# Fluctuating winds and the ocean circulation

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ABSTRACT

The fluctuating and steady response of a simple bounded ocean model to a surface wind stress which oscillates at or near one of the Rossby wave resonance frequencies of the basin is computed under the assumption that the response is limited by finite amplitude effects.

It is shown that oscillations of the normal modes can never rectify in a way to produce intense boundary currents, but that motions driven not at a resonance frequency are able to produce such steady boundary layers.

## 1. Introduction

In an earlier paper (PEDLOSKY, 1965), hereafter referred to as I, the response of a simple bounded ocean model to the action of an oscillating wind stress was studied. Both the direct oscillating response and the steady rectified currents produced by certain non-linear interactions were studied as functions of the frequency of the forcing wind stress. In order to investigate the entire relevant frequency range, it was assumed that the dissipative effect of bottom friction dominated the inertial nonlinearities. The consequence of this assumption was that the amplitudes of both the fluctuating and rectified steady circulations were limited only by the dissipation when the frequency of the wind forcing approached a natural frequency of oscillation of the basin.

One of the purposes of this paper is to remove the frictional assumption and analyze the response of the basin near resonance assuming now the alternate possibility, namely, that the response be limited primarily by non-linear effects.

This new alternative may be regarded as simply an attempt to discuss the possible ocean circulations under a more complete range of parametric situations (i.e. primarily viscous or inertial). In addition, the amplitude of the response in the inertial case is not sensitive to poorly understood parameterizations of the dissipative process.

Another purpose of this paper is to study the possible rectified circulations produced by the non-linearities, with a discussion of possible boundary current phenomena. It is shown that although the rectified circulation satisfies a dynamical equation of only first order (a Sverdrup equation), the steady circulations produced by the normal modes *never* lead to a boundary current phenomena, contrary to what is found in the theory of the steady oceanic circulation. This is shown for an arbitrarily shaped basin.

## 2. The model

The simplified ocean model used for the analysis is the same as in (I); it is a uniform and homogeneous sheet of fluid on the $\beta$ plane, on which the fluid motion is two dimensional and incompressible, while the viscous dissipation, assumed to be due to bottom friction, is modeled by a simple velocity drag law. With these assumptions the governing equation for the stream function is the statement of vorticity balance,

$$\nabla^2\psi_t + \beta\psi_x + \psi_x\nabla^2\psi_y - \psi_y\nabla^2\psi_x = \frac{\mathbf{k}\cdot\nabla_x\tau}{\varrho_0 H} - R\nabla^2\psi, \tag{2.1}$$

$\tau$ is the forcing wind stress, $\mathbf{k}$ is a unit vector normal to the plane of motion, $\varrho_0$ is the density of the fluid, $H$ its constant depth while $R$ is the coefficient of viscous drag. $\beta$ is the northward gradient of the Coriolis parameter. The coordinate system is oriented so that $x$ measures eastward, $y$ measures northward. The eastward velocity $u$ and the northward velocity $v$ are derived from $\psi$ by the relations

$$u = -\psi_y, \tag{2.2a}$$

$$v = \psi_x, \tag{2.2b}$$

As in (I) the basin is assumed to be square of side length $L$, on the boundaries of which a zero normal flow condition is imposed i.e.

$$\psi = 0 \quad \text{on} \quad x = 0, L, \qquad y = 0, L, \tag{2.3}$$

This assumption is made for simplicity of analysis only. The extension to an arbitrarily shaped basin introduces no new qualitative features, as will be seen later.

