# The damping of gravity waves in shallow water by energy dissipation in a turbulent boundary layer

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ABSTRACT

The method of matched asymptotic expansions is applied to the determination of the damping of gravity waves propagating in turbulent conditions. The effect of the turbulence is introduced by a general system of coefficients of eddy viscosity, whilst the turbulence itself is supposed to be confined to boundary layers adjacent to a rigid impermeable bottom and the free surface. The lowest order damping in the system is found to be independent of surface turbulence and computations are made for a physically meaningful distribution of eddy viscosity in the lower boundary layer.

## Introduction

A boundary layer method has recently been given by Johns (1967) for the determination of viscous wave damping in highly complex situations. The method depends upon the use of matched asymptotic expansions as a means of solving the governing equations when the viscosity of the fluid is small.

In the present paper, the method is suitably adapted with a view to an application to the determination of the spatial attenuation of small amplitude gravity waves propagating in turbulent conditions. The effect of the turbulence is introduced in a general manner by three independent coefficients of eddy viscosity. Initially, the only physical assumption made about the coefficients is that they be independent of time and the horizontal spatial coordinate.

In the development of the theory, the turbulence is assumed to be confined to boundary layers adjacent to an impermeable rigid bottom and the free surface. Suitable perturbation series are proposed for the solutions in terms of a reference eddy viscosity and the spatial damping is calculated in terms of a general distribution of eddy viscosity. The lowest order damping in the system is found to be independent of surface turbulence in all physically realistic situations and results solely from energy dissipation in the lower boundary layer. This is evaluated by prescribing a suitable variation for the eddy viscosity in the lower boundary layer which is consistent with that used by Johns (1966) in a tidal flow problem.

## Formulation

All spatial conditions are referred to rectangular Cartesian axes $(x, y)$ fixed in the undisturbed free surface of an incompressible homogeneous fluid of constant depth with the $y$-axis directed vertically upwards. The equation of the oscillating surface is $y = \eta$ whilst the impermeable rigid bottom is given by $y = -h$.

Assuming small amplitude two-dimensional gravity waves at the free surface, all governing equations are linearised and boundary conditions applied at the mean position of oscillating levels. Appropriately averaged velocity components, denoted by $(u, v)$, and the pressure $p$ are therefore solutions of

$$\frac{\partial u}{\partial t} = -\frac{1}{\varrho}\frac{\partial p}{\partial x} - \frac{1}{\varrho}\left(\frac{\partial \tau_{xn}}{\partial x} + \frac{\partial \tau_{yn}}{\partial y}\right), \qquad (2.1)$$

$$\frac{\partial v}{\partial t} = -g - \frac{1}{\varrho}\frac{\partial p}{\partial y} - \frac{1}{\varrho}\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}\right), \qquad (2.2)$$

where $\varrho$ denotes the fluid density and the $\tau$'s the components of stress resulting from internal friction and the turbulent transfer of momentum. Although the origin of the Reynolds stresses lies within the neglected nonlinear inertia terms, the gradients thereof are, in general, expected to be significant and are accordingly retained in the present formulation.

The equation of continuity of mass for an incompressible fluid yields

$$\frac{\partial u}{dx} + \frac{\partial v}{\partial y} = 0. \tag{2.3}$$

At this juncture, appropriate expressions must be proposed for the Reynolds stresses in order to relate them to the other averaged properties of the flow. We therefore introduce coefficients of eddy viscosity by writing

$$\tau_{yx} = \tau_{xy} = -\varrho N_{xy}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tag{2.4}$$

$$\tau_{xx} = -2\varrho N_{xx}\frac{\partial u}{\partial x}; \quad \tau_{yy} = -2\varrho N_{yy}\frac{\partial v}{\partial y}. \tag{2.5}$$

Until assumptions be made about the coefficients of eddy viscosity (which also include the molecular viscosity), the above scheme does not involve the physics of the system.

