The operational solution of the balance equation

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ABSTRACT

The balance equation is commonly used to provide accurate non-divergent initial wind fields for prognostic models. The most important technical operational problems occurring in the numerical solution of the equation are examined. Methods of convergence are proposed, which are accurate and economical of computer time. The specific points treated are the convergence criterion, the relaxation factor, a correction technique for the residual error and a simple method of selective relaxation. Typical error fields at 500 mb illustrate the accuracy of the proposed method.

1. Introduction

The balance wind equation was incorporated into the operational numerical weather prediction program of the Central Analysis Office in 1964. It is applied at 500 mb to derive the initial stream function analysis for the barotropic model and at other levels to provide initial input for baroclinic models. Previously, a linearized form of the balance equation had been used but the frequent occurrence of serious errors, involving mainly excessive anticyclogenesis and overestimate of vorticity in regions of strong curvature, proved it unsatisfactory for operational use.

Starting with BOLIN (1955), many authors have published papers on the solution and the application of the balance equation. Here, it is intended to resolve some of the technical difficulties arising in the operational use of a method of solution similar to Bolin's. The conditions for convergence of the relaxation scheme are investigated and some techniques are outlined for improving the accuracy and decreasing the overall computing time.

2. Definitions and basic equations

The complete balance equation is

$$f\nabla^2\Psi + \nabla f \cdot \nabla\Psi + 2m^2 J\left(\frac{\partial\Psi}{\partial x}, \frac{\partial\Psi}{\partial y}\right) = g\nabla^2 Z \quad (1\,\text{a})$$

where ∇ is the two-dimensional del operator, J is the Jacobian operator, f the Coriolis parameter, g the acceleration due to gravity, x and y the map coordinates, m the map scale factor, Z the geopotential height of the pressure surface, and Ψ the stream function. It represents, in the absence of horizontal divergence, a diagnostic relationship between the height field and the balanced wind field defined by the stream function. Starting from an initial height analysis, the problem encountered operationally is to solve equation (1a) in as short a time as possible and to provide a satisfactory stream function for numerical prediction models.

First it is convenient to define a new stream function

$$\psi = \frac{\bar{f}}{g}\Psi \quad (2)$$

having the same units as Z, and two fields of constants

$$F = \frac{f}{\bar{f}}, \quad M^2 = \frac{m^2 g}{2\bar{f}^2 d^2} \quad (3)$$

where $\bar{f}$ is the value of f at latitude 45°, and d is the grid length at the standard latitude of the map projection used. With these definitions, equation (1a) can be written as

$$F\nabla^2\psi + \nabla F \cdot \nabla\psi + 4M^2 d^2 J\left(\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}\right) = \nabla^2 Z. \quad (1\,\text{b})$$

Equation (1b) can be elliptic, parabolic or hyperbolic (see for example ARNASON 1958). But objective height analyses usually consist of fields that are elliptic everywhere except for a few small isolated regions of hyperbolic points. Since hyperbolic points offer special

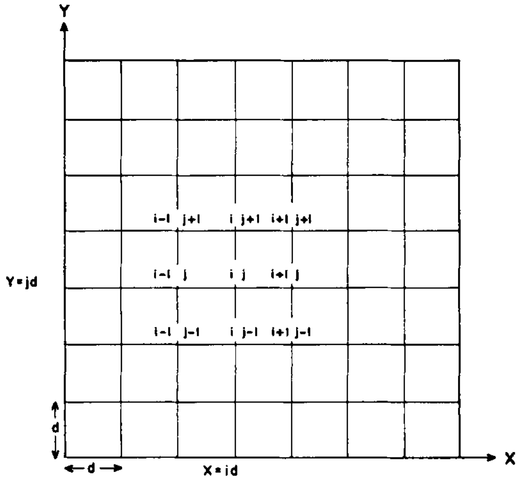

FIG. 1. Arrangement of gridpoints.

