NOTE

On similarity and the outer boundary conditions for flow in plumes

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This note discusses the relationship between similarity flow in turbulent jets and plumes and the outer radial boundary conditions. The lateral inflow into such cores *must* be determined as part of the solution and cannot be treated as a disposable constant. The discussion provides an explanation of the flow profiles given previously in this journal by Schmidt, and shows that they can apply only to plumes in conical or wedgeshaped regions of moderately small semi-angle.

Schmidt (1957) has given a treatment for turbulent plumes in which he generalises earlier work by Priestley & Ball (1955), and shows that the main features of flow development with axial distance in a similarity region of two-dimensional or axisymmetric plume can be deduced without introducing special assumptions about the shapes of velocity and temperature profiles. These results were obtained using equations for turbulent motion integrated over a normal section of the core. It is not possible to deduce profile forms from such integrated equations, and the further assumption of similarity means that the effect of particular profile shapes is concentrated entirely into the magnitudes of certain constant coefficients. Hence the integration of the equations can be completed without further specification of the profiles. Although unable to deduce profile shapes, Schmidt gave some discussion of likely forms and concluded that in general the flow field of a buoyant plume would consist of a central core of upflow surrounded by an annular downflow with magnitude a rapidly decreasing function of radial distance. This conclusion is so surprising that it clearly merits further comment. We shall demonstrate that it follows from the over-constraint of the outer boundary condition that lateral inflow at each level is zero.

Plumes and jets, whether laminar or turbulent, are long narrow flows in the sense that the region of relatively vigorous motion subtends a small semi angle (usually some 10°) at the virtual source. Thus, except in a neighborhood of the source, lateral gradients are very much larger than longitudinal gradients, and provided that the flow is effectively unconstricted by boundaries it is free to develop slowly along its axis under a balanced system of forces. In such an asymptotic region both turbulent and mean parts of the flow are likely to settle into selfpreserving or similarity form in which the changing structure can be represented fully by gradually varying scales of velocity, temperature and radius. Axial momentum flux is conserved in a jet, but increases in a plume because of the work done by the conserved buoyancy. In both cases there is slow lateral diffusion (turbulent or viscous) communicating longitudinal momentum to ambient fluid which consequently moves forward into the core and must thenceforth be identified as core fluid. The acceleration of this entrained fluid is associated with a weak external pressure field which produces a slow replacement inflow.

In taking radial gradients as large relative to axial gradients we are making a boundary-layer type of assumption and this corresponds with a subdivision of the whole flow field into two parts: a narrow inner core or shear layer in which the flow disturbance is large but which occupies only a small part of the whole field; and an extensive outer region in which the flow disturbance is small. The purpose of a boundary-layer type analysis is to decouple

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these two flow regions in so far as this is possible. The following argument is given in terms of axisymmetric (or axi symmetric-in-the-mean) flows referred to polar coordinates (r, z) with velocity components (u, w). The radius of an axisymmetric *laminar* jet is proportional to $\mathrm{Re}^{-\frac{1}{2}}$, where Re is a Reynolds number for the jet as a whole and $\varrho(\nu\,\mathrm{Re})^2$ is the constant momentum flux transported by the flow. At large values of the Reynolds number (ignoring questions of flow stability) the "width" of the jet is negligible in terms of the r-scale and we introduce the "stretched" coordinates ($r_i = r\,\mathrm{Re}^{\frac{1}{2}}$; $z_i = z$) in order that r_i may have unit order variation across the core. Inner boundary conditions for the outer region may be applied at $r = 0$ in the sense that the ambient motion arises from the communication of stress across the jet boundary and may be well-approximated through almost all the region by an appropriate driving mechanism concentrated at $r = 0$; outer boundary conditions for the core flow should be applied outside the actual core boundaries where r is small but r_i large, normally represented as $r_i \to \infty$ though not to be confused with the outside of the unconfined ambient region $r \to \infty$. We note that there is no sharply defined (mean) boundary to an actual core, though we find it convenient to define a mean radius and sometimes to refer to a mean boundary as though it were a precise physical entity.

The laminar plume is characterised as a whole by a Rayleigh number Ra proportional to the heat flux and depending also on a scale length l characteristic of the source, and the corresponding inner coordinates are ($r_i = r\,\mathrm{Ra}^{\frac{1}{4}}$; $z_i = z$). Thus the laminar jet (or plume) may in principle be made as narrow as we please by selection of sufficiently large Re (or Ra), but in fact become unsteady at an early stage and fully turbulent at quite moderate Reynolds (or Rayleigh) numbers. At large values we should expect Reynolds (or Rayleigh) number similarity with core flow independent of further increases in Re (or Ra). Experimental turbulent jets and plumes are found to be conical with semiangle of spread roughly 10° (though rather less for plumes than jets, Morton 1959). The appropriate inner variables are ($r_i = r/\alpha$; $z_i = z$) where α is the semi-angle of spread, or alternatively ($r_i = r/E$; $z_i = z$) where E is the entrainment constant; both stretching factors are roughly *10* and *independent of* Re (or Ra). Thus turbulent jets and plumes have universal forms which are independent of Re or Ra, but are sufficiently narrow to allow a meaningful separation of core and outer flow regions.

The relationship between core and outer flows may be understood very simply in terms of entrainment. The flux of plume fluid (recognisable either by the measure of its axial velocity or its excess temperature) increases progressively with distance because of the outward radial diffusion of longitudinal momentum and buoyancy. Viewed from large radial distance r, the outer flow is that due to a distribution of *sinks* with strength $-q(z) = (2\pi r u)_{r\,\mathrm{small},\ r_i\,\mathrm{large}}$ per unit length spread over the core "surface", or approximately to the distribution of sinks $-q(z)$ along the core axis. Thus the radial velocity $u \sim -l/r$ and ru is a finite function of z which remains non-zero at all radial distances. Taylor (1958) has used this approach to find the patterns of entrainment into jets and plumes. As long as the outer flow is able freely to feed the entrainment requirements of the core, an inner similarity flow is possible; however, the entrainment flux $q(z)$ is *unknown and must be obtained as part of the* core solution, and cannot be imposed arbitrarily as one of the boundary conditions on the inner flow. Thus we picture a narrow core extracting fluid from and embedded in a weak ambient flow which serves to organise the entrainment flux in conformity with the external boundary conditions. Rigid outer walls need not restrict the possibility of a similarity structure for jets and plumes provided that they are sufficiently far from the core so as not to constrain entrainment seriously. If they are too close, however, similarity flow will be possible only when the walls form part of the similarity geometry, and the profiles given by Schmidt should apply most nearly to plumes rising in conically bounded regions with spread angle a little larger than that for an unrestricted plume (but not too large, or an outer flow will form between plume and cone).

We may note also that the forced plume rising above a mixed source of heat and momentum behaves first as a jet and ultimately as a plume and, as these have different semi-angles, cannot exhibit simple conical spread throughout as postulated by Schmidt (Morton, 1959).

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