Upon use of (2.3), a streamfunction $\psi$ is defined by writing

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \tag{2.6}$$

and the pressure distribution eliminated between (2.1) and (2.2). Introducing (2.4) and (2.5) into the resultant, the equation for $\psi$ is readily found to be

$$\frac{\partial}{\partial t}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) = \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\left\{N_{xy}\left(\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2}\right)\right\} + 2\frac{\partial^2}{\partial x \partial y}\left\{(N_{xx} + N_{yy})\frac{\partial^2\psi}{\partial x \partial y}\right\}. \tag{2.7}$$

The foregoing formulation is quite general, but we now suppose that the eddy viscosity is independent of time and the horizontal spatial coordinate and write

$$\left.\begin{aligned} N_{xy} &= \nu F(y), \\ N_{xx} + N_{yy} &= 2\nu G(y), \end{aligned}\right\} \tag{2.8}$$

where $\nu$ is a reference viscosity and $F(y)$ and $G(y)$ describe the vertical structure of the turbulence. Such an assumption has been usefully employed by Johns (1966) in a tidal problem.

The boundary conditions are that both components of velocity vanish at the impermeable bottom:

$$\psi = \frac{\partial \psi}{\partial y} = 0 \quad \text{at} \quad y = -h. \tag{2.9}$$

At the free surface, there is no applied tangential stress:

$$\tau_{xy} = 0 \quad \text{at} \quad y = 0, \tag{2.10}$$

whilst, in the absence of surface tension, the normal stress must equal the atmospheric pressure (taken as zero):

$$p - g\varrho\eta + \tau_{yy} = 0 \quad \text{at} \quad y = 0. \tag{2.11}$$

Finally, the kinematical condition at the free surface yields

$$\frac{\partial \eta}{\partial t} = -\frac{\partial \psi}{\partial x} \quad \text{at} \quad y = 0. \tag{2.12}$$

## Solution of equations

The present method of solution depends upon the existence of turbulent boundary layers adjacent to the rigid bottom and the free surface whilst the interior of the flow is free of turbulence.

Prescribing the oscillatory surface in the form

$$\eta = a e^{i(kx - \sigma t)}, \tag{3.1}$$

the flow exterior to the boundary layers is derived from a streamfunction $\Psi$. In the case of a spatially attenuated wave, we write

$$\psi = \Psi = \frac{a\sigma h}{\sinh k_0 h}\{\Psi_0(Y) + \varepsilon\Psi_1(Y) + \dots\}e^{i(kx - \sigma t)}, \tag{3.2}$$

where

$$\left.\begin{aligned} yh &= Y, \\ \Psi_0'(-1) &= 1 \end{aligned}\right\} \tag{3.3}$$

and

$$\varepsilon^2 = \frac{2\nu}{h^2\sigma} < 1. \tag{3.4}$$

The wave number $k$ is developed in the form

$$k = k_0 + \varepsilon k_1 + \dots, \tag{3.5}$$

whilst the angular frequency $\sigma$ is prescribed. The quantity $k_1$, therefore, will determine the spatial attenuation of the wave by energy dissipation in the system.

Throughout, only real parts of complex quantities have significance.

From (2.7) and (2.8), the basic equation for $\psi$ is

$$\frac{\partial}{\partial t}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)=\frac{h^2\sigma}{2}\varepsilon^2\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)\times\left\{F(y)\left(\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial y^2}\right)\right\}+2h^2\sigma\varepsilon^2\frac{\partial^2}{\partial x\partial y}\left\{G(y)\frac{\partial^2\psi}{\partial x\partial y}\right\}, \quad (3.6)$$

having substituted for $v$ from (3.4). Upon introduction of (3.2) and (3.5) into (3.6) and identification of terms $o(1)$ and $o(\varepsilon)$ we readily find that

$$\frac{d^2\Psi_0}{dY^2}-(k_0h)^2\Psi_0=0, \quad (3.7)$$

$$\frac{d^2\Psi_1}{dY^2}-(k_0h)^2\Psi_1=2k_0k_1h^2\Psi_0. \quad (3.8)$$

Within the boundary layer adjacent to $y=-h$, it is well known (in the non-dimensional sense) that

$$\psi=o(\varepsilon)$$

and we write

$$\psi=\frac{a\sigma h}{\sinh k_0h}\varepsilon\psi_0(s)e^{i(kx-\sigma t)}+o(\varepsilon^2), \quad (3.9)$$

where $$y+h=\varepsilon hs. \quad (3.10)$$

For a rapidly changing intensity of turbulence, the derivatives of $F(y)$ and $G(y)$ will be $o(1/\varepsilon)$ within the boundary layers and so, on introduction of (3.9) and (3.10) into (3.6) and identification of the lowest order terms,