difficulties in the solution of the balance equation, which have so far proved insurmountable, the usual procedure of artificially restricting equation (1b) to the elliptic case is followed. The height field is modified in the critical regions so that the ellipticity criterion,

$$F\nabla^2\psi + 4M^2d^2J\left(\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}\right) > -\frac{F^2}{4M^2d^2} \tag{4}$$

or equivalently,

$$\nabla^2 Z - \nabla F \cdot \nabla\psi > -\frac{F^2}{4M^2d^2} \tag{4b}$$

is satisfied everywhere. In which case equation (1b) can be solved for ψ as a boundary value problem. The boundary conditions used are those suggested by Bolin, namely

$$\frac{\partial\psi}{\partial s} = \frac{1}{F}\frac{\partial z}{\partial s} - \frac{\oint_S \frac{1}{F}\frac{\partial z}{\partial s}ds}{\oint_S ds}. \tag{5}$$

A first trial field, ψ^0, is obtained similarly by applying the geostrophic relationship across each row of the grid, and then adjusting the interior values slightly so that the boundary points remain unchanged.

Now define some finite difference approximations, the prime in $(\delta^2\psi)/(\delta' x^2)$ indicating the special approximation used for the Jacobian (axes rotated by 45°). The indices refer to Fig. 1.

$$\frac{\partial^2\psi}{\partial x^2} \approx \frac{1}{d^2}(\psi_{i-1j} + \psi_{i+1j} - 2\psi_{ij}) = \frac{1}{d^2}\frac{\delta^2\psi}{\delta x^2} \tag{6a}$$

$$\frac{\partial^2\psi}{\partial y^2} \approx \frac{1}{d^2}(\psi_{ij+1} + \psi_{ij-1} - 2\psi_{ij}) = \frac{1}{d^2}\frac{\delta^2\psi}{\delta y^2} \tag{6b}$$

$$\frac{\partial^2\psi}{\partial x^2} \approx \frac{1}{2d^2}(\psi_{i+1j+1} + \psi_{i-1j-1} - 2\psi_{ij}) = \frac{1}{2d^2}\frac{\delta^2\psi}{\delta' x^2} \tag{6c}$$

$$\frac{\partial^2\psi}{\partial y^2} \approx \frac{1}{2d^2}(\psi_{i+1j-1} + \psi_{i-1j+1} - 2\psi_{ij}) = \frac{1}{2d^2}\frac{\delta^2\psi}{\delta' y^2} \tag{6d}$$

$$\frac{\delta^2\psi}{\partial x\partial y} \approx \frac{1}{2d^2}(\psi_{ij+1} + \psi_{ij-1} - \psi_{i-1j} - \psi_{i+1j}) = \frac{1}{2d^2}\frac{\delta^2\psi}{\delta' x\delta' y} \tag{6e}$$

$$\begin{aligned}\nabla F \cdot \nabla\psi &\approx \frac{1}{4d^2}[(F_{i+1j} - F_{i-1j})(\psi_{i+1j} - \psi_{i-1j}) \\ &\quad + (F_{ij+1} - F_{ij-1})(\psi_{ij+1} - \psi_{ij-1})] \\ &= \frac{1}{4d^2}\nabla F \cdot \nabla\psi\end{aligned} \tag{6f}$$

The balance equation (1b) can now be written in the finite difference form

$$F_{ij}\left(\frac{\delta^2\psi}{\delta x^2} + \frac{\delta^2\psi}{\delta y^2}\right) + \frac{\nabla F \cdot \nabla\psi}{4} + M_{ij}^2\left[\frac{\delta^2\psi}{\delta' x^2}\frac{\delta^2\psi}{\delta' y^2} - \left(\frac{\delta^2\psi}{\delta' x\delta' y}\right)^2\right] - \left(\frac{\delta^2 z}{\delta x^2} + \frac{\delta^2 z}{\delta y^2}\right) = 0. \tag{7}$$

Since the Coriolis term will not be computed at each iteration, it must be evaluated separately from the $F\nabla^2\psi$ term. The effect of using a longer base length should be small since the term is most important for long waves. The finite difference form of the Jacobian is not defined at the first inside points of the diagonal sides of the octagon; the best results have been obtained by simply replacing it by zero at the first inside points of all sides.