To investigate the response near resonance, non-dimensional variables are introduced in the following manner. If $\tau_0$ is the characteristic amplitude of the wind stress, the non-dimensional variables are defined by the relations

$$\frac{k \cdot \nabla_x \tau}{\varrho_0 H} = \frac{\tau_0}{\varrho_0 HL} T(x, y, t), \tag{2.4}$$

$$(x, y) = L(x', y'), \quad t = (\beta L)^{-1} t',$$

$$\psi = \varepsilon\, \beta L^3\, \Phi, \quad R = \varepsilon^2\, \beta L\, \delta, \quad \delta = 0\,(1).$$

In the above scaling, the parameter $\varepsilon$ which is the relevant Rossby number for the flow is chosen as

$$\varepsilon = \left(\frac{\tau_0}{\varrho_0 H \beta^2 L^3}\right)^{1/3} \tag{2.5}$$

The last scaling relation in (2.4) asserts that the ratio of a Rossby wave period in the basin to the frictional dissipation time is of order $\varepsilon^2$ or less. This assertion can be taken as the precise statement that the response of the model ocean be inertially controlled.

The vorticity equation rewritten after dropping the prime notation for the dimensionless variables is:

$$\left.\begin{aligned} \nabla^2\Phi_t + \Phi_x &= -\varepsilon(\Phi_x \nabla^2 \Phi_y - \Phi_y \nabla^2 \Phi_x) \\ &\quad + \varepsilon^2 (T(x, y, t) - \delta \nabla^2 \Phi), \\ \Phi = 0 \quad &\text{on} \quad x = 0, 1, \qquad y = 0, 1. \end{aligned}\right\} \tag{2.6}$$

## 3. Method of solution

The Rossby number $\varepsilon$ is assumed to be small and the solution to (2.6) will be sought as a series in $\varepsilon$

$$\Phi = \varphi_0 + \varepsilon \varphi_1 + \varepsilon^2 \varphi_2 + \dots. \tag{3.1}$$

In addition, the forcing stress curl, $T(x, y, t)$, is assumed to be purely oscillatory, with a frequency of oscillation $\Omega$. The forcing frequency $\Omega$ is supposed, in the case of resonance, to be near one of the natural frequencies $\omega_{mn}$ of the basin where

$$w_{mn} = \frac{1}{2\pi} \sqrt{m^2 + n^2}, \quad n, m = 1, 2, \cdots \tag{3.2}$$

as was shown in (I). It is convenient to anticipate that as a condition for a uniformly valid expansion of the type (3.1) to exist, a certain relation between $\Omega$ and $\omega_{mn}$ must exist of the form

$$\Omega = \omega_{mn}(1 + \varepsilon^2 \alpha_1 + \varepsilon^4 \alpha_2 + \dots), \tag{3.3}$$

where $\alpha_1, \alpha_2, \dots$ are functions of the amplitude of the motion. Once the $\alpha$'s are determined, (3.3) will serve as the amplitude response curve for driving frequencies $\Omega$ near $\omega_{mn}$. In the theory presented here only $\alpha_1$ will be determined. The higher order effects could in principle be similarly computed.

To determine $\alpha_1$, we proceed in the spirit of LIGHTHILL (1949). $T(x, y, t)$ will be taken in the form

$$T = F_0 \cos(\Omega t - kx + \theta) \sin n\pi y. \tag{3.4}$$

The driving stress is thus a traveling wave moving toward the east, vanishing on the northern and southern boundaries of the ocean. Following Lighthill we introduce a new time scale $\tau$ (not to be confused with the dimensional wind stress) defined by

$$\tau = t(1 + \varepsilon^2 \alpha_1 + \dots). \tag{3.5}$$

The basic technique is to express (2.6) in terms of $\tau$ instead of $t$. This new stretched variable will allow a flexibility which will be needed later. If (3.1), (3.4) and (3.5) are substituted into (2.6) and the coefficients of like powers of $\varepsilon$ are equated, the resulting sequence of linear problems for $\varphi_0$, $\varphi_1$, and $\varphi_2$ are

$$\frac{\partial}{\partial \tau} \nabla^2 \varphi_0 + \varphi_{0x} = 0, \tag{3.7a}$$

$$\frac{\partial}{\partial \tau} \nabla^2 \varphi_1 + \varphi_{1x} = -J(\varphi_0, \nabla^2 \varphi_0), \tag{3.7b}$$

$$\frac{\partial}{\partial\tau}\nabla^2\varphi_2+\varphi_{2x}=-J(\varphi_0,\nabla^2\varphi_1)-J(\varphi_1,\nabla^2\varphi_0)$$
$$+F_0\cos(\omega_{mn}\tau-kx+\theta)\sin\pi y$$
$$-\delta\nabla^2\varphi_0-\alpha_1\frac{\partial}{\partial\tau}\nabla^2\varphi_0. \qquad (3.7\,c)$$