$$\frac{d^2}{ds^2}\{F\psi_0''(s)\}+2i\psi_0''(s)=0, \quad (3.11)$$

the prime denoting a differentiation with respect to $s$. Within the surface boundary layer,

$$\psi=o(1)$$

and we write

$$\psi=\frac{a\sigma h}{\sinh k_0h}\{\Psi_0(r)+\varepsilon\Psi_1(r)+\varepsilon^2\Psi_2(r)+\dots\}e^{i(kx-\sigma t)}, \quad (3.12)$$

where $$y=-\varepsilon hr. \quad (3.13)$$

Upon introduction of (3.12) and (3.13) into (3.6) and identification of terms of like order, it is readily found that

$$\frac{d^2}{dr^2}\{F\Psi_n''(r)\}+2i\Psi_n''(r)=0, \quad (n=0,1), \quad (3.14)$$

$$\frac{d^2}{dr^2}\{F\Psi_2''(r)\}+2i\Psi_2''(r)=2i(k_0h)^2\Psi_0(r)-(k_0h)^2F\Psi_0''(r)-(k_0h)^2\frac{d^2}{dr^2}\{F\Psi_0(r)\}+4(k_0h)^2\frac{d}{dr}\{G\Psi_0'(r)\}, \quad (3.15)$$

the prime denoting a differentiation with respect to $r$.

The solution of (3.7) satisfying (3.3) is

$$\Psi_0=\Psi_0(-1)\cosh k_0h(1+Y)+\frac{1}{k_0h}\sinh k_0h(1+Y). \quad (3.16)$$

By virtue of (2.9), the solution of (3.11) must satisfy

$$\psi_0(0)=\psi_0'(0)=0, \quad (3.17)$$

and, characterising the outer limit of the boundary layer by $s=s_{00}$, the vorticity and its derivative must decay outwards from the impermeable surface:

$$\psi_0''(s_{00})=\psi_0'''(s_{00})=0. \quad (3.18)$$

A single integration of (3.11) from $s$ to $s_{00}$ therefore yields

$$\frac{d}{ds}\{F\psi_0''(s)\}+2i\psi_0'(s)=2i\psi_0'(s_{00}), \quad (3.19)$$

whilst a further integration from $o$ to $s_{00}$ gives

$$\psi_0(s_{00})=s_{00}\psi_0'(s_{00})-\frac{i}{2}F_{s=0}\psi_0''(0). \quad (3.20)$$

The boundary conditions to accompany (3.14) and (3.15) result from the surface requirements (2.10) and (2.12). The first of these leads to

$$\Psi_n''(0) = 0, \quad (n = 0, 1), \tag{3.21}$$

$$\Psi_2''(0) = -(k_0 h)^2 \Psi_0(0), \tag{3.22}$$

whilst the second yields

$$\Psi_0(0) = \frac{1}{k_0 h} \sinh k_0 h, \tag{3.23}$$

$$\Psi_1(0) = -\frac{k_1}{k_0} \Psi_0(0). \tag{3.24}$$

Characterising the outer limit of the boundary layer by $r = r_{00}$, the appropriate vorticity requirements demand that

$$\Psi_n''(r_{00}) = \Psi_n'''(r_{00}) = 0 \quad (n = 0, 1), \tag{3.25}$$

$$\Psi_2''(r_{00}) = (k_0 h)^2 \Psi_0(r_{00}); \quad \Psi_2'''(r_{00}) = (k_0 h)^2 \Psi_0'(r_{00}). \tag{3.26}$$

Integrating (3.14) from $r$ to $r_{00}$ and using (3.25),

$$\frac{d}{dr}\{F\Psi_n''(r)\} + 2i\Psi_n'(r) = 2i\Psi_n'(r_{00}), \tag{3.27}$$

whilst a second integration from $r$ to $r_{00}$ gives

$$F\Psi_n''(r) + 2i\Psi_n(r) = 2i\Psi_n(r_{00}) + 2i\Psi_n'(r_{00})(r - r_{00}). \tag{3.28}$$