Before proceeding with the solution of equation (7), a second trial field, ψ^1, may be obtained by solving the Poisson equation,

$$\frac{\delta^2\psi^1}{\delta x^2} + \frac{\delta^2\psi^1}{\delta y^2} = \frac{1}{F_{ij}}\left(\frac{\delta^2 z}{\delta x^2} + \frac{\delta^2 z}{\delta y^2} - \frac{\nabla F \cdot \nabla\psi^0}{4}\right) \tag{8}$$

which is precisely the linearized balance equation previously in operational use at the Central Analysis Office (Kwizak & Robert, 1963). The function ψ^1 is quasi-geostrophic and only in regions of strong curvature does it differ noticeably from the solution ψ of (7). The residual of equation (7) at the point (i, j) is defined as R_{ij}^{ν}, for any trial stream function ψ^{ν}.

The essential feature of the Bolin approach consists of linearizing equation (7) by writing,

$$\psi = \psi^{\nu} + \psi^{*} \tag{9}$$

and assuming the trial field ψ^{ν} is sufficiently close to the true solution that second order terms in ψ^{*} may be neglected. Equation (7) becomes:

$$F_{ij}\left(\frac{\delta^2\psi^*}{\delta x^2} + \frac{\delta^2\psi^*}{\delta y^2}\right) + M_{ij}^2 \frac{\delta^2\psi^{\nu}}{\delta' y^2}\frac{\delta^2\psi^*}{\delta' x^2} - 2M_{ij}^2 \frac{\delta^2\psi^{\nu}}{\delta' x\delta' y}\frac{\delta^2\psi^*}{\delta' x\delta' y} + M_{ij}^2 \frac{\delta^2\psi^{\nu}}{\delta' x^2}\frac{\delta^2\psi^*}{\delta' y^2} + \frac{\nabla F\cdot\nabla\psi^*}{4} + R_{ij}^{\nu} = 0. \tag{10}$$

Bolin was able to solve equation (10) by the application of a Liebmann sequential relaxation scheme in which the latest available values of ψ were always used for ψ^{ν} and where the relaxation factor was computed at each point. The convergence rate of this rechnique is slow when the number of gridpoints is very large. Bolin, nevertheless, continued the relaxation until a ψ of the desired accuracy was obtained.

It is essential to minimize computing time for routine operational forecasting procedures. Upon investigation, the author has found that, when a certain stage is attained in the relaxation, further increases in accuracy are most economically achieved by applying a simple correction technique (described below) to the stream function field, rather than by continuing with the relaxation scans. However, an analysis of the convergence of the relaxation process is discussed first.

3. The Convergence of the Liebmann relaxation process

The method used by the author consists of applying directly to equation (7) a Liebmann relaxation scheme defined as follows:

$$\psi_{ij}^{\nu+1} = \psi_{ij}^{\nu} + \alpha R_{ij}^{\nu} \tag{11}$$

where α is the relaxation factor. The analysis of the convergence of this scheme on a non-linear equation such as (7) is difficult. Some conclusions derived from the linear equation (10) might none the less be helpful. Re-write equation (10) in the form

$$F_{ij}\left(\frac{\delta^2\psi^{*K}}{\delta x^2} + \frac{\delta^2\psi^{*K}}{\delta y^2}\right) + A_{ij}^{\nu}\frac{\delta^2\psi^{*K}}{\delta' x^2} - B_{ij}^{\nu}\frac{\delta^2\psi^{*K}}{\delta' x\delta' y} + C_{ij}^{\nu}\frac{\delta^2\psi^{*K}}{\delta' y^2} + S_{ij}^{\nu} = Q_{ij}^{K} \tag{12}$$

where A, B, and C are known functions of F, M, ψ^{ν} and ν is constant; where, for convenience, the slowly varying term $\frac{1}{4}\nabla F\cdot\nabla\psi^{*K}$ is assumed constant and is combined with R_{ij}^{ν} to form S_{ij}^{ν}; where Q_{ij}^{K} is the residual from the linear equation (10) when some trial stream function ψ^{*K} is substituted for ψ^*; and where K is the iteration index for the linear equation.

