Some dynamical aspects of atmospheric convection

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ABSTRACT

Some dynamical characteristics of atmospheric convection are investigated by obtaining an analytical solution for a most simplified set of difference equations which represent a two-layer model of an incompressible buoyant fluid. Several invariants are obtained as a function of initial values of vertical and horizontal potential temperature differences and kinetic energy. Then the analytical solution is obtained in using these invariants. One of the major advantages of using such a simplified model is to make it possible to discuss the nonlinear processes based on an analytical solution. The major disadvantage is caused by the low resolution in the space derivatives (time derivatives are exact). However, invariant quantities obtained for this simple model are equivalent to those obtained for the original differential equations, so that the important characteristics of convection are probably described in the present simplified model. Major results obtained from the model are presented.

1. Introduction

Problems of atmospheric convection have been investigated theoretically by methods based on parcel theories, entrainment theories, bubble theories, similarity analyses, perturbation theories, and numerical experiments. One of the major differences between the above theories is the various treatments of interaction between the buoyant element and its environment.

Parcel theories assume that an interaction between a buoyant element and the motionless environment is caused only by the temperature differences between them. Entrainment theories (STOMMEL 1947) have employed the assumption that the quantities of the environmental air such as temperature and moisture are entrained at a rate proportional to the upward velocity in the buoyant element and mix with those of the element. However, the environment is assumed to be motionless and not influenced from the buoyant element. Similar treatments have been employed for the bubble theories and plume models (SCORER & LUDLUM, 1953). Cumulus clouds have been understood in the bubble theories as groups of bubbles rising successively. Plume models have been used for the investigations of motion of locally heated air, such as due to ground fires and explosions. The similarity analysis was used by BATCHELOR (1954) for the problems treated in the above theories, and valuable conclusions were obtained concerning the dimensional relationships between the quantities of representing thermals.

The above concepts do describe mechanisms of the atmospheric convection, however, the nonlinear interactions between a buoyant element and the environment have been little clarified. One of such interactions is caused by the change of stabilities of the element and environment. The static stability will increase inside the buoyant element due to upward heat flux and influence not only on the stability of the buoyant element but also of the environment. Changes of the stability of environmental air should change the motion of the buoyant element. Such interaction is mainly represented by the terms of temperature gradient (both vertical and horizontal) and of continuity of mass. Heat transfer in the vertical and horizontal directions is the important process for the interaction. Momentum transfer is likely less important. The interaction is essentially a nonlinear process, which was not treatable in the perturbation theories. Recently, however, progress has been made via numerical experiments which have solved the nonlinear hydrodynamical equations for convection problems (OGURA 1962). The results agree to a certain extent with those previously obtained from the other theories.

In this article, the author attempts to investigate convection mechanisms by using a most simplified model. The approach in this study is somewhat similar to those used by LORENZ (1962) and A. Arakawa in their studies of the general circulation.

2. Basic equations

The basic equations used in this study are the same as those derived by OGURA & PHILLIPS (1962) for shallow convection. In x, z and t coordinates, the equations are written as

$$\frac{\partial u}{\partial t}+\frac{\partial uu}{\partial x}+\frac{\partial uw}{\partial z}=-\frac{\partial \pi}{\partial x}, \qquad (2.1)$$

$$\frac{\partial w}{\partial t}+\frac{\partial uw}{\partial x}+\frac{\partial ww}{\partial z}=-\frac{\partial \pi}{\partial z}+\beta\theta, \qquad (2.2)$$

$$\frac{\partial \theta}{\partial t}+\frac{\partial u\theta}{\partial x}+\frac{\partial w\theta}{\partial z}=0, \qquad (2.3)$$

and

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0. \qquad (2.4)$$

All variables in the above equations are nondimensional , u, w, θ and π are corresponding, respectively, to the notations u_{00}, w_{00}, θ_{10} and π_{11} used in the previously mentioned article. The relations between the above nondimensional quantities and the original and dimensional quantities u', w', θ' and π' are

$$u'=(d/\tau)\,u, \qquad (2.5)$$

$$w'=(d/\tau)\,w, \qquad (2.6)$$

$$\theta'=\Theta(1+\varepsilon\theta), \qquad (2.7)$$

and

$$\pi'=1-\beta z+\varepsilon\beta\pi \qquad (2.8)$$

where d and τ are the length and time scales of disturbance, Θ represents a constant mean value of the actual potential temperature, and π is a sort of Exner function defined by using pressure p and a reference pressure P and the ratio $\varkappa(\equiv(c_p-c_v)/c_p)$ as