The solution of (3.28) may therefore be written in the form

$$\Psi_n = \Psi_n(r_{00}) + \Psi_n'(r_{00})(r - r_{00}) + f_n(r), \tag{3.29}$$

where $$Ff_n''(r) + 2if_n(r) = 0. \tag{3.30}$$

Integrating (3.15) from $o$ to $r_{00}$ and using (3.26),

$$\begin{aligned} F_{r=0}\Psi_2'''(0) + 2i\Psi_2'(0) = &-\left(\frac{dF}{dr}\right)_{r=0} \Psi_2''(0) \\ &+ 2i\Psi_2'(r_{00}) - 2i(k_0 h)^2 \int_0^{r_{00}} \Psi_2(r)\,dr \\ &+ (k_0 h)^2 \int_0^{r_{00}} F\Psi_0''(r)\,dr + (k_0 h)^2 \left[\frac{d}{dr}\{F\Psi_0(r)\}\right]_0^{r_{00}} \\ &- 4(k_0 h)^2 [G\Psi_0'(r)]_0^{r_{00}}. \end{aligned} \tag{3.31}$$

## Matching of the outer and boundary layer flows

The matching technique to be used here has been fully described by Johns (1967). In the present treatment, the matching conditions are

$$(\Psi)_{Y=-1+\varepsilon s_{00}} = (\psi)_{s=s_{00}}; \quad \left(\frac{\partial \Psi}{\partial Y}\right)_{Y=-1+\varepsilon s_{00}} = \frac{1}{\varepsilon}\left(\frac{\partial \psi}{\partial s}\right)_{s=s_{00}}, \tag{4.1}$$

$$(\Psi)_{Y=-\varepsilon r_{00}} = (\psi)_{r=r_{00}}; \quad \left(\frac{\partial \Psi}{\partial Y}\right)_{Y=-\varepsilon r_{00}} = -\frac{1}{\varepsilon}\left(\frac{\partial \psi}{\partial r}\right)_{r=r_{00}}. \tag{4.2}$$

The expansion of these conditions yields the requirements

$$\Psi_0(-1) = 0; \quad \psi_0'(s_{00}) = \Psi_0'(-1), \tag{4.3}$$

$$\Psi_1(-1) + s_{00}\Psi_0'(-1) = \psi_0(s_{00}), \tag{4.4}$$

$$\Psi_0(0) = \Psi_0(r_{00}); \quad \Psi_0'(r_{00}) = 0, \tag{4.5}$$

$$\Psi_1(0) - r_{00}\Psi_0'(0) = \Psi_1(r_{00}), \tag{4.6}$$

$$\Psi_1'(r_{00}) = -\Psi_0'(0), \tag{4.7}$$

$$\Psi_2'(r_{00}) = -\Psi_1'(0) + r_{00}\Psi_0''(0). \tag{4.8}$$

Application of these conditions will determine the unknown factors in the solutions of sec. (3).

Upon use of (4.3) in (3.16) and (3.20),

$$\Psi_0 = \frac{1}{k_0 h} \sinh k_0 h(1 + Y) \tag{4.9}$$

and $$\psi_0(s_{00}) = s_{00} - \frac{i}{2} F_{s=0}\psi_0''(0). \tag{4.10}$$

Putting $n = 0$ in (3.29) and using (3.23) and (4.5),

$$\Psi_0 = \frac{1}{k_0 h} \sinh k_0 h + f_0(r), \tag{4.11}$$

where $$f_0(0) = f_0(r_{00}) = f_0'(r_{00}) = 0.$$

For physically acceptable functions $F$, it is shown in the appendix that the only solution for $f_0$ is

$$f_0 = 0, \tag{4.12}$$

and so $$\Psi_0 = \frac{1}{k_0 h} \sinh k_0 h. \tag{4.13}$$

Putting $n=1$ in (3.29) using (3.24) and (4.7),

$$\Psi_1 = -\frac{k_1}{k_0^2 h}\sinh k_0 h - r\cosh k_0 h + f_1(r), \quad (4.14)$$

where $\quad f_1(0) = f_1(r_{00}) = f_1'(r_{00}) = 0.$

Again, the only solution for $f_1$ is

$$f_1 = 0, \quad (4.15)$$

and so

$$\Psi_1 = -\frac{k_1}{k_0^2 h}\sinh k_0 h - r\cosh k_0 h. \quad (4.16)$$

Upon introduction of (3.22), (4.8), (4.13) and (4.16) into (3.31) we therefore obtain

$$F_{r=0}\Psi_2'''(0) + 2i\Psi_2'(0) = -2i\Upsilon_1'(0) + k_0 h \sinh k_0 h \left(\frac{dF}{dr}\right)_{r=r_{00}}. \quad (4.17)$$