The relaxation scheme to solve (12) is then the analogue to (11)

$$\psi_{ij}^{*K+1} = \psi_{ij}^{*K} + \alpha Q_{ij}^{K}. \tag{13}$$

The latest available values of ψ^* are always used during the relaxation.

$$\text{e.g.}\quad \frac{\delta^2\psi^{*K}}{\delta x^2} = \psi_{i-1j}^{*K-1} + \psi_{i+1j}^{*K} - 2\psi_{ij}^{*K}. \tag{14}$$

It is generally accepted that the ellipticity criterion is a sufficient condition for the convergence of the solution of a linear equation of the type of (12) with the Liebmann scheme. The convergence is obviously also related to the choice of the relaxation factor α. Furthermore it can be verified easily that for a given equation and a given size of grid the optimum relaxation factor is not fixed but depends on the wavelength composition of the solution. In an attempt to find such a relationship for the solution of (12), re-write (12) for iteration $K+1$ and eliminate ψ^* from it by substituting (12) and (13) for all values of i, j, K. The result is the following

$$Q_{ij}^{K+1} = Q_{ij}^{K} + \alpha\left[F_{ij}\left(\frac{\delta^2 Q^K}{\delta x^2} + \frac{\delta^2 Q^K}{\delta y^2}\right) + A_{ij}^{\nu}\frac{\delta^2 Q^K}{\delta' x^2} - B_{ij}^{\nu}\frac{\delta^2 Q^K}{\delta' x\delta' y} + C_{ij}^{\nu}\frac{\delta^2 Q^K}{\delta' y^2}\right]. \tag{15}$$

Using a Fourier term for Q

$$Q^K = q^K \exp\left(2\pi\sqrt{-1}\left(\frac{x}{md} + \frac{y}{nd}\right)\right) \tag{16}$$

and substituting (16) into (15) gives, after some manipulation

$$\frac{q^{K+1}}{q^K} = \frac{1 - \alpha\eta - \alpha v + \sqrt{-1}\,\alpha\omega}{1 - \alpha\eta + \alpha v + \sqrt{-1}\,\alpha\omega} \tag{17}$$

where $\eta = 2F + A + C$

$$v = \eta - F\left(\cos\frac{2\pi}{n} + \cos\frac{2\pi}{m}\right) - A\cos 2\pi\left(\frac{1}{n} + \frac{1}{m}\right)$$

$$+ B\left(\cos\frac{2\pi}{n} - \cos\frac{2\pi}{m}\right) - C\cos 2\pi\left(\frac{1}{n} - \frac{1}{m}\right)$$

and

$$\omega = F\left(\sin\frac{2\pi}{n} + \sin\frac{2\pi}{m}\right) + A\sin 2\pi\left(\frac{1}{n} + \frac{1}{m}\right)$$

$$- B\left(\sin\frac{2\pi}{n} - \sin\frac{2\pi}{m}\right) + C\sin 2\pi\left(\frac{1}{n} - \frac{1}{m}\right).$$

For the sequence of ψ^{*K} to converge to ψ^*, the inequality

$$\left|\frac{q^{K+1}}{q^K}\right| < 1 \tag{18}$$

must hold for all sufficiently large values of K. Substitution of the complex expression (17) into (18) gives

$$v\left(\frac{1}{\alpha} - \eta\right) > 0. \tag{19}$$

Examination of v reveals that it is mainly positive (it can be shown that the ellipticity condition implies that v is positive); consequently

$$0 < \alpha < \frac{1}{2F + A + C}. \tag{20}$$

The trial field used by the author, which is obtained from (8), is generally not good enough to satisfy the conditions under which (7) becomes similar to (10). Consequently the convergence of (7) is theoretically not certain under the conditions for the convergence of (12). The author, like many others, has suspected that the scheme might nevertheless be convergent under (20) and if the ellipticity criterion (4b) was satisfied at all times (see next section on this subject).

So far, approximately five thousand runs have been made without failure. Although the changes necessitated by the ellipticity criterion are small after the first application, it was found that convergence was generally not attained if (4b) was not satisfied continually. For want of a rigid proof these results should be accepted as statistical evidence of the convergence of the author's method.

4. Effect of the Coriolis term

The ellipticity criterion (4b) involves Z as the main parameter but ψ also appears in the less important Coriolis term. There are various ways to show that this term is smaller in magnitude than the other terms in the balance equation and that it varies only slightly from iteration to iteration during the relaxation. It is also of interest to note that the relaxation scheme (11) is divergent when applied to an equation where the variable appears only in $\nabla F \cdot \nabla\psi$ and that the presence of this term in equation (7) would slow down the convergence. For these reasons, it is found advantageous to recompute it only every twenty iterations or so, thus reducing the computing time considerably. Consequently, it now becomes necessary only to apply the ellipticity criterion (4b) initially using a trial value for ψ, and subsequently each time a new value of $\nabla F \cdot \nabla\psi$ is computed.