Here $J(f, g)$ is read as the Jacobian of the functions $f$ and $g$ with respect to $x$ and $y$.

The method of solution proceeds in the following manner.

The appropriate solution to (3.7a) satisfying the necessary boundary conditions will be the free normal mode of frequency $\omega_{mn}$, oscillating with the stretched time $\tau$ and can be written

$$\varphi_0=A\cos\left(\omega_{mn}\tau+\frac{x}{2\omega_{mn}}\right)\sin m\pi x\sin n\pi y. \qquad (3.8)$$

The amplitude, $A$, of the solution is not as yet determined, and in fact will not be determined until the problem for $\varphi_2$ is considered. Having found $\varphi_0$ the problem for $\varphi_1$ becomes

$$\nabla^2\varphi_{1\tau}+\varphi_{1x}=\frac{A^2}{4}\left\{\frac{m^2n\pi^3}{\omega_{mn}}\cos\left(2\omega_{mn}\tau+\frac{x}{\omega_{mn}}\right)-\frac{mn\pi^2}{2w_{mn}^2}\sin 2m\pi x\right\}\sin 2n\pi y. \qquad (3.9)$$

The solution for $\varphi_1$ is

$$\varphi_1=\frac{-A^2n\pi}{8\omega_{mn}^2}\sin^2 m\pi x\sin 2n\pi y+\frac{A^2m^2n\pi^3}{4\omega_{mn}^2\varkappa_n^2}\left[\cos\left(\frac{x}{\omega_{mn}}+2\omega_{mn}\tau\right)+\cos\left(2\omega_{mn}\tau+\frac{x}{4\omega_{mn}}\right)\frac{\sinh\alpha_n(x-1)}{\sinh\alpha_n}-\cos\left(2\omega_{mn}\tau+\frac{x-1}{4\omega_{mn}}+\frac{1}{\omega_{mn}}\right)\frac{\sinh\alpha_n x}{\sinh\alpha_n}\right]\times\sin 2n\pi y, \qquad (3.10)$$

where

$$\varkappa_n^2=4n^2\pi^2+(2\omega_{mn}^2)^{-1},\quad \alpha_n^2=4n^2\pi^2-(16\omega_{mn}^2)^{-1}.$$

Now that $\varphi_0$ and $\varphi_1$ are known, the problem for $\varphi_2$ must be examined. In the problem for $\varphi_2$ the inhomogeneity on the right hand side of (3.7c) contains the effects of the non-linear interactions between $\varphi_1$ and $\varphi_0$, the forcing wind stress, the effect of dissipation and a term involving $\alpha_1$. The interactions between $\varphi_1$ and $\varphi_0$ will produce inhomogeneous terms in (3.7c) oscillating at frequencies $\omega_{mn}$ and $3\omega_{mn}$. The forcing field also oscillates at frequency $\omega_{mn}$, as does the viscous term and the term involving $\alpha_1$. Now the operator on the left hand side of (3.7c) has, as one of its characteristic frequencies, the normal mode frequency $\omega_{mn}$. Thus if that part of the inhomogeneity of (3.7c) which oscillates with frequency $\omega_{mn}$ is not orthogonal to the normal modes of the operator on the left hand side of (3.7c) a resonance will occur in the problem for $\varphi_2$. Such an occurrence would invalidate our expansion procedure, for while $\varphi_0$ has a time dependence of the form $\cos(\omega_{mn}\tau)$ the part of the solution for $\varphi_2$ which is forced at frequency $\omega_{mn}$ would have a time dependence of the form $\tau\cos\omega_{mn}\tau$, a so-called secular term. Thus the ratio $\varepsilon^2\varphi_2/\varepsilon\varphi_1=0(\varepsilon\tau)$ and within a time $\tau=\varepsilon^{-1}$ the contribution to the full solution for $\Phi$ of $\varphi_2$ would be as large as that due to $\varphi_1$ and our expansion in the small parameter $\varepsilon$ would be invalid.