$$\pi=(p/P)^{\varkappa}. \qquad (2.9)$$

The parameters ε and β are respectively the percentage variation of potential temperature and the ratio of d to the depth of an isentropic atmosphere $H(\equiv c_p\Theta/g)$,

$$\varepsilon=\Delta\theta'/\Theta \qquad (2.10)$$

and

$$\beta=d/H. \qquad (2.11)$$

Equations (2.1) through (2.8) are valid if the ratio ε and β are less than 1 (about $\frac{1}{10}$), and if the terms of ε^2 or higher and β^2 or higher are negligible. For such case the time scale τ relates to the Brunt-Väisälä frequency N, $(g\ \partial \ln\theta'/\partial z')^{\frac{1}{2}}$, as

$$\tau\sim N^{-1}\sim(d/g\varepsilon)^{\frac{1}{2}}. \qquad (2.12)$$

The above approximation is believed to be acceptable for cumulus convection because d may be less than several kilometers and H is about 30 km.

The equations (2.1) through (2.4) will be solved in the subsequent chapters by using a special boundary condition, i.e., normal component of velocity vanishes at the boundary

$$v_n=0. \qquad (2.13)$$

3. Difference equations

Using the variables arranged at the grid points as shown in Fig. 1, a set of differential equations (2.1) through (2.9) will be written as follows:

$$\frac{du_2}{dt}-\frac{\alpha}{4\gamma}(u_1+u_2)(w_1+w_2)=-\alpha(\pi_4-\pi_3), \qquad (3.1')$$

$$\frac{du_1}{dt}-\frac{\alpha}{4\gamma}(u_1+u_2)(w_1+w_2)=-\alpha(\pi_2-\pi_1), \qquad (3.2')$$

$$\frac{dw_1}{dt}+\frac{\alpha}{4}(u_1+u_2)(w_1+w_2)=-\frac{\alpha}{\gamma}(\pi_3-\pi_1)+\frac{\beta}{2}(\theta_3+\theta_1), \qquad (3.3')$$

$$\frac{dw_2}{dt}-\frac{\alpha}{4}(u_1+u_2)(w_1+w_2)=-\frac{\alpha}{\gamma}(\pi_4-\pi_2)+\frac{\beta}{2}(\theta_4+\theta_2), \qquad (3.4')$$

$$\frac{d\theta_1}{dt}+\frac{\alpha}{2}u_1(\theta_1+\theta_2)+\frac{\alpha}{2\gamma}w_1(\theta_1+\theta_3)=0, \qquad (3.5)$$

$$\frac{d\theta_2}{dt}-\frac{\alpha}{2}u_1(\theta_1+\theta_2)+\frac{\alpha}{2\gamma}w_2(\theta_2+\theta_4)=0, \qquad (3.6)$$

$$\frac{d\theta_3}{dt} + \frac{\alpha}{2}u_2(\theta_3+\theta_4) - \frac{\alpha}{2\gamma}w_1(\theta_1+\theta_3) = 0, \quad (3.7)$$

$$\frac{d\theta_4}{dt} - \frac{\alpha}{2}u_2(\theta_3+\theta_4) - \frac{\alpha}{2\gamma}w_2(\theta_2+\theta_4) = 0, \quad (3.8)$$

and the continuity equation is represented by

$$\gamma u_2 = -\gamma u_1 = w_1 = -w_2 (\equiv w), \quad (3.9)$$

where the coefficients α and γ are defined as

$$\alpha = \frac{1}{2\Delta x} \quad \text{and} \quad \gamma = \frac{\Delta z}{\Delta x}. \quad (3.10)$$

The grid increments Δx and Δz are all non-dimensional and measured in the unit of d.