Upon use of (4.4), (4.6), (4.9), (4.10) and (4.16) it is readily seen that the solution of (3.8) must satisfy

$$\Upsilon_1(-1) = -\frac{i}{2}F_{s=0}\psi_0''(0), \quad (4.18)$$

and $$\Upsilon_1(0) = -\frac{k_1}{k_0^2 h}\sinh k_0 h. \quad (4.19)$$

It follows, therefore, that

$$\Upsilon_1 = \left\{\frac{i}{2}F_{s=0}\psi_0''(0) - \frac{k_1}{k_0}\right\}\operatorname{cosch} k_0 h \sinh k_0 h Y - \frac{k_1}{k_0^2 h}\sinh k_0 h(1+Y) + \frac{k_1}{k_0} Y\cosh k_0 h(1+Y), \quad (4.20)$$

use of which in (4.17) yields

$$F_{r=0}\Psi_2'''(0) + 2i\Psi_2'(0) = k_0 h\left\{F_{s=0}\psi_0''(0) + 2i\frac{k_1}{k_0}\right\}\operatorname{cosch} k_0 h + k_0 h \sinh k_0 h \left(\frac{dF}{dr}\right)_{r=r_{00}}. \quad (4.21)$$

## Determination of the damping

The relation which determines the wave number and spatial attenuation factor is the stress condition (2.11). Eliminating $p$, $\eta$ and $\tau_{yy}$ through use of (2.1), (2.4), (2.5) and (2.12), it is readily established that

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial\psi}{\partial y}\right) - 4\nu\frac{\partial^2}{\partial x\partial t}\left(G\frac{\partial^2\psi}{\partial x\partial y}\right) + \nu\frac{\partial^2}{\partial y\partial t}\left\{F\left(\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2}\right)\right\} - g\frac{\partial^2\psi}{\partial x^2} = 0 \quad \text{at} \quad y=0. \quad (5.1)$$

Introducing (3.5), (3.12) and (3.13) into (5.1) and expressing $\nu$ in terms of $\varepsilon$, the identification of quantities $o(1)$ gives

$$\Psi_1'(0) - \frac{i}{2}\left[\frac{d}{dr}\{F\Psi_1''(r)\}\right]_{r=0} + \frac{gh}{\sigma^2}k_0^2\Psi_0(0) = 0. \quad (5.2)$$

Substituting from (4.13) and (4.16),

$$\sigma^2 = gk_0 \tanh k_0 h, \quad (5.3)$$

in accordance with established theory for a perfect fluid.

The identification of terms $o(\varepsilon)$ yields

$$\Psi_2'(0) - \frac{i}{2}F_{r=0}\Psi_2'''(0) - \frac{i}{2}\left(\frac{dF}{dr}\right)_{r=0}\Psi_2''(0) + 2i(k_0 h)^2 G_{r=0}\Psi_0'(0) - \frac{i}{2}(k_0 h)^2\left[\frac{d}{dr}\{F\Psi_0(r)\}\right]_{r=0} + \frac{gh}{\sigma^2}\{k_0^2\Psi_1(0) + 2k_0 k_1\Psi_0(0)\} = 0. \quad (5.4)$$

Substituting the various quantities in this equation from (3.22), (4.13) and (4.16) it reduces to

$$F_{r=0}\Psi_2'''(0) + 2i\Psi_2'(0) = -\frac{2igk_1}{\sigma^2}\sinh k_0 h. \quad (5.5)$$

Calculating the left hand side by use of (4.21), the condition reduces to an equation for $k_1$ yielding

$$k_1 = \frac{ik_0^2 h\left\{F_{s=0}\Psi_0''(0) + \sinh^2 k_0 h\left(\frac{dF}{dr}\right)_{r=r_{00}}\right\}}{2k_0 h + \sinh 2k_0 h}. \quad (5.6)$$

