10	6	7	7	7	0	0	0	0	0	−20	5	17	14	9	34	6	−15°
17	11	9	0	11	3	8	6	0	0	0	9	2	−5	7	15	27	15	−25°
2	4	6	−5	0	0	6	4	0	5	−3	−11	−9	−1	13	0	4	24	−35°
0	0	0	0	0	0	3	3	0	0	0	0	0	0	0	0	2	10	−45°
0	0	0	0	0	0	3	2	0	0	0	0	0	0	0	9	3	0	−55°
0	0	0	12	3	0	0	0	0	0	0	0	0	0	2	0	0	0	−65°
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	5	5	−75°
−14	−14	−14	−14	−14	−14	−14	−14	−14	−14	−9	−14	−14	−14	−14	−14	−14	−14	−85°

Unit: Milligal.

185°	195°	205°	215°	225°	235°	245°	255°	265°	275°	285°	295°	305°	315°	325°	335°	345°	355°	Lat.
−1	−2	5	18	16	10	8	5	12	14	4	−12	0	−2	0	−1	10	9	+85°
0	1	0	1	0	−6	6	0	0	−2	−10	16	1	13	17	3	9	8	+75°
1	7	16	35	9	−5	−8	−12	−5	0	6	−5	−9	0	15	7	7	8	+65°
−13	4	4	9	1	1	3	−1	−14	−29	−25	4	1	−2	9	−1	0	20	+55°
5	4	7	−10	−8	0	14	18	3	−7	−5	−7	−13	12	17	18	6	2	+45°
−7	−9	2	−9	−15	−16	−4	6	−3	0	−20	−18	−19	4	24	14	0	31	+35°
−6	15	8	−2	−10	−21	−20	5	−2	7	−19	−24	−24	8	−7	−10	18	2	+25°
2	5	17	−2	−2	−10	−9	−11	10	8	−8	−32	−31	−20	−16	8	19	7	+15°
−6	−1	0	0	0	0	0	−4	1	6	23	7	−14	−24	−4	2	1	9	+ 5°
−8	0	7	4	2	0	0	0	0	−7	11	4	−5	−22	−3	−9	0	−3	− 5°
2	5	4	0	0	0	0	0	0	−2	3	39	2	−9	−15	0	−3	12	−15°
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0	0	0	0	0	0	0	0	0	0	0	3	0	0	0	0	0	0	−45°
0	0	0	0	0	0	0	0	0	0	9	12	−3	0	0	0	0	0	−55°
0	0	0	0	0	0	0	0	0	3	0	12	2	0	0	0	0	0	−65°
0	−2	−4	−2	−8	−2	0	−8	0	6	0	−3	−6	−3	−5	0	0	0	−75°
−14	−14	−17	−13	−14	−15	−20	−18	−3	−12	−14	−19	−24	−22	−16	−14	−14	−14	−85°

Unit: Milligal.

Free air gravity anomalies (International Formula)

Terrestrial data with missing data from satellites

Longitude east Greenwich

5°	15°	25°	35°	45°	55°	65°	75°	85°	95°	105°	115°	125°	135°	145°	155°	165°	175°	Lat.
15	−2	14	0	1	8	−15	−10	−3	3	7	7	7	−2	7	7	7	7	+85°
−5	3	13	4	3	6	−3	−5	−2	−3	0	3	2	−10	7	10	18	17	+75°
19	6	−1	2	12	−2	−6	−15	2	−5	0	2	5	14	20	23	23	18	+65°
7	6	9	11	7	10	2	−6	0	−26	−8	5	4	2	−3	5	0	−6	+55°
13	23	31	8	1	−10	−15	−23	−14	0	0	1	7	5	0	−3	−2	−2	+45°
12	5	−12	12	22	8	0	0	11	0	0	−11	11	24	−2	−6	−6	−2	+35°
12	12	1	2	1	−19	6	−4	−16	−5	−4	0	1	0	1	0	−8	−8	+25°
6	3	2	9	0	−9	−5	−27	−43	−12	−10	−3	12	−1	−8	−6	−6	0	+15°
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5	−6	−9	5	−10	0	0	−13	−16	0	9	21	0	0	−11	3	35	−14	− 5°
15	10	6	7	7	7	−4	−15	−22	−12	−9	−20	5	17	14	9	34	6	−15°
17	11	9	0	11	3	8	6	−10	−16	−18	9	2	−5	7	15	27	15	−25°
2	4	6	−5	0	0	6	4	0	5	−3	−11	−9	−1	13	0	4	24	−35°
5	9	9	14	19	25	3	3	14	1	−12	−16	−15	−10	−4	3	2	10	−45°
10	11	16	20	25	27	3	2	16	8	0	−5	−5	−3	−3	9	3	−4	−55°
10	10	12	12	3	18	18	16	15	13	8	6	3	0	2	−7	−14	−17	−65°
10	12	13	12	13	17	18	17	17	15	11	8	3	1	−6	−8	5	5	−75°
−14	−14	−14	−14	−14	−14	−14	−14	−14	−14	−9	−14	−14	−14	−14	−14	−14	−14	−85°

Unit: Milligal.

185°	195°	205°	215°	225°	235°	245°	255°	265°	275°	285°	295°	305°	315°	325°	335°	345°	355°	Lat.
−1	−2	5	18	16	10	8	5	12	14	4	−12	0	−2	0	−1	10	9	+85°
13	1	0	1	0	−6	6	0	0	−2	−10	16	1	13	17	3	9	8	+75°
1	7	16	35	9	−5	−8	−12	−5	0	6	−5	−9	0	15	7	7	8	+65°
−13	4	4	9	1	1	3	−1	−14	−29	−25	4	1	−2	9	−1	0	20	+55°
5	4	7	−10	−8	0	14	18	3	−7	−5	−7	−13	12	17	18	6	2	+45°
−7	−9	2	−9	−15	−16	−4	6	−3	0	−20	−18	−19	4	24	14	0	31	+35°
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2	5	17	−2	−2	−10	−9	−11	10	8	−8	−32	−31	−20	−16	8	19	7	+15°
−6	−1	−3	−7	−10	−14	−14	−4	1	6	23	7	−14	−24	−4	2	1	9	+ 5°
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−18	1	1	0	−5	0	−2	−3	−3	−2	1	11	2	−7	0	15	9	6	−35°
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−20	−18	−16	−13	−10	0	5	9	9	3	0	12	2	19	19	13	12	12	−65°
0	−2	−4	−2	−8	−2	0	−8	0	6	0	−3	−6	−3	−5	2	8	8	−75°
−14	−14	−17	−13	−14	−15	−20	−18	−3	−12	−14	−19	−24	−22	−16	−14	−14	−14	−85°

Unit: Milligal.