From (3.9), we can easily see that the momentum transfer term $(u_1+u_2)(w_1+w_2)$ is zero. Therefore, (3.1′) through (3.4′) become

$$\frac{du_1}{dt} = -\alpha(\pi_2-\pi_1), \quad (3.1)$$

$$\frac{du_2}{dt} = -\alpha(\pi_4-\pi_3), \quad (3.2)$$

$$\frac{dw_1}{dt} = -\frac{\alpha}{\gamma}(\pi_3-\pi_1) + \frac{\beta}{2}(\theta_1+\theta_3), \quad (3.3)$$

and

$$\frac{dw_2}{dt} = -\frac{\alpha}{\gamma}(\pi_4-\pi_2) + \frac{\beta}{2}(\theta_2+\theta_4). \quad (3.4)$$

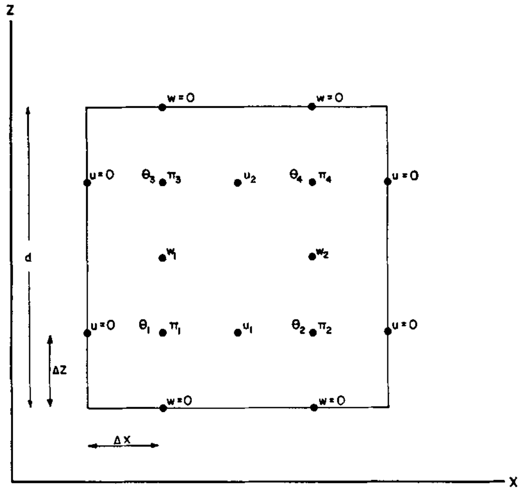

FIG. 1. Model in a grid system.

4. Invariants

From the difference equations, (3.1) through (3.9), we may derive several important conservative quantities such as total mass, total potential temperature, square of potential temperature, differential static stability and total energy.

(*a*) Total mass

The equation (3.9) naturally indicates the conservation of total mass,

$$\gamma u_1 + \gamma u_2 + w_1 + w_2 = 0. \quad (4.1)$$

From (3.5) through (3.8) the following invariants (*b*) through (*d*) are obtained.

(*b*) Total potential temperature

$$\theta_1+\theta_2+\theta_3+\theta_4 = \text{constant} \equiv 4\theta_0 \quad (4.2)$$

(*c*) Square potential temperature

$$\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2 = \text{constant} \quad (4.3)$$

(*d*) Differential static stability

$$(\theta_3-\theta_1)-(\theta_4-\theta_2) = \text{constant} \equiv \Delta S_0. \quad (4.4)$$

The left hand side of (4.4) represents a horizontal difference of static stabilities between inner and outer air.

(*e*) Total energy

Conservation of total energy is obtained from the following two equations

$$\frac{d}{dt}[\tfrac{1}{2}(u_1^2+u_2^2+w_1^2+w_2^2)] = \frac{\beta}{2}w_1(\theta_1-\theta_2+\theta_3-\theta_4), \quad (4.5)$$

which is derived from (3.1′) through (3.4′) and (3.9), and

$$\frac{d}{dt}(\theta_1+\theta_2-\theta_3-\theta_4) = -\frac{\alpha}{\gamma}w_1(\theta_1-\theta_2+\theta_3-\theta_4) \quad (4.6)$$

which is obtained from (3.5) and (3.6) by a simple manipulation. From (4.5) and (4.6) we can derive

$$\tfrac{1}{2}(u_1^2+u_2^2+w_1^2+w_2^2) + \frac{\beta\gamma}{2\alpha}(\theta_1+\theta_2-\theta_3-\theta_4) = \text{constant} \equiv E_0. \quad (4.7)$$

Equation (4.7) will be written in a more convenient form by using (4.2) and (4.4),

$$\tfrac{1}{2}(u_1^2+u_2^2+w_1^2+w_2^2)-\frac{\beta\gamma}{\alpha}\left(\theta_3-\theta_1-\frac{\Delta S_0}{2}\right)=E_0 \qquad (4.8)$$

or by use of (3.9), (4.8) is rewritten as

$$\frac{(1+\gamma^2)}{\gamma^2}w^2-\frac{\beta\gamma}{\alpha}(\theta_3-\theta_1)=E_0-\frac{\beta\gamma}{2\alpha}\Delta S_0. \qquad (4.9)$$

5. Equations of $\bar{\theta}$ and S

For convenience, we will use the notations $\bar{\theta}$ and S defined respectively as

$$\bar{\theta}\equiv(\theta_1+\theta_3)/2 \qquad (5.1)$$

and

$$S\equiv\theta_3-\theta_1. \qquad (5.2)$$

It is clear that $\bar{\theta}$ is the mean temperature and S is the static stability of the air inside cumulus.