The assumed boundary layer structure at the free surface indicates, in physically realistic situations, that $dF/dr$ will vanish at the outer edge of the layer. This reveals, to the present order, that the spatial attenuation will be unaffected by surface turbulence and results from energy dissipation in the lower boundary layer. The value of the damping factor will depend upon that of $\psi_1''(0)$ which in turn depends upon the solution of (3.19) subject to (3.17) and (4.3). Consequently, a numerical evaluation is only possible if the functional form of $F$ be prescribed within the lower boundary layer. Appeal is made at this stage to the arguments advanced by Johns (1966) concerning this variation and, in the present treatment, we suppose that $\nu$ denotes the kinematical viscosity of the fluid whilst $F$ is defined by

$$F=\begin{cases}(1+\alpha s)^2 & \text{for} \quad 0\leqslant s\leqslant s_1\\ \left[1+\dfrac{\alpha s_1(s_2-s)}{s_2-s_1}\right]^2 & \text{for} \quad s_1\leqslant s\leqslant s_2\\ 1 & \text{for} \quad s\geqslant s_2\end{cases} \qquad (5.7)$$

This form of variation is consistent with a scale of turbulence which increases with height above the impermeable bottom and, after attaining a prescribed maximal value, decreases towards the edge of the boundary layer. Moreover, the necessary integrations are readily performed by the method used by Johns (1966) therefore avoiding the familiar complications associated with the numerical solution of a two-point boundary value problem. The quantities $\alpha$, $s_1$ and $s_2$ are disposable parameters which determine the overall intensity of the turbulence and the geometrical scale of the boundary layer.

The details of the integration of (3.19) with $F$ given by (5.7) are described briefly in the appendix whilst we give here the results of the numerical computations.

In the work of Biesel (1949), and others, the value of $k_1$ for a homogeneous fluid in which the oscillatory motion is laminar (with kinematical viscosity $\nu$) is given by

$$k_1=\frac{k_0^2 h(1+i)}{2k_0 h+\sinh 2k_0 h}, \qquad (5.8)$$

and so, in order to determine the damping effect of the turbulence, it is sufficient to compute the value of $i\psi_0''(0)$ with the distribution (5.7). Denoting the maximum value of the function $F$ in the lower boundary layer by $F_{max}$, the real and imaginary parts of the required quantity have been computed as functions of $F_{max}$ for $s_2=10$ with $s_1=2, 4, 6$. The results are presented in Table 1.

Table 1

| | $F_{max}$ | Rl $i\psi_0''(0)$ | Im $i\psi_0''(0)$ |
|---|---|---|---|
| $s_1=2.0$ | | | |
| | 10 | 0.9835 | 1.691 |
| | 100 | 1.370 | 4.645 |
| | 500 | 1.961 | 10.86 |
| | 1000 | 2.046 | 15.58 |
| $s_1=4.0$ | | | |
| | 10 | 0.9750 | 1.288 |
| | 100 | 0.7906 | 2.648 |
| | 500 | 0.6455 | 5.634 |
| | 1000 | 0.5834 | 7.930 |
| $s_1=6.0$ | | | |
| | 10 | 0.9926 | 1.188 |
| | 100 | 0.8215 | 1.928 |
| | 500 | 0.5360 | 3.799 |
| | 1000 | 0.4311 | 5.309 |

With the exception of $s_1=2$, it is apparent that the turbulence results in a decrease of the real wave number in comparison with its laminar significance. That is, the turbulence produces a small lengthening of the waves when the maximum eddy viscosity is attained at a point well within the boundary layer.

The most important conclusions are concerned with the effect of the turbulence on the spatial damping of the wave amplitude. This is significantly increased in comparison with energy dissipation by molecular processes. If, for example, $s_1=4$, $F_{max}=500$, it is seen that for a laminar flow to result in the same damping it would be necessary for the kinematical viscosity to be enhanced by a factor of 31.

In conclusion, it is interesting to observe that a small value of $F_{max}$ attained close to the bottom is as efficient as regards energy dissipation as a large value of $F_{max}$ attained further from the boundary.