To avoid such a circumstance, and to therefore keep our expansion in $\varepsilon$ uniformly valid for large time, it is only necessary to choose $\alpha_1$, which is still at our disposal, to cancel that portion of the inhomogeneity in (3.7c) which oscillates at frequency $\omega_{mn}$ and is not orthogonal to the normal mode with frequency $\omega_{mn}$ of the operator of the left hand side. This freedom to choose $\alpha_1$ is in fact the reason the stretched time $\tau$ was originally introduced. The normal modes of (3.7c), which are the free $\beta$ modes of the basin, were given in (I) and are

$$\varphi_{nm}=\left\{\begin{matrix}\sin(\omega_{mn}\tau+x/2\omega_{mn})\\ \cos(\omega_{mn}\tau+x/2\omega_{mn})\end{matrix}\right\}\sin m\pi x\sin n\pi y. \qquad (3.11)$$

Either of the two forms in (3.11) is a proper normal mode and the inhomogeneity in (3.7c) will contain terms proportional to both. If both such terms in the inhomogeneity are set equal to zero two equations result. The details are straightforward and only the result is quoted here. If the resonant terms (or secular terms) are set equal to zero in the problem for $\varphi_2$ we obtain

$$\mathfrak{F}_{nm}\cos\theta = \frac{\delta A}{8\omega_{mn}^2}, \tag{3.12 a}$$

$$\mathfrak{F}_{nm}\sin\theta = \frac{A^3 N_{mn}}{2} - \frac{\alpha_1 A}{8\omega_{mn}}. \tag{3.12 b}$$

The quantity $\theta$ is the phase between the forcing wind stress and the linear response. The parameter $N_{mn}$ is the coefficient of the contribution to the resonant inhomogeneity of (3.7c) due to the non-linear interactions between $\varphi_0$ and $\varphi_1$. A lengthy calculation shows that $N_{nm}$, which depends only on the structure of the normal mode with frequency $\omega_{nm}$, can be written

$$N_{nm} = \tfrac{1}{2}\left\{\frac{n^2 m^4 \pi^6}{16\omega_{mn}^3 \varkappa_n^2} + \frac{3}{32}\frac{n^2(m^2-n^2)\pi^4}{\omega_{mn}^3}\right.$$
$$+\frac{m^2 n\pi^3}{8\omega_{mn}^2\varkappa_n^2}\left[\frac{\cosh\alpha_n - \cos\dfrac{3}{4\omega_{mn}}}{2\sinh\alpha_n}\right]$$
$$\times\left\{\frac{3\alpha_n n\pi}{8\omega_{mn}^2}\left[\frac{m\pi - 1/2\omega_{mn}}{\alpha_n^2 + \left(2m\pi - \dfrac{3}{4\omega_{mn}}\right)^2}\right.\right.$$
$$\left.-\frac{m\pi + 1/2\omega_{mn}}{\alpha_n^2 + \left(2m\pi + \dfrac{3}{4\omega_{mn}}\right)^2}\right]$$
$$\left.\left.+\frac{2\alpha_n n\pi}{\omega_{mn}(\alpha_n^2 + 9/16\omega_{mn}^2)}\left[\frac{3}{16\omega_{mn}^2} - m^2\pi^2\right]\right\}\right\}. \tag{3.13}$$

The parameter $\mathfrak{F}_{nm}$ is a measure of the projection of the stress function $T$ on the normal mode $\varphi_{nm}$ and it can be shown that $\mathfrak{F}_{nm}$ is related to $F_0$ simply as

$$\mathfrak{F}_{nm} = \frac{F_0}{(m^2\pi^2 - \gamma^2)}[1-(-1)^m\cos\gamma]^{1/2}, \tag{3.14}$$

where $\gamma = k + 1/2\omega_{mn}$.