The equations of $\bar{\theta}$ and S are easily obtained from (3.5) through (3.7) as

$$\frac{d}{dt}\bar{\theta}=-\frac{\alpha}{2\gamma}\left(S-\frac{\Delta S_0}{2}\right)w \qquad (5.3)$$

and

$$\frac{dS}{dt}=\frac{2\alpha}{\gamma}(\bar{\theta}-\theta_0)w. \qquad (5.4)$$

6. Pressure

For convenience, the notations a, b, c and d are used for the pressure gradients,

$$\left.\begin{aligned} a&\equiv\pi_2-\pi_1,\\ b&\equiv\pi_4-\pi_3,\\ c&\equiv\pi_3-\pi_1,\\ \text{and}\quad d&\equiv\pi_4-\pi_2.\end{aligned}\right\} \qquad (6.1)$$

Relations between them are derived from (3.1) through (3.4) and (3.9) as

$$a+b=0, \qquad (6.2)$$

$$c+d=2\beta\gamma/\alpha\,\theta_0, \qquad (6.3)$$

$$\gamma^2a+c=\beta\gamma/\alpha\,\bar{\theta}, \qquad (6.4)$$

and from (6.1) we also get

$$a-b-c+d=0. \qquad (6.5)$$

The pressure gradients a, b, c and d may be solved from (6.2) through (6.5) as follows

$$a=\frac{\beta\gamma}{\alpha(1+\gamma^2)}(\bar{\theta}-\theta_0), \qquad (6.6)$$

$$b=\frac{-\beta\gamma}{\alpha(1+\gamma^2)}(\bar{\theta}-\theta_0), \qquad (6.7)$$

$$c=\frac{\beta\gamma}{\alpha(1+\gamma^2)}(\bar{\theta}+\gamma^2\theta_0), \qquad (6.8)$$

and

$$d=\frac{-\beta\gamma}{\alpha(1+\gamma^2)}(\bar{\theta}-(2+\gamma^2)\theta_0). \qquad (6.9)$$

7. Equation of w

Substitution of the pressure gradient c, (6.8) into (3.3) with (3.9), (4.2) and (5.1) may lead to the equation,

$$\frac{dw}{dt}=\frac{\beta\gamma^2}{(1+\gamma^2)}(\bar{\theta}-\theta_0). \qquad (7.1)$$

Differentiating (7.1) with respect to t, and substituting $d\bar{\theta}/dt$ of (5.3) into it, we get

$$\frac{d^2w}{dt^2}=-\frac{\alpha\beta\gamma}{2(1+\gamma^2)}\left(S-\frac{\Delta S_0}{2}\right)w. \qquad (7.2)$$

To get the w-equation, we need to eliminate S from (7.2). This will be done by substitution of (4.9) into (7.2) and the result will be

$$\frac{d^2w}{dt^2}=\frac{\alpha^2}{2(1+\gamma^2)}E_0w-\frac{\alpha^2}{2\gamma^2}w^3. \qquad (7.3)$$

8. Discussion on the w-equation

The equations (7.2) and (7.3) represent non-linear oscillation because the static stability S of (7.2) varies with w and $\bar{\theta}$ as shown in (5.3) and (5.4) and because (7.3) contains the term of w^3.

It is important to note that if the static stability is uniform with respect to time and space (7.2) reduced to an equation familiar in the linear stability theory, that is, if we assume that

$$S\doteq\text{constant everywhere}\equiv S_0$$

and therefore $\Delta S_0\doteq 0,$

(7.2) becomes

$$\frac{d^2w}{dt^2} \doteq -\frac{\alpha\beta\gamma}{(1+\gamma^2)} S_0 w. \tag{8.1}$$

Solution of (8.1) will be an exponential function of t for $S_0 < 0$ (unstable) and will be a harmonic function for $S_0 > 0$ (stable).

It is also important to quote that the nonlinear equation (7.3) is approximated by a linear equation of w,

$$\frac{d^2w}{dt^2} \doteq \frac{\alpha^2}{2(1+\gamma^2)} E_0 w \tag{8.2}$$

if w is small or $\gamma > 1$. The condition of small w is the same as that of the perturbation method, and the condition $\gamma (\equiv \Delta z/\Delta x) > 1$ is somewhat similar to that of the parcel method.

From the above discussion, we can speculate the motion of cumulus developed under the condition of instability ($S_0 < 0$) in the following way: initially the vertical and horizontal velocities, while they are all small, will increase exponentially due to conversion of thermal energy into kinetic energy, and during the mature stage, when the velocities are sufficiently large, the motion will become periodical. The periodical motion at and after the mature stage is due to the change of stabilities inside and outside the air of a cumulus. The outside air is acting something like spring attached to the cumulus and the stability of the air is similar to the tension of the spring.