## Appendix

The results of the foregoing sections depend critically upon the solution of the system

$$Ff_n''(r) + 2if_n(r) = 0 \quad (6.1)$$

subject to

$$f_n(0) = f_n(r_{00}) = f_n'(r_{00}) = 0. \quad (6.2)$$

It is shown here that if $F$ be bounded and nonzero for $0 \leq r \leq r_{00}$, then the only solution is

$$f_n = 0. \quad (6.3)$$

Writing $\quad f_n = P + iQ,$

and separating (6.1) into real and imaginary parts,

$$FP'' - 2Q = 0, \quad (6.4)$$

$$FQ'' + 2P = 0. \quad (6.5)$$

Multiplying (6.4) by $P$ and (6.5) by $Q$ and adding

$$F\{P''P + Q''Q\} = 0,$$

and, since $F \neq 0$,

$$P''P + Q''Q = 0. \quad (6.6)$$

Integrating by parts from $r = 0$ to $r = r_{00}$,

$$[P'P + Q'Q]_0^{r_{00}} - \int_0^{r_{00}} (P'^2 + Q'^2)dr = 0. \quad (6.7)$$

The first term in (6.7) vanishes by virtue of (6.2) and so

$$\int_0^{r_{00}} (P'^2 + Q'^2)dr = 0, \quad (6.8)$$

whereupon

$$P'^2 + Q'^2 = 0, \quad (6.9)$$

since the integrand can never be negative.

We therefore obtain

$$P' = Q' = 0, \quad (6.10)$$

and, since both $P$ and $Q$ must vanish at $r = 0$, the result is established.

The second objective of the appendix is to indicate the method of solution of the equation

$$\frac{d}{ds}\left(F\frac{d\psi_0'}{ds}\right) + 2i\psi_0' = 2i, \quad (6.11)$$

with $F$ given by (5.7), subject to

$$\left.\begin{array}{lll}\psi_0' = 0 & \text{at} & s = 0,\\ \psi_0' = 1 & \text{at} & s = \infty.\end{array}\right\} \quad (6.12)$$

For $0 \leq s \leq s_1$, it is readily shown that

$$\psi_0' = 1 + A_1\xi_1^{m_1} + A_2\xi_1^{m_2}, \quad (6.13)$$

where $\quad \xi_1 = 1 + \alpha s,$

and $\quad m_1, m_2 = -\tfrac{1}{2} \pm \left(\frac{1}{4} - \frac{2i}{\alpha^2}\right)^{\frac{1}{2}}. \quad (6.14)$

For $\quad s_1 \leq s \leq s_2,$

$$\psi_0' = 1 + B_1\xi_2^{n_1} + B_2\xi_2^{n_2}, \quad (6.15)$$

where $\quad \xi_2 = 1 + \dfrac{\alpha s_1(s_2 - s)}{s_2 - s_1},$

and $\quad n_1, n_2 = -\tfrac{1}{2} \pm \left\{\frac{1}{4} - \frac{2i(s_2 - s_1)^2}{\alpha^2 s_1^2}\right\}^{\frac{1}{2}}, \quad (6.16)$

whilst for $s \geq s_2$ the appropriate "boundary layer type solution" is

$$\psi_0' = 1 + Ce^{-(1-i)s}. \quad (6.17)$$

The various constants in (6.13), (6.15) and (6.17) are determined by using (6.12) and enforcing the continuity of $\psi_0'$ and $\psi_0''$ at $s = s_1$ and $s = s_2$. The numerical scheme for this process has been programmed for automatic computation.

Finally, upon use of (6.13), it follows that

$$\psi_0''(0) = \alpha(A_1 m_1 + A_2 m_2), \quad (6.18)$$

which, together with (5.6), determines the damping.

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## ЗАТУХАНИЕ ГРАВИТАЦИОННЫХ ВОЛН В МЕЛКОХ ВОДЕ ВСЛЕДСТВИЕ ДИССИПАЦИИ ЭНЕРГИИ В ТУРБУЛЕНТНОМ ПОГРАНИЧНОМ СЛОЕ

Метод асимптотических разложений применяется к определению затухания гравитационных волн, распространяющихся в турбулентных условиях. Эффект турбулентности вводится общей системой коэффициентов турбулентной вязкости, в то же время предполагается, что сама турбулентность ограничена турбулентными слоями, прилегающими к твердому непроницаемому дну и к свободной поверхности. Найдено, что затухание самого низкого порядка в системе не зависит от турбулентности у поверхности. Проделаны вычисления для физически правдоподобного распределения турбулентной вязкости в нижнем пограничном слое.