5. An efficient method of increasing the accuracy

The problem of the convergence limits remains to be discussed. The round-off errors inherent in any machine computation make it impossible to carry out the relaxation until the equation is exactly satisfied at every interior grid-point. In practice the process is terminated when the change in ψ from one iteration to the next is less than a certain value ε for all points of the grid. A resulting error E may be defined as the difference between the original height field (corrected for ellipticity) Z_e, and the height field which can be recovered from a solution of the balance equation in reverse, solving for Z using ψ as input. Namely,

$$E = Z_e - Z_{\psi-1}. \tag{21}$$

Applying the Laplacian operator to (21) and assuming that the solution for $Z_{\psi-1}$ is exact, gives:

$$\nabla^2 E = -R^f \tag{22}$$

where R^f is the residual from the balance equation when the final value ψ^f is used.

The error E is zero on the boundary and experience shows that it usually grows to a single maximum (or minimum) at the centre of the grid, when ε is chosen small enough. Then E can be assumed to have its main component wavelength of twice the total dimensions of the grid. Assuming a square grid of 45×45 points, one obtains

$$E \approx W \sin\frac{\pi x}{45}\sin\frac{\pi y}{45} \tag{23}$$

where x and y are in grid lengths and W is the amplitude of the error. Then

$$-R^f = \nabla^2 E \approx -.01E.$$

Since $\quad |\psi^{f-1}-\psi^f| = |\alpha R^f| \leqslant \varepsilon$

and using a value of 0.25 for α, it follows that

$$\varepsilon \geqslant \frac{E}{400}.$$

In order to reduce the maximum error to 4 or 5 meters by relaxation alone, the convergence limit would have to be set at $\varepsilon = 0.01$ metre, which would greatly prolong the relaxation process. Instead, the limit is set at a much higher value and the final degree of accuracy is attained indirectly using the knowledge of the error pattern provided by (22). Determine β such that

$$\psi = \psi^f + \beta E + (\Delta\psi) \tag{24}$$

where $\Delta\psi$ is an error function, whose amplitude must be made as small as possible. Substituting (24) into (1b), and collecting all the error due to $\Delta\psi$ into a residual D, gives

$$F\nabla^2(\psi^f+\beta E) + \nabla F\cdot\nabla(\psi^f+\beta E) + 4M^2d^2 J\left[\frac{\partial(\psi^f+\beta E)}{\partial x}, \frac{\partial(\psi^f+\beta E)}{\partial y}\right] - \nabla^2 Z = D. \tag{25}$$

The Jacobian in (25) expands to

$$J\left(\frac{\partial\psi^f}{\partial x}, \frac{\partial\psi^f}{\partial y}\right) + \beta^2 J\left(\frac{\partial E}{\partial x}, \frac{\partial E}{\partial y}\right) + \beta\left(\frac{\partial^2\psi^f}{\partial x^2}\frac{\partial^2 E}{\partial y^2} + \frac{\partial^2\psi^f}{\partial y^2}\frac{\partial^2 E}{\partial x^2} - 2\frac{\partial^2\psi^f}{\partial x\partial y}\frac{\partial^2 E}{\partial x\partial y}\right). \tag{26}$$

The relative magnitudes for the terms in (25) and (26) are given by L^{-n} where n is the order of the derivative and L is the wavelength of the quantity derived. The wavelength of E is fixed at about 90 grid-lengths while that of ψ may range from 2 to 90 grid-lengths. From this it is evident that E may be neglected in the Jacobian term, so that (25) reduces to

$$D \approx \beta(\nabla F\cdot\nabla E + F\nabla^2 E) - \nabla^2 E.$$

The value of β can be computed experimentally with the requirement that the RMS value of D be a minimum. Hence,

$$\beta = -\frac{\sum_{ij}[\nabla^2 E(\nabla F\cdot\nabla E + F\nabla^2 E)]_{ij}}{\sum_{ij}(\nabla F\cdot\nabla E + F\nabla^2 E)^2_{ij}}.$$

This value does not vary very much for different situations, so that it can be computed once and for all. In the NWP program currently in use at the Central Analysis Office, a value of 1.72 is used for β at 500 mb (together with a convergence limit $\varepsilon = 0.15$ meter) and the values selected for other levels range between 1 and 2.