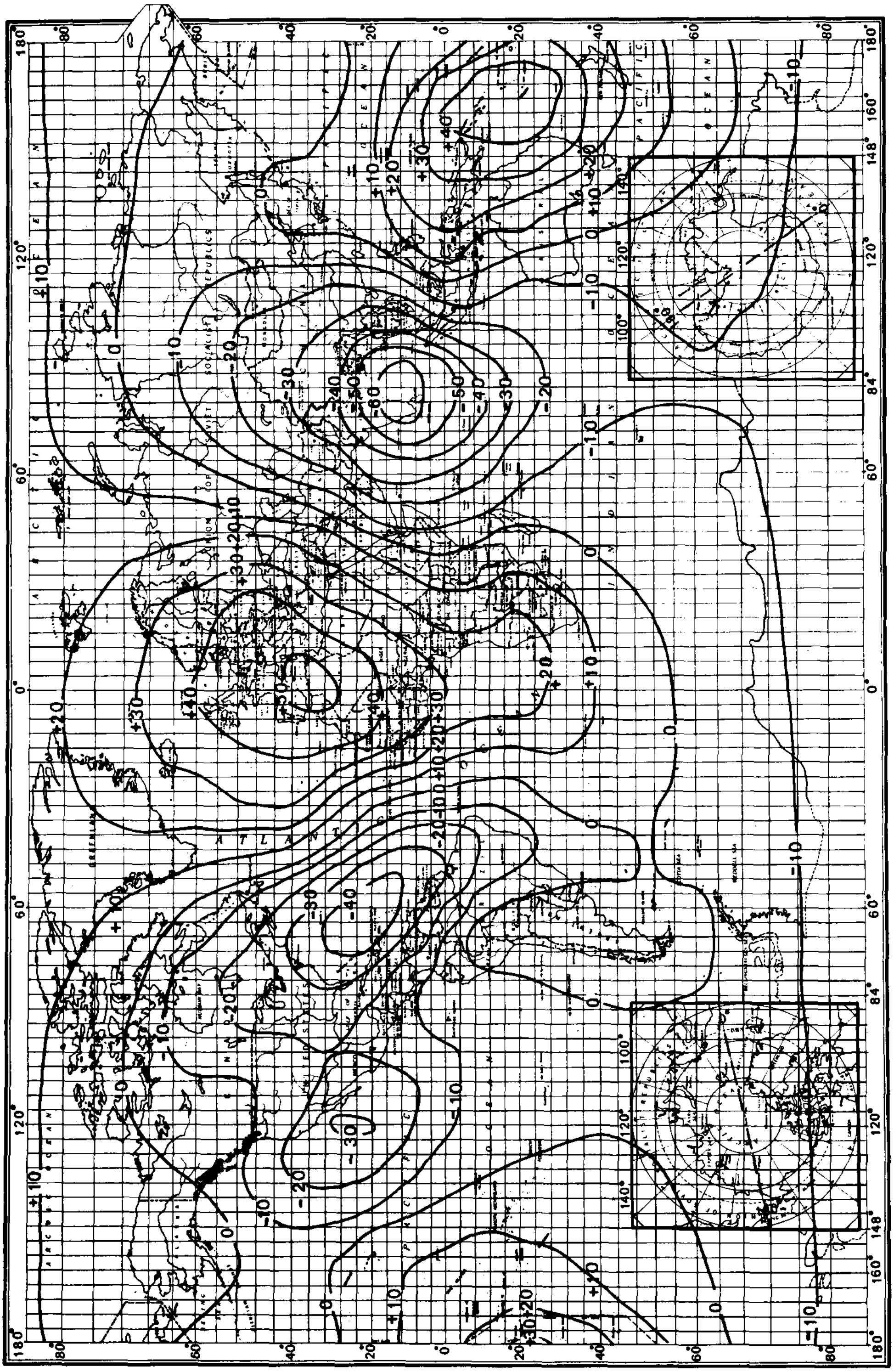

Fig. 1. Gravimetric geoid: Surface elements $10° \times 10°$ for gravity data. International reference ellipsoid $f = 1{:}287$. Zero anomaly for unsurveyed surface elements.