It should be noted that if the lateral boundary is open, different from the present case, $v_n = 0$, the energy dissipation will occur because of the dispersion of gravity waves occurring in the outside stable air.

9. Existence of solution

The w-equation (7.3) is written as follows

$$\frac{dw}{dt} = (c_0 + a_0 w^2 - b_0 w^4)^{\frac{1}{2}} \equiv [g(w)]^{\frac{1}{2}} \tag{9.1}$$

where

$$a_0 = \frac{\alpha^2}{2(1+\gamma^2)} E_0, \tag{9.2}$$

$$b_0 = \frac{\alpha^2}{4\gamma^2}, \tag{9.3}$$

$$c_0 = \left(\frac{dw_0}{dt}\right)^2 - a_0 w_0^2 + b_0 w_0^4 \tag{9.4}$$

and the subscript 0 indicates value at $t = 0$. Solution (real value) of w exists only if $g(w)$ is positive.

(*i*) $c_0 > 0$.

In this case, $g(w) = 0$ has two real and two imaginary roots. The two real roots w_a and w_b are

$$w_a = \left[\frac{a_0 + (a_0^2 + 4b_0 c_0)^{\frac{1}{2}}}{b_0}\right]^{\frac{1}{2}} \equiv w_M^*, \tag{9.5}$$

and $$w_b = -w_M^*. \tag{9.6}$$

Fig. 2 shows $g(w)$ schematically as a function of w. The motion exists only in the region

$$-w_M^* \leqslant w \leqslant w_M^*. \tag{9.7}$$

The condition (9.7) means that the motion can vary continuously from upward to downward and vice versa. This important characteristic is significantly different from the case of $c_0 < 0$ as will be seen in (*ii*). The condition $c_0 > 0$, that is, the first and third terms of (9.4) are predominant, may correspond to the following real atmospheric conditions:

(*a*) external lifting of air or external acceleration such as those associated with fronts, being stronger than the effects of thermal instability, i.e., the terms involving w are larger than the term containing S,

(*b*) weak instability in the well-developed cumulus and cumulonimbus clouds, while the motion is strong.

This case may agree to the empirical fact that a downward current inside a cumulonimbus frequently exists after the mature stage.

(*ii*) $c_0 < 0$.

Roots of $g(w)$ are four reals, w_1, w_2, w_3, and w_4 ($w_1 > w_2 > w_3 > w_4$) as follows:

$$w_1 = -w_4 = \left[\frac{a_0 + (a_0^2 - 4b_0|c_0|)^{\frac{1}{2}}}{b_0}\right]^{\frac{1}{2}} \equiv w_M, \tag{9.8}$$

and

$$w_2 = -w_3 = \left[\frac{a_0 - (a_0^2 - 4b_0|c_0|)^{\frac{1}{2}}}{b_0}\right]^{\frac{1}{2}} \equiv w_m. \tag{9.9}$$

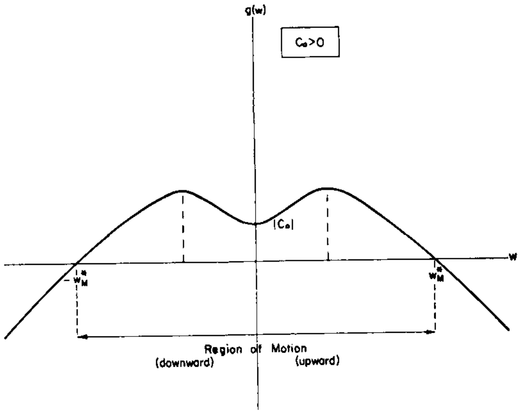

FIG. 2. Existence of solution for the case of $c_0 > 0$.

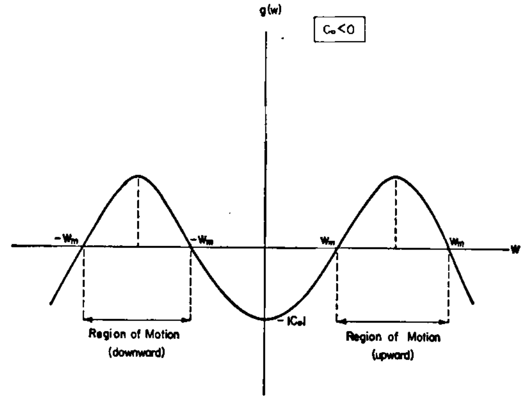

FIG. 3. Existence of solution for the case of $c_0 < 0$.