Fig. 2 shows the residual error after the straight relaxation of the balance equation with a convergence limit of $\varepsilon = 0.075$ metre. For Fig. 3, the limit was only 0.15 metre but the correction technique was applied to the stream function leaving the much smaller error displayed. The computing time saved in this instance was 20 percent as compared to the case of Fig. 2.

The correction is simple to apply in practice. After the last relaxation scan, the residuals R^f are stored in a separate field and the error E is obtained by solving the Poisson equation

$$\nabla^2 E = -R^f. \tag{22}$$

Because of the particularly long wave composition of the field of R^f, a large overrelaxation

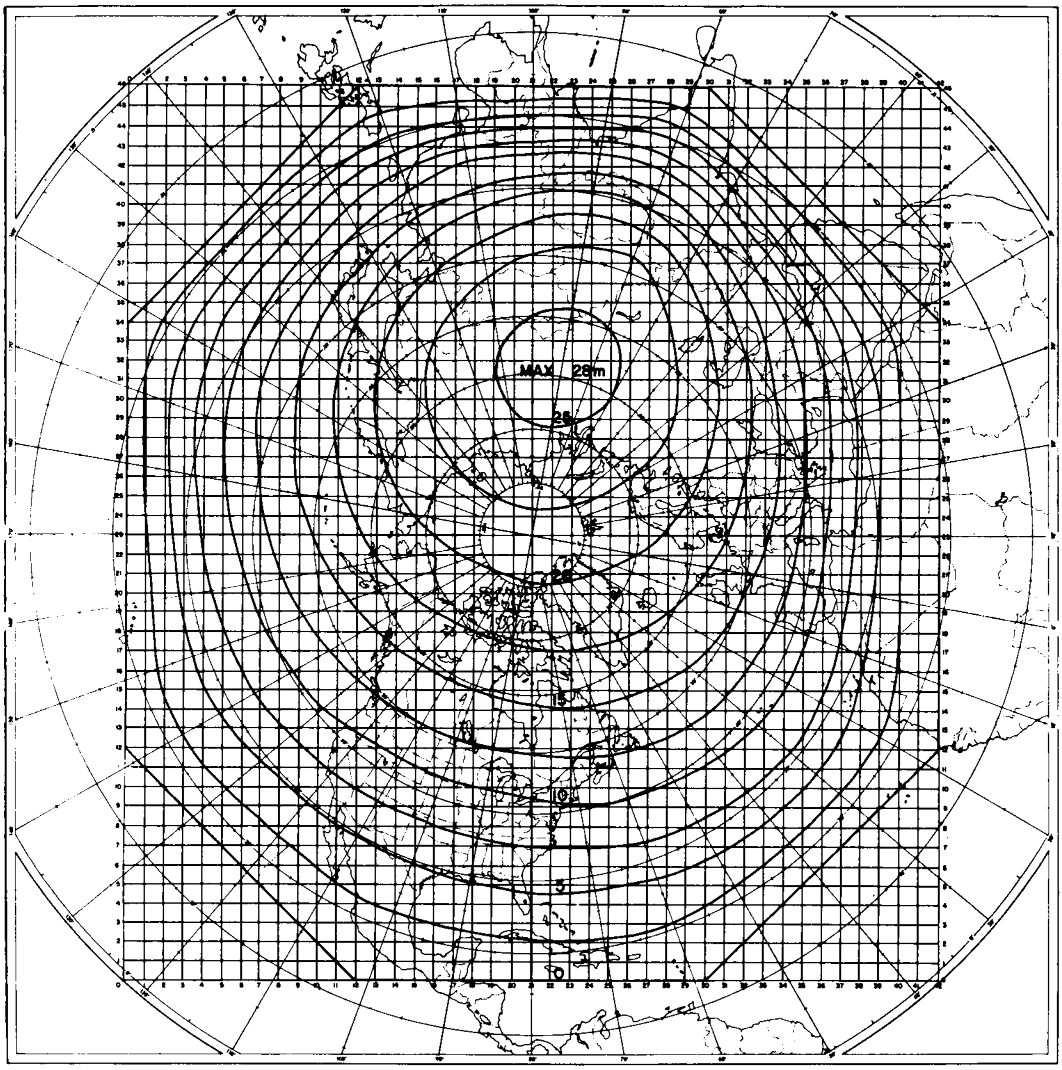

FIG. 2. Typical error pattern resulting from the solution of the balance equation with a convergence limit $\varepsilon = 0.075$ meters.

factor can be used with rapid convergence. Then βE is added to ψ^f to give the improved stream function.