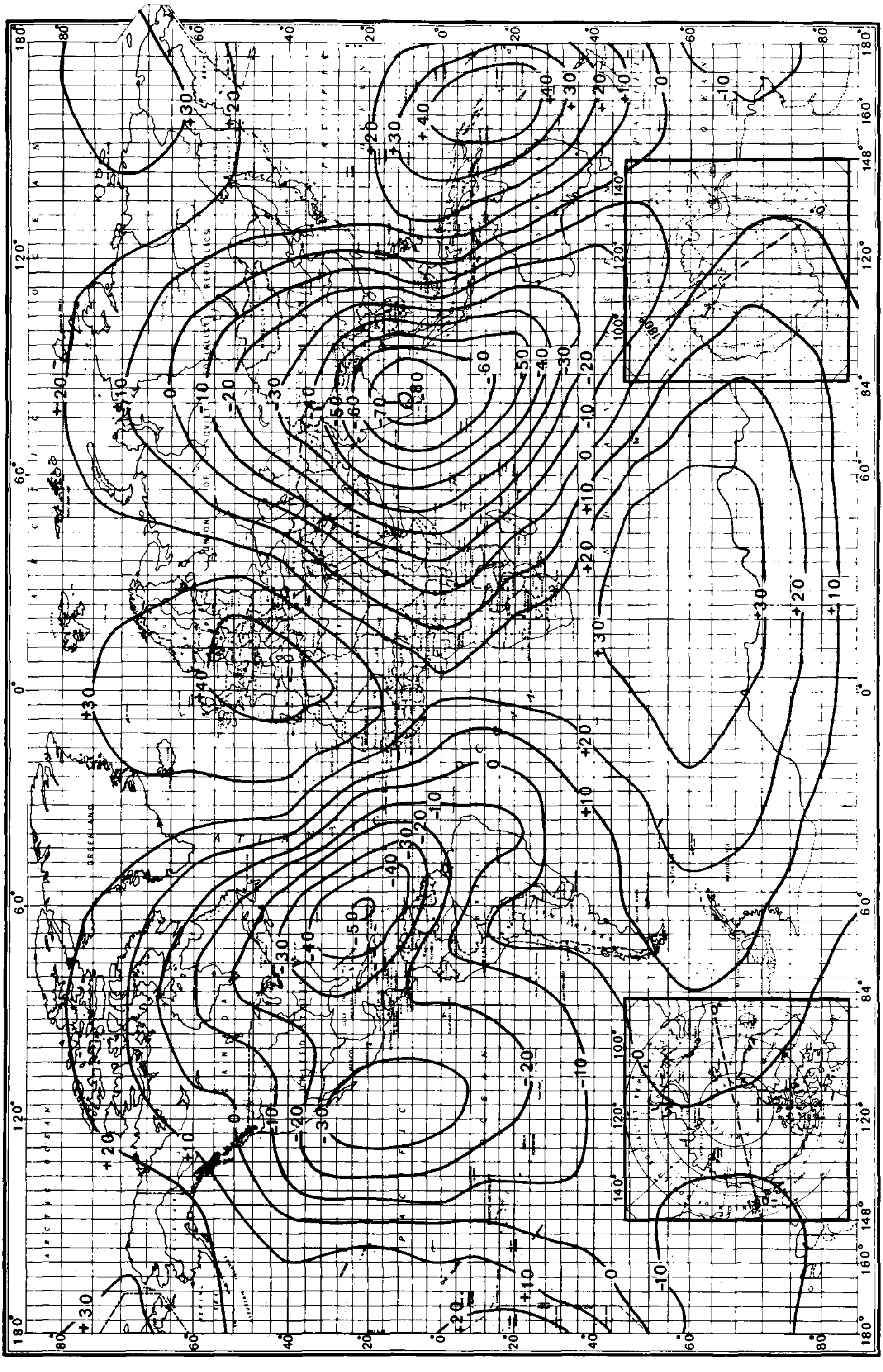

Fig. 2. Gravimetric geoid: Surface elements 10° × 10° for gravity data. International reference ellipsoid $f = 1:297$. Satellite data used for unsurveyed surface elements.