Fig. 3 shows a schematic characteristic of $g(w)$ as a function of w for $c_0 < 0$. Since the motion only exists for positive values of $g(w)$, the regions of real motion are given by

$$w_m \leqslant w \leqslant w_M \quad \text{(upward motion)}, \tag{9.10}$$

or

$$-w_M \leqslant w \leqslant -w_m \quad \text{(downward motion)}. \tag{9.11}$$

Fig. 3 also shows that the motion always remains in the same direction. In the case of instability, w is always positive. Therefore, only the condition (9.10) is of interest, and the velocity w will oscillate between the values of w_m and w_M, being like a pulsating cumulus-type motion. The quantity c_0 depends on the relative magnitudes among $(dw/dt)^2$, $b_0(w^4)$ and $-a_0(w^2)$ at $t=0$. The condition $c_0 < 0$ is, in general, satisfied if S is a large negative. This condition is similar to the real atmospheric situations of

(*a*) the incipient stage of cumulus growth, or

(*b*) the strong instability due to latent heat release even at the mature stage of cumulus or cumulonimbus.

It should be noted that there is no root if a_0 is negative, i.e., no motion exists under the conditions of negative values of both c_0 and a_0.

10. Analytical solution

Equation (8.1) is written in the standard form of elliptic integral by appropriate manipulation as shown below.

(*i*) $c_0 > 0$.

Equation (8.1) is expressed in terms of b_0, w_M and w_m as

$$t = \int_0^w \frac{dw}{[b_0(w_M^2 - w_m^2)(w^2 + w_m^2)]^{\frac{1}{2}}}. \tag{10.1}$$

Using the new variable ϕ and parameter k defined by

$$w = w_M \cos\phi, \tag{10.2}$$

and

$$k^2 = \frac{w_M^2}{w_M^2 + w_m^2}, \tag{10.3}$$

(9.1) becomes

$$\frac{b_0^{\frac{1}{2}} w_M}{k} t = -\int_0^\phi \frac{d\phi}{(1 - k^2 \sin^2\phi)^{\frac{1}{2}}}. \tag{10.4}$$

The right hand side of (9.4) is a standard form of elliptic integral, the numerical values of which may be easily found in mathematical tables (for instance Yamke and Emde). From (9.4), the period of oscillation T is also calculated

$$T = \frac{4}{[b_0(w_M^2 + w_m^2)]^{\frac{1}{2}}} \int_0^{\pi/2} \frac{d\phi}{(1 - k^2 \sin^2\phi)^{\frac{1}{2}}} \tag{10.5}$$

(*ii*) $c_0 < 0$.

Equation (8.1) in this case is written as

$$t = \int_0^w \frac{dw}{[b_0(w_M^2 - w^2)(w^2 - w_m^2)]^{\frac{1}{2}}}. \tag{10.6}$$

Employing new parameters ϕ and k defined by

$$w = (w_M^2 \cos^2\phi + w_m^2 \sin^2\phi)^{\frac{1}{2}}, \tag{10.7}$$

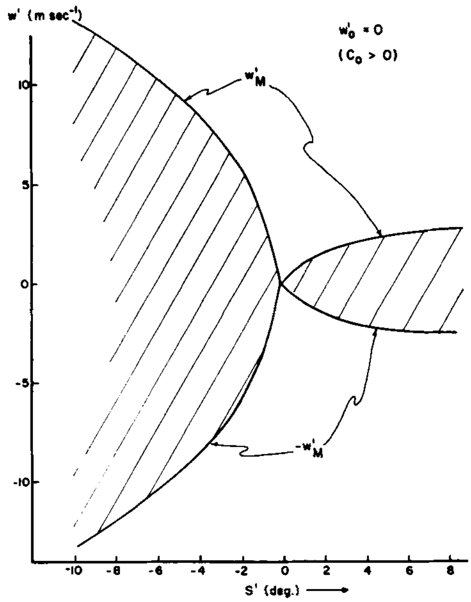

Fig. 4. Dimensional vertical velocity w' as a function of S' (dimensional value of S) for the initial condition of $w_0' = 0$ as a special case of $c_0 > 0$.