Equations (3.12a) and (3.12b) can be considered as yielding relations between the phase of the forcing and the amplitude of the response as a function of $\alpha_1$.

If $\theta$ is eliminated from (3.12) a single relation between $\alpha_1$ and $A$ can be found and since $\alpha_1 = (\Omega - \omega_{mn})/\varepsilon^2\omega_{mn}$ we obtain

$$\Omega - \omega_{mn} = \varepsilon^2 A^2 N_{nm} 4\omega_{mn}^2 - \frac{\varepsilon^2 8\omega_{mn}^2}{A}\sqrt{2\mathfrak{F}_{nm}^2 - \frac{A^2\delta^2}{64\omega_{mn}^4}} \tag{3.15}$$

which serves as our frequency response relation. Equation (3.15) may be thought of as yielding a value for the amplitude $A$ as a function of forcing frequency $\Omega$, forcing amplitude $\mathfrak{F}_{nm}$ and viscosity $\delta$.

## 4. Discussion of results

The solution for the stream function $\Phi$, expressed in terms of the real time $t$, may be written:

$$\Phi = A\cos\left(\Omega t + \frac{x}{2\omega_{mn}}\right)\sin m\pi x \sin n\pi y$$
$$-\frac{\varepsilon A^2 n\pi}{8\omega_{mn}^2}\sin^2 m\pi x \sin 2n\pi y$$
$$+\frac{\varepsilon A^2 m^2 n\pi^3}{4\omega_{mn}\varkappa_n^2}\left[\cos\left(2\Omega t + \frac{x}{\omega_{mn}}\right)\right.$$
$$+\cos\left(2\Omega t + \frac{x}{4\omega_{mn}}\right)\frac{\sinh\alpha_n(x-1)}{\sinh\alpha_n}$$
$$\left.-\cos\left(2\Omega t + \frac{x-1}{4\omega_{mn}} + \frac{1}{\omega_{mn}}\right)\frac{\sinh\alpha_n x}{\sinh\alpha_n}\right]\sin 2n\pi y. \tag{4.1}$$

The amplitude $A$ is determined as a function of the forcing frequency $\Omega$ from (3.15).

The structure of the solution is very interesting. The largest part of the solution, the part corresponding to $\varphi_0$, has the spatial structure of the Rossby $\beta$ wave normal mode. The frequency of oscillation, however, is identical to the forcing frequency $\Omega$. This fluctuating part of the solution has lines of constant phase which move to the west with velocity $2\Omega\omega_{mn}$. As discussed in (I) this fluctuating field consists of moving cells of vortical circulation which alternately contract and expand as a moving node of $\varphi_0$ approaches and then passes a fixed node. In addition to this fluctuating response there is a steady response of $O(\varepsilon)$ smaller, in which we are particularly interested, since it arises from a purely oscillating forcing. *There is no east-west asymmetry in the steady circulation, and it does not possess a boundary layer character.* This result is quite general and as will be shown later,

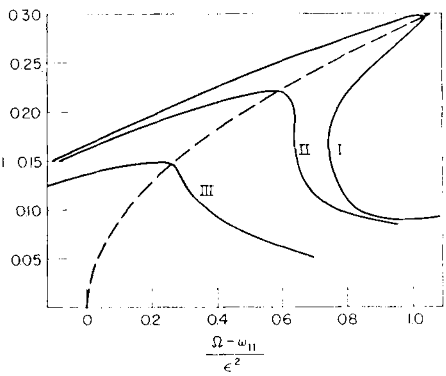


*Fig. 1.* Amplitude response curves for