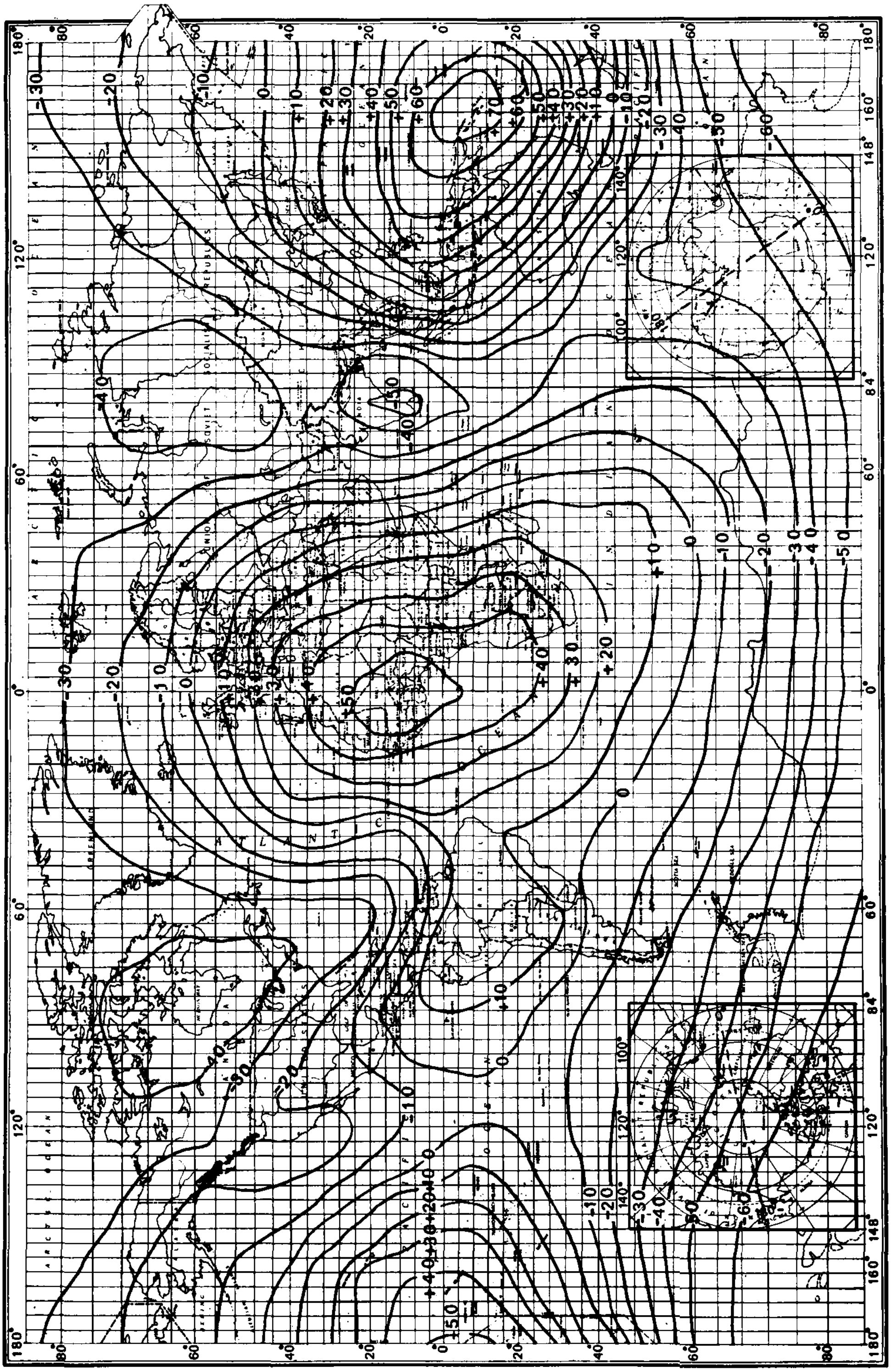

Fig. 3. Gravimetric geoid: Surface elements 10° × 10° for gravity data. Reference ellipsoid $f = 1:298.3$. Satellite data used for unsurveyed surface elements.

Pure gravimetric solution

Primary data. Gravity data from free air anomalies (10° × 10°). Observation equations are excluded for missing data. Solution with *spherical harmonics.* Reference sphere $a = 6378165$ m ("Geosphere"). $GM = 3.986032 \cdot 10^{20}$ cm^3 sec^{-2}. $\omega^2 = 5.317489 \cdot 10^{-9}$ sec^{-2}. Gravity reduction: Eq. (3:9). (Spherical gravity anomalies.)

Spherical harmonic solution

Pure gravimetric data

C(2,0) =	−4.88324093 − 004	S(2,0) =	0.00000000 + 000
C(3,0) =	1.00463230 − 006	S(3,0) =	0.00000000 + 000
C(4,0) =	4.77443604 − 007	S(4,0) =	0.00000000 + 000
C(5,0) =	−7.11436294 − 008	S(5,0) =	0.00000000 + 000
C(6,0) =	−1.18543266 − 007	S(6,0) =	0.00000000 + 000
C(7,0) =	2.32765416 − 007	S(7,0) =	0.00000000 + 000
C(8,0) =	−4.54567193 − 008	S(8,0) =	0.00000000 + 000
C(2,1) =	5.20177528 − 007	S(2,1) =	5.56063725 − 007
C(3,1) =	9.40371587 − 007	S(3,1) =	−4.47485680 − 007
C(4,1) =	−5.12375317 − 008	S(4,1) =	2.00755997 − 007
C(5,1) =	−3.50007070 − 007	S(5,1) =	−2.23334857 − 007
C(6,1) =	−2.81951745 − 009	S(6,1) =	−5.72711736 − 008
C(7,1) =	2.09822815 − 008	S(7,1) =	7.45098639 − 008
C(8,1) =	3.26181211 − 008	S(8,1) =	−3.59297526 − 008
C(2,2) =	2.88555272 − 006	S(2,2) =	−5.26252227 − 007
C(3,2) =	8.01944460 − 007	S(3,2) =	1.57239586 − 007
C(4,2) =	2.21339244 − 007	S(4,2) =	1.04405599 − 007
C(5,2) =	1.89119171 − 007	S(5,2) =	−2.50611172 − 009
C(6,2) =	5.01285612 − 008	S(6,2) =	−1.58996016 − 008
C(7,2) =	1.50638902 − 008	S(7,2) =	−8.18079699 − 009
C(8,2) =	1.27850908 − 007	S(8,2) =	1.06118534 − 007
C(3,3) =	9.63229912 − 007	S(3,3) =	1.11220623 − 006
C(4,3) =	5.28824452 − 007	S(4,3) =	−2.60926282 − 007
C(5,3) =	5.14848725 − 008	S(5,3) =	8.88935720 − 008
C(6,3) =	−1.96063381 − 008	S(6,3) =	−1.36212922 − 007
C(7,3) =	2.87472249 − 008	S(7,3) =	1.12480202 − 007
C(8,3) =	8.09088839 − 008	S(8,3) =	−9.23203016 − 008
C(4,4) =	−7.40460520 − 008	S(4,4) =	1.31701707 − 007
C(5,4) =	1.38661579 − 009	S(5,4) =	2.55595026 − 007
C(6,4) =	4.98197854 − 008	S(6,4) =	−4.25275016 − 007
C(7,4) =	−1.67320536 − 007	S(7,4) =	1.31714081 − 007
C(8,4) =	−1.58074661 − 007	S(8,4) =	1.03906572 − 008
C(5,5) =	3.43300879 − 007	S(5,5) =	−3.33415874 − 007
C(6,5) =	−2.07649626 − 007	S(6,5) =	−6.58648815 − 007
C(7,5) =	−1.52314230 − 007	S(7,5) =	6.71918245 − 008
C(8,5) =	−4.55755634 − 008	S(8,5) =	8.32693800 − 008
C(6,6) =	1.05201691 − 007	S(6,6) =	−3.40873421 − 007
C(7,6) =	−2.85595546 − 007	S(7,6) =	2.49422597 − 007
C(8,6) =	1.32951454 − 009	S(8,6) =	1.46033503 − 007
C(7,7) =	3.84525602 − 008	S(7,7) =	6.73253542 − 008
C(8,7) =	7.07786954 − 008	S(8,7) =	1.04233303 − 007
C(8,8) =	−1.48921228 − 007	S(8,8) =	1.12874547 − 008

All coefficients are fully normalized.

Reference surface: Sphere of radius 6,378,165 m.

Primary data: Free air gravity 10° × 10°.

Gravity reduction: Free air gravity anomalies are reduced to the *reference sphere* (the new "theory").

Extrapolation: None. (Observation equations for missing data are excluded.)