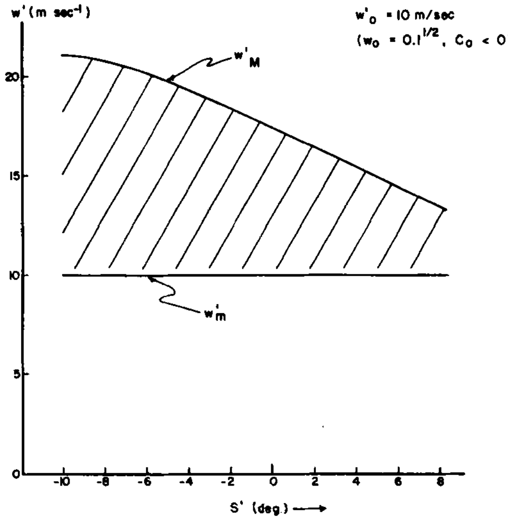

Fig. 5. Dimensional vertical velocity w' as function of S' for the initial condition of $w_0' = 10$ cm sec^{-1} as a special case of $c_0 < 0$.

and
$$\tilde{k}^2 = \frac{w_M^2 - w_m^2}{w_M^2}, \qquad (10.8)$$

(9.6) becomes

$$b_0^{\frac{1}{2}} w_M t = -\int_0^{\phi} \frac{d\phi}{\sqrt{1-\tilde{k}^2 \sin^2\phi}}. \qquad (10.9)$$

The right hand side integral of (9.9) is the same as that of (9.4) except the constant k. The period T for the case of $c_0 < 0$ therefore, is given as

$$T = \frac{4}{b_0^{\frac{1}{2}} w_M} \int_0^{\pi/2} \frac{d\phi}{(1-\tilde{k}^2 \sin^2\phi)^{\frac{1}{2}}}. \qquad (10.10)$$

11. Numerical examples

To estimate the order of magnitude of the maximum and minimum vertical velocities w_M and w_m and the period of oscillation, the following examples are given. Values of constants used in the examples are given below

$$H = 30 \text{ km}, \; d = 1 \text{ km}, \; \tau = 30 \text{ sec},$$

and non dimensional constants are

$$\Delta x = \tfrac{1}{4}, \; \Delta z = \tfrac{1}{4}, \; \alpha = 2, \; \beta = \tfrac{1}{30}, \; \gamma = 1.$$

The values of d and τ satisfy the frequency relationship of Brunt-Väisälä oscillation and the value of β, less than one, satisfies the condition of shallow convection and justifies the basis of (2.1) through (2.4) of this article (Ogura & Phillips 1962).

For simplicity, the initial values are chosen as follows:

$$-b < (\theta_1')_{t=0} < 10 \text{ degree},$$
$$(\theta_2')_{t=0}, \; (\theta_3')_{t=0}, \; (\theta_4')_{t=0} = 0,$$
$$w_0' = 0 \quad \text{or} \quad \pm 10 \text{ m/sec}.$$

The values of w_0 are chosen in view of the observation made by Anderson (1960) who found that the vertical velocity was fluctuating between 6.3 m/sec and -1.2 m/sec. The initial acceleration dw_0/dt is calculated by using (4.2), (5.1) and (7.1) for various values of the initial $(\theta_1)_{t=0}$ as

$$\frac{dw_0}{dt} = [(\bar{\theta})_{t=0} - \theta_0] \qquad (11.1)$$

and
$$(\bar{\theta})_{t=0} = \tfrac{1}{2}[(\theta_1)_{t=0} + (\theta_3)_{t=0}]. \qquad (11.2)$$

Figs. 4 and 5 show the vertical velocities for the cases of $w' = 0$ and $w_0' = 10$ m/sec, respec-

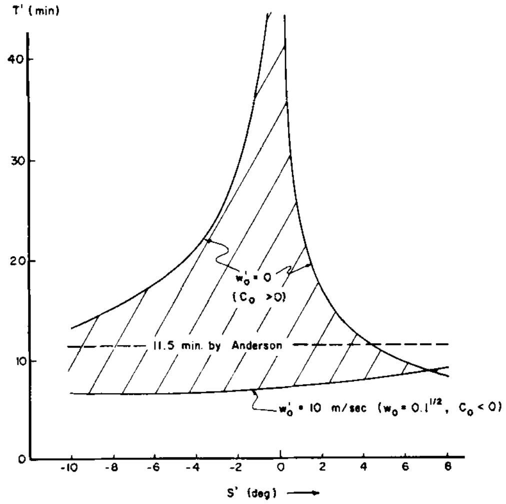

FIG. 6. Dimensional period of oscillation T' versus S' for the cases of $w_0' = 0$ and 10 m sec^{-1}. The period observed by Anderson for cumulus, 11.5 min, is shown by a broken line for a comparison.