6. The relaxation factor

It was found in section 3 that the relaxation factor α must obey the following inequality

$$0 < \alpha < \frac{1}{2F + A + C}. \qquad (20)$$

Replacing A and C by their definitions, which are implicit in (10) and (12) gives,

$$0 < \alpha < \frac{1}{2F + M^2\left(\dfrac{\delta^2\psi^\nu}{\delta' x^2} + \dfrac{\delta^2\psi^\nu}{\delta' y^2}\right)}. \qquad (27)$$

It is desirable to use a value of α which does not depend on ψ. Consequently, using (6c) and (6d), and replacing the relative vorticity

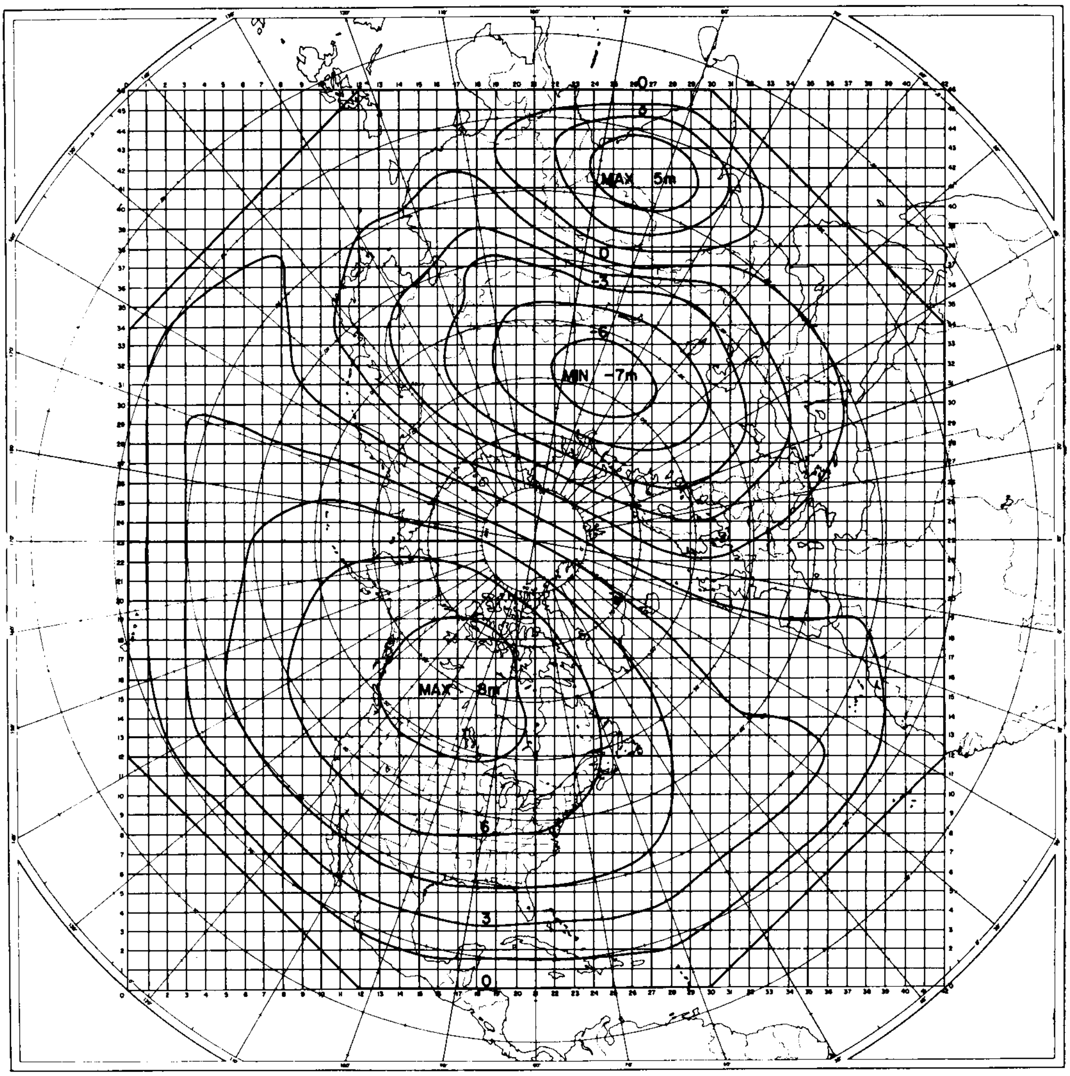

FIG. 3. Error field obtained for the same initial height field as for Fig. 2, when a convergence limit $\varepsilon = 0.15$ meters was used and the correction technique applied.

by its geostrophic counterpart, the inequality (27) becomes

$$0 < \alpha < \frac{1}{2\left[F + \frac{M^2}{F}\left(\frac{\delta^2 Z}{\delta x^2} + \frac{\delta^2 Z}{\delta y^2}\right)\right]}. \qquad (28)$$

The maximum value of the denominator in (28) can be evaluated before the beginning of the relaxation, and the corresponding value $\alpha_{\min}$ is guaranteed to ensure the convergence of the process at all points. Overrelaxation is obtained by starting the relaxation with $\alpha' = \alpha_{\min} + 0.15$, $\alpha' \leqslant 0.35$ and decreasing this value by equal steps every fifty iterations, using a final value not smaller than $\alpha_{\min}$; a method which proves to be very efficient.

7. Streamlining the relaxation process

One last technique also has proved to be time saving. At any stage in the relaxation, when the change $|\psi_{ij}^{\nu-1} - \psi_{ij}^{\nu}|$ at a particular point is less than the convergence limit ε, this point is "flagged"; so that in the subsequent iterations, it can simply be skipped over. Of course, since changes at the surrounding points affect the value of the residual at a flagged point, all flags must be ignored from time to time (10 iterations is a good figure). But in any case, before the flagging technique can be applied at all, at least half of the points in the grid must satisfy the convergence criterion; otherwise, ultimate convergence may not be obtained. The technique can also be successfully applied in the solutions of the Poisson equations (8) and (22) and, with some modification, in the application of the ellipticity criterion (4b). The time reduction obtained from this technique alone was of the order of 30 to 40 percent and the final accuracy was almost entirely unaffected.