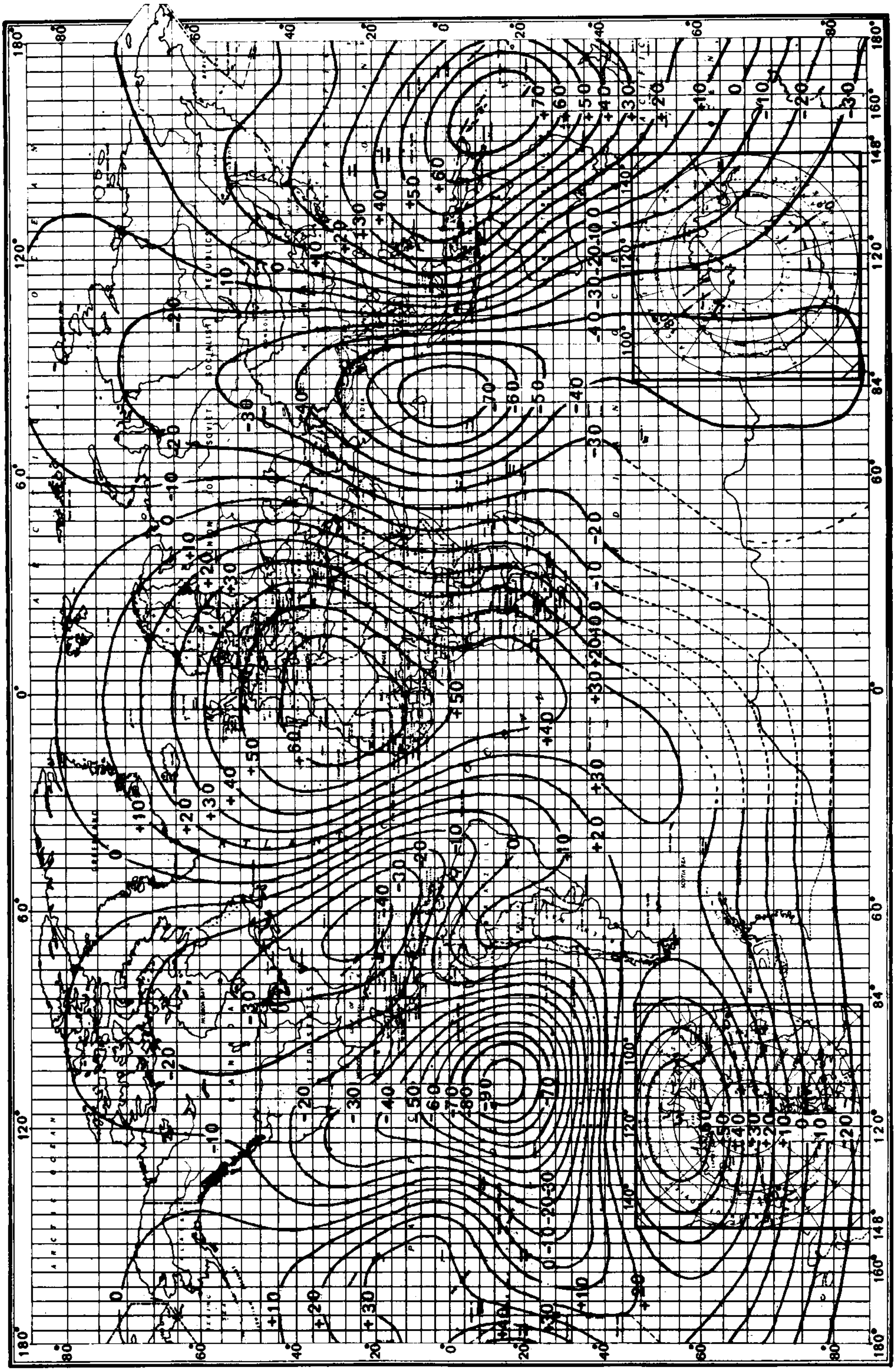

Fig. 4. Gravimetric geoid: Surface elements $10° \times 10°$. Preassigned value for $C_{20} = 484.17 \cdot 10^{-6}$. Reference ellipsoid $f = 1:298.3$. Major axis of the earth $= 6{,}378{,}165$ m.

Spherical harmonic solution with preassigned value

C(2,0) =	−4.84170000 − 004	S(2,0) =	0.00000000 + 000
C(3,0) =	−2.95239290 − 008	S(3,0) =	0.00000000 + 000
C(4,0) =	2.45026893 − 007	S(4,0) =	0.00000000 + 000
C(5,0) =	3.85721510 − 007	S(5,0) =	0.00000000 + 000
C(6,0) =	−4.40518500 − 007	S(6,0) =	0.00000000 + 000
C(7,0) =	2.40329891 − 007	S(7,0) =	0.00000000 + 000
C(8,0) =	−2.35018327 − 008	S(8,0) =	0.00000000 + 000
C(2,1) =	6.80689219 − 007	S(2,1) =	9.11376205 − 007
C(3,1) =	4.32248373 − 007	S(3,1) =	−1.17879840 − 006
C(4,1) =	2.42172568 − 007	S(4,1) =	7.94831649 − 007
C(5,1) =	−4.12028601 − 007	S(5,1) =	−3.11760249 − 007
C(6,1) =	−1.64750547 − 007	S(6,1) =	−2.45793108 − 007
C(7,1) =	1.49098387 − 007	S(7,1) =	2.85112585 − 007
C(8,1) =	−1.91594676 − 008	S(8,1) =	−1.23405233 − 007
C(2,2) =	4.14996031 − 006	S(2,2) =	−1.50376502 − 006
C(3,2) =	3.57615465 − 007	S(3,2) =	3.62597698 − 007
C(4,2) =	9.49497066 − 008	S(4,2) =	3.96219519 − 007
C(5,2) =	4.29608090 − 007	S(5,2) =	−4.22084312 − 007
C(6,2) =	1.58596158 − 008	S(6,2) =	1.38291845 − 007
C(7,2) =	−7.18317995 − 008	S(7,2) =	2.87663383 − 008
C(8,2) =	2.11759803 − 007	S(8,2) =	1.33832994 − 008
C(3,3) =	3.92375455 − 007	S(3,3) =	1.00638692 − 006
C(4,3) =	7.61938691 − 007	S(4,3) =	−1.32632670 − 007
C(5,3) =	2.76222611 − 007	S(5,3) =	−7.25153799 − 008
C(6,3) =	−3.92837811 − 007	S(6,3) =	5.61301684 − 008
C(7,3) =	2.72795028 − 007	S(7,3) =	−2.64000559 − 008
C(8,3) =	6.38538551 − 009	S(8,3) =	−1.87552442 − 008
C(4,4) =	−9.23869287 − 008	S(4,4) =	5.32526016 − 007
C(5,4) =	1.39833896 − 008	S(5,4) =	1.11605827 − 008
C(6,4) =	5.02291123 − 008	S(6,4) =	−4.65453721 − 007
C(7,4) =	−1.59249112 − 007	S(7,4) =	2.69598763 − 007
C(8,4) =	−1.63165587 − 007	S(8,4) =	−8.32026167 − 008
C(5,5) =	5.34908826 − 007	S(5,5) =	−4.91091405 − 007
C(6,5) =	−3.37776462 − 007	S(6,5) =	−5.24533830 − 007
C(7,5) =	−1.38338746 − 007	S(7,5) =	4.30097035 − 008
C(8,5) =	−1.78785302 − 008	S(8,5) =	5.95397229 − 008
C(6,6) =	1.51692073 − 008	S(6,6) =	−4.00348650 − 007
C(7,6) =	−2.39253287 − 007	S(7,6) =	2.72623118 − 007
C(8,6) =	−6.17707240 − 010	S(8,6) =	1.51321651 − 007
C(7,7) =	2.44155159 − 008	S(7,7) =	1.68772077 − 007
C(8,7) =	8.27130452 − 008	S(8,7) =	2.54162093 − 008
C(8,8) =	−1.19510006 − 007	S(8,8) =	−8.93924952 − 009