tively. In these figures, the ordinate is $(S')_{t=0}$ and the abscissa is w. Domains where the solutions are existent, i.e., $-w_M \leqslant w \leqslant w_M$ in Fig. 4 and $w_m \leqslant w \leqslant w_M$ in Fig. 5 are shaded. Figs. 4 and 5 show two extreme cases; the former is the case where $c_0 > 0$ and the termal energy due to static stability is predominate initially and the latter is the case where $c_0 < 0$ and the kinetic energy is initially predominate. The conditions of the real atmosphere may lie between the above two cases.

Fig. 6 is made to show the period of oscillation for both the cases of $w_0' = 0$ $(c_0 > 0)$ and $w_0' = 10$ m/sec $(c_0 < 0)$. Infinity of T at $(S')_{t=0}$ for the case of $w_0 = 0$ represents no motion forever. Since the realistic initial condition may be between the two cases, the period observed in the real atmosphere may be also between the periods calculated for the above two cases, as shown by shade in Fig. 6. For a comparison the period 11.5 min of cumulus oscillation observed by ANDERSON (1960) is shown by a broken line in Fig. 6. It is interesting to quote that the observed value lies well between the theoretical period calculated for the realistic initial values of $(\theta_1)_{t=0}$ and w_0 except the neighborhood values of $(\theta_1)_{t=0} = 0$ and $w_0 = 0$. He also obtained another significant period of about 1.2 min which may be due to the motion of an individual thermal as an element of a cumulus. The scale of individual thermals may be about 200 meters. It is also expected to obtain the period of a few minutes from (10.5) and (10.10), if we take d to be 200 meters instead of 1 km.

8. Summary and comments

Some major results obtained from the study are listed.

1. In the case $\gamma > 1$, if vertical velocity is not large, the solution becomes equivalent to those obtained from the parcel method.
2. If stability is negative, the disturbance will initially grow exponentially as expected from linear perturbation theories.
3. If stability is positive the motion of disturbances is described by Brunt-Väisälä oscillation.
4. If vertical velocity is large, a nonlinear effect becomes predominant and acts to dampen the vertical motion.
5. In the case of $\gamma < 1$, the condition is nearly hydrostatic, the nonhydrostatic acceleration being proportional to the ratio of the vertical to horizontal scale of disturbances.
6. Throughout all the above cases the most important characteristic of nonlinear interaction processes is due to the change of static stability which is a result of vertical and horizontal heat transport processes.
7. Periodical oscillation of a cumulus is obtained, primarily due to the initial differential stability both inside and outside of the thermal and initial vertical velocity. The period is on the order of ten minutes for $d = 1$ km.
8. The boundary condition is influential in the change of stability in the simplified model. This is somewhat unrealistic; however, the upper and lateral solid boundaries could be interpreted as the condition for which the area considered is surrounded by very stable air.
9. The development of a downward current is possible after a certain time inside a convective system under certain initial conditions of static stability and air motion. Under other conditions, the upward and downward motion will alternate.

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НЕКОТОРЫЕ ДИНАМИЧЕСКИЕ АСПЕКТЫ АТМОСФЕРНОЙ КОНВЕКЦИИ

Исследуются некоторые динамические характеристики атмосферной конвекции с помощью аналитического решения наиболее упращеных дифференциальных уравнений, описывающих двухслойную модель из двух несжимаемых жидкостей. Получено несколько инвариантов, как функции начального значения разницы вертикальной и горизонтальной потенциальной температуры и кинетической энергии. Аналитическое решение получено с использованием этих инвариантов. Одиним из главных приимуществ использования такой упущенной модели является возможность изучения не линейных процессов с помощью аналитического решения. Главным недостатком является низкая разрешимость пространственных производных (временные производные являются точными). Однако инварианты, найденные для этой простой модели, эквивалентны инвариантам полной системы, так что существенные характеристики конвекции описываются, вероятно, рассмотренной упрощенной моделью. Представлены главные результаты, полученные с помощью этой модели.