8. Conclusion

The problems investigated in the study of the finite difference approximation of the balance equation were concerned with convergence, accuracy, and economy of computer time.

Convergence of the proposed relaxation process was obtained without difficulty if the ellipticity criterion was satisfied and if the relaxation factor was kept within the limits of 0 to .35.

A practical method for obtaining the necessary degree of accuracy was developed involving the use of experimental data and theoretical formulae. With the 1709 point octagonal grid, it was possible to obtain, in approximately 100 seconds for a 500 mb field, a solution with a RMS error always less than five metres; the solution being executed on the Control Data G-20 computer which has an addition time of 12 microseconds and a core memory of 24,576 32-bit words. The solution can be obtained at other pressure levels as well (one minute for 1000 and 850 mb levels and three to four minutes for the 200 mb level).

The solution was examined from the viewpoint of economy of computing time required for operational procedures. Keeping the Coriolis term constant for approximately 25 scans and computing a new value of the function at a grid-point only if the residual at that point was large in the previous scan (flagging technique) resulted in savings of as much as 40 % of the computer time. To obtain convergence of the long waves, it was found advisable to terminate the complete relaxation early and to use another method, such as that described in the text.

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ОПЕРАТИВНОЕ РЕШЕНИЕ УРАВНЕНИЯ БАЛАНСА

Уравнение баланса обычно используется для получения точного бездивергентного начального поля ветра в прогностических моделях. Изучаются наиболее важные технические проблемы, возникающие при численном решении этого уравнения. Предлагаются сходящиеся методы, которые являются достаточно точными и экономичными с точки зрения времени вычисления. Особо рассматриваются такие вопросы как критерий сходимости, множитель релаксации, техника коррекции для невязки и простой метод селективной релаксации. Типичные поля ошибок на 500 мб иллюстрируют точность предложенного метода.