Data: As solution on page 542.

Preassigned value: $C_{20} = -484.17$ (from Kozai 1964).

Note: Mostly all remaining significant harmonics are impaired when using the preassigned value. The new coefficients differ considerably from corresponding satellite values.

Combined solution

Primary data. Gravity data from free air anomalies ($10° \times 10°$). Kozai's observation equations (1967). Anderle's observation equations (1965). Gaposchkin's observation equations (1966). Reference sphere: $a = 6{,}378{,}165$ m. $GM = 3.986032 \cdot 10^{20}$ cm^3 sec^{-2}. $\omega^2 = 5.317489 \cdot 10^{-9}$ sec^{-2}. Gravity reduction: Eq. (3:9). Spherical gravity anomalies.

The combined solution is obtained from reduced normal equations according to Eq. (5:13).

Final combined gravity satellite solution

C(2,0) =	−4.84170000 −004	S(2,0) =	0.00000000 +000
C(3,0) =	9.98147325 −007	S(3,0) =	0.00000000 +000
C(4,0) =	4.76088343 −007	S(4,0) =	0.00000000 +000
C(5,0) =	8.22751243 −009	S(5,0) =	0.00000000 +000
C(6,0) =	−1.22461212 −007	S(6,0) =	0.00000000 +000
C(7,0) =	1.34351113 −007	S(7,0) =	0.00000000 +000
C(8,0) =	−4.71438507 −008	S(8,0) =	0.00000000 +000
C(2,1) =	4.72725175 −009	S(2,1) =	−6.02535109 −008
C(3,1) =	1.84377356 −006	S(3,1) =	1.11256720 −007
C(4,1) =	−4.98508281 −007	S(4,1) =	−4.61333098 −007
C(5,1) =	−2.11068792 −007	S(5,1) =	−1.03059497 −007
C(6,1) =	−4.67464911 −008	S(6,1) =	1.42737436 −008
C(7,1) =	8.99214939 −008	S(7,1) =	5.85537332 −009
C(8,1) =	4.14119203 −008	S(8,1) =	4.66840421 −008
C(2,2) =	2.38938684 −006	S(2,2) =	−1.31588582 −006
C(3,2) =	6.85209403 −007	S(3,2) =	−4.95962938 −007
C(4,2) =	3.29268222 −007	S(4,2) =	6.72314792 −007
C(5,2) =	4.88654375 −007	S(5,2) =	−2.00630234 −007
C(6,2) =	2.15597414 −008	S(6,2) =	−2.71076302 −007
C(7,2) =	2.13993203 −007	S(7,2) =	1.69590318 −007
C(8,2) =	5.47848052 −008	S(8,2) =	−4.01761971 −009
C(3,3) =	5.77712670 −007	S(3,3) =	1.59815473 −006
C(4,3) =	8.30154588 −007	S(4,3) =	−2.14581614 −007
C(5,3) =	−3.52772865 −007	S(5,3) =	4.37533679 −008
C(6,3) =	−1.41855257 −007	S(6,3) =	−3.82780721 −008
C(7,3) =	2.14981984 −007	S(7,3) =	9.61243804 −008
C(8,3) =	−6.11461078 −008	S(8,3) =	−1.02052240 −007
C(4,4) =	−3.29561300 −008	S(4,4) =	3.64854409 −007
C(5,4) =	−2.86460260 −007	S(5,4) =	6.46043051 −008
C(6,4) =	−1.26638754 −008	S(6,4) =	−4.00616071 −007
C(7,4) =	−1.20963931 −007	S(7,4) =	−4.16210766 −009
C(8,4) =	−1.82952801 −007	S(8,4) =	4.83002849 −008
C(5,5) =	1.91347325 −007	S(5,5) =	−5.29530130 −007
C(6,5) =	−2.44632680 −007	S(6,5) =	−4.35793847 −007
C(7,5) =	−6.14528931 −008	S(7,5) =	−6.16349371 −009
C(8,5) =	−6.65180650 −008	S(8,5) =	8.27333442 −008
C(6,6) =	−2.88951010 −008	S(6,6) =	−3.39540172 −007
C(7,6) =	−1.85519298 −007	S(7,6) =	1.57600488 −007
C(8,6) =	−4.24522228 −008	S(8,6) =	1.61211998 −007
C(7,7) =	9.20110722 −008	S(7,7) =	8.13543305 −008
C(8,7) =	2.06309144 −008	S(8,7) =	5.84535059 −008
C(8,8) =	−1.30971257 −007	S(8,8) =	−9.44830661 −009

This study is a simultaeous solution of all available gravity and satellite data.

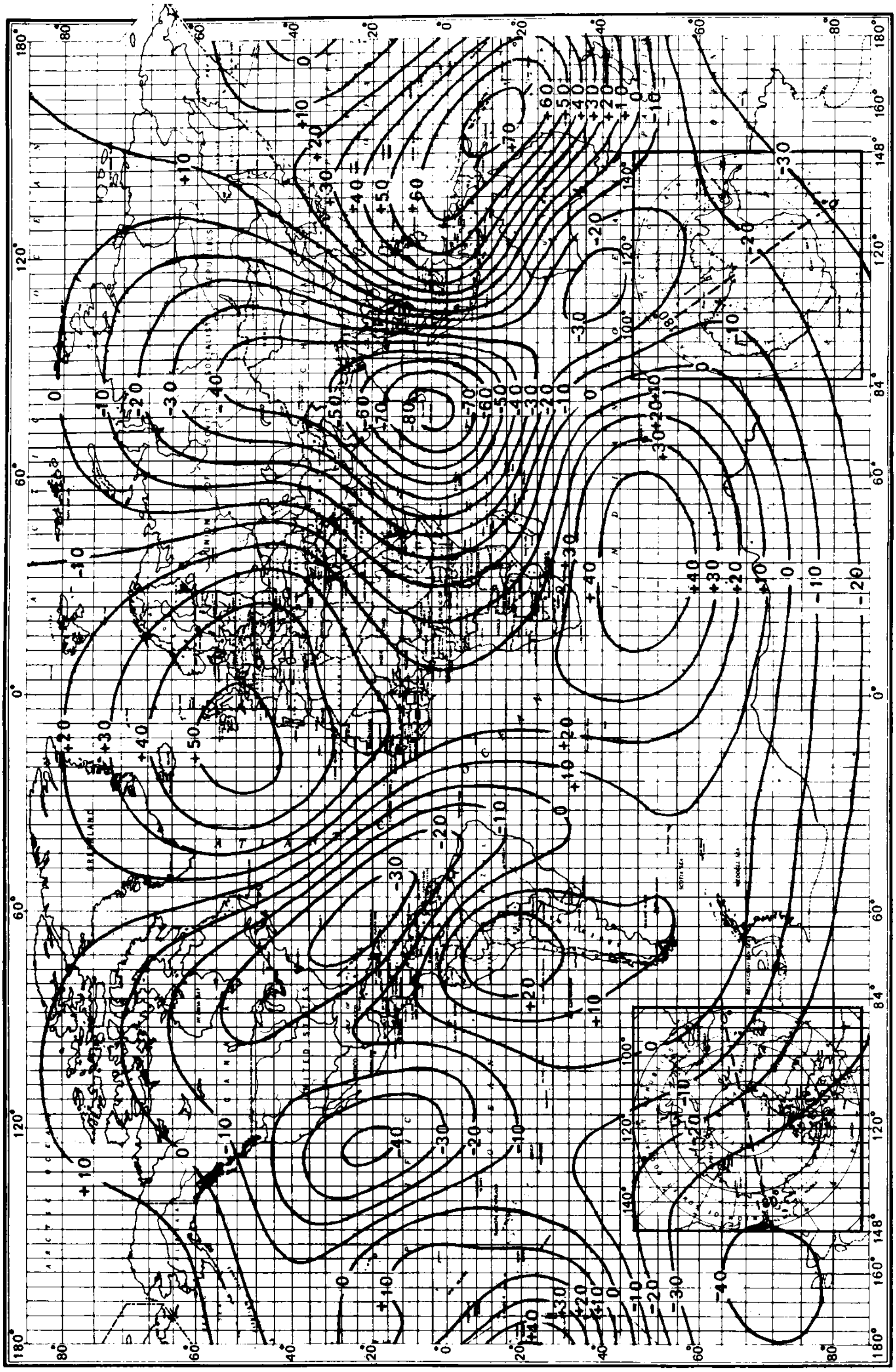

Fig. 5. Satellite—gravimetric geoid: Gravimetric surface elements 10° × 10°. Reference ellipsoid $f = 1:298.3$. Major axis of the earth = 6,378,165 m.

Comparison satellite and terrestrial gravity

							Bjerhammar	
$\bar{C}\bar{S}$	Kozai (1964)	Kinghele (1965)	Guier & Newton (1966)	Anderle (1965)	Gaposchkin (1966)	Kaula (1966)	Pure gravity (1967)	Satellites & gravity (1967)
C20	−484.17	−484.17		−484.20			−488.38	−484.17
C30	0.96	0.97	1.02	0.98			1.01	1.00
C40	0.55	0.51		0.51			0.47	0.47
C50	0.06	0.05	0.00	0.04			−0.06	0.01
C60	−0.18	−0.16		−0.22			−0.12	−0.12
C70	0.09	0.11	0.16	0.11			0.24	0.13
C80	0.07	−0.11		—			−0.05	−0.05
C22			2.38	2.45	2.38	2.42	2.90	2.39
S22			−1.20	−1.52	−1.35	−1.39	−0.51	−1.32
C31			1.84	2.15	1.94	1.90	0.94	1.84
S31			0.21	0.27	0.27	0.11	−0.45	0.11
C32			1.22	0.98	0.73	0.69	0.79	0.69
S32			−0.68	−0.91	−0.54	−0.78	0.15	−0.50
C33			0.66	0.58	0.56	0.55	0.96	0.58
S33			0.98	1.62	1.62	1.29	1.11	1.60
C41			−0.56	−0.49	−0.57	−0.59	−0.05	−0.50
S41			−0.44	−0.57	−0.47	−0.48	0.20	−0.46
C42			0.42	0.27	0.33	0.28	0.23	0.33
S42			0.44	0.67	0.66	0.69	0.11	0.67
C43			0.84	1.03	0.85	0.89	0.53	0.83
S43			0.00	−0.25	−0.19	0.19	−0.25	−0.21
C44			−0.21	−0.41	−0.05	−0.32	−0.08	−0.03
S44			0.19	0.34	0.23	0.00	0.14	0.36
C51			0.14	0.03	−0.08	−0.01	−0.35	−0.21
S51			−0.17	−0.12	−0.10	0.02	−0.23	−0.10
C52			0.27	0.64	0.63	0.68	0.18	0.49
S52			−0.34	−0.33	−0.23	−0.25	0.00	−0.20
C53			0.09	−0.39	−0.52	−0.67	0.05	−0.35
S53			0.10	−0.12	0.01	0.12	0.08	0.04
C54			−0.49	−0.55	−0.26	0.08	0.00	−0.29
S54			−0.26	0.15	0.06	0.37	0.25	0.06
C55			−0.03	0.21	0.16	−0.45	0.34	0.19
S55			−0.67	−0.59	−0.59	−0.21	−0.33	−0.53
C61			0.	−0.08	−0.05	−0.19	0.00	−0.05
S61			0.10	0.19	−0.03	0.13	−0.05	0.01
C62			−0.16	0.13	0.07	0.08	0.06	0.02
S62			−0.16	−0.46	−0.37	−0.41	−0.02	−0.27
C63			0.53	−0.02	−0.05	0.10	−0.02	−0.14
S63			0.05	−0.13	0.03	0.46	−0.13	−0.04
C64			−0.31	−0.19	−0.04	0.08	0.05	−0.01
S64			−0.51	−0.32	−0.52	−0.43	−0.43	−0.40
C65			−0.18	−0.09	−0.31	−0.04	−0.21	−0.24
S65			−0.50	−0.79	−0.46	−0.38	−0.66	−0.44
C66			0.01	−0.32	−0.04	0.15	0.10	−0.03
S66			−0.23	−0.36	−0.16	−0.15	−0.34	−0.34
C71			0.13	0.33	0.20	0.06	0.02	0.09
S71			0.09	0.08	0.16	0.06	0.07	0.01
C72			0.46	0.35	0.36	0.31	0.01	0.21
S72			0.06	−0.19	0.16	0.26	−0.01	0.17
C73			0.39	0.32	0.25	−0.03	0.03	0.21
S73			−0.21	0.04	0.02	−0.32	0.11	0.10
C74			−0.14	−0.47	−0.15	−0.41	−0.17	−0.12
S74			0.00	−0.24	−0.10	0.15	0.13	0.00
C75			−0.06	0.05	0.08	0.22	−0.15	−0.06
S75			−0.19	0.02	0.05	−0.31	0.07	−0.01

Table (*cont.*)

							Bjerhammar	
$\bar{C}\bar{S}$	Kozai (1964)	Kinghele (1965)	Guier & Newton (1966)	Anderle (1965)	Gaposchkin (1966)	Kaula (1966)	Pure gravity (1967)	Satellites & gravity (1967)
C76			−0.45	−0.48	−0.21		−0.29	−0.19
S76			−0.75	−0.24	0.06		0.25	0.16
C77			0.09		0.06		0.04	0.09
S77			−0.14		0.10		0.07	0.08
C81			−0.15		−0.08	−0.06	0.03	0.04
S81			−0.05		0.07	0.06	−0.04	0.05
C82			0.09		0.03	0.08	0.13	0.05
S82			−0.04		0.04	−0.07	0.11	−0.00
C83			−0.05		−0.04	0.08	0.08	−0.06
S83			0.22		0.00	0.22	−0.09	−0.10
C84			−0.07		−0.21	0.08	−0.16	−0.18
S84			−0.04		−0.01	0.04	0.01	0.05
C85			0.08		−0.05	0.03	−0.05	−0.07
S85			0.00		0.12	−0.34	0.08	0.08
C86			−0.02		−0.02	0.10	0.00	−0.04
S86			0.67		0.32	0.12	0.15	0.16
C87			0.17		−0.01		0.07	0.02
S87			−0.07		0.03		0.10	0.06
C88			−0.15		−0.25		−0.15	−0.13
S88			0.09		0.10		0.01	−0.01

Note. The final complete solution is found on page 65 for gravity combined with all available satellite data in a joint solution. Unit: 10^{-6}.

Comments

The present study using pure gravity and a gravity reduction by spherical harmonics gives rather interesting results for a comparison with satellite solutions. Perhaps it is surprising to find that C_{20} from terrestrial data still gives a value closer to the international ellipsoid. For the remaining larger data we note for the gravimetric solution

Zonal harmonics

C_{30} gives good agreement with satellite data

C_{40} gives good agreement with satellite data

C_{50} small uncertain value

C_{60}, C_{70} fair agreement

C_{80} between Kozai's and King-Hele's values

Sectorial harmonics

S_{33}, C_{44}, S_{44}, S_{55}, C_{66}, C_{77} and C_{88} are all inside the limits defined by the satellite solutions.

Tesseral harmonics

Agreement between different satellite solutions is poor and a comparison is hard to make.

Finally, it should be noted that a fair comparison can only be made with solutions that are making use of the same truncation. Furthermore, we find that most of the satellite solutions are made with preassigned values of the zonal harmonics. A direct comparison can therefore be questioned.

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ИССЛЕДОВАНИЕ ОБЪЕДИНЁННОЙ МИРОВОЙ ГЕОДЕЗИЧЕСКОЙ СИСТЕМЫ

Изучено гравитационное поле Земли при использовании совместного решения для аномалий свободной атмосферы и спутниковых данных. В совместном решении вычислены сферические гармоники вплоть до порядка 8.8. В этом решении были использованы спутниковые данные систем допплеровского слежения (NWL) и оптических наблюдений (SAO) с подробным рассмотрением полной корреляционной матрицы. Эта работа основывается на наиболее полном наборе гравитационных данных, до сих пор имеющихся (около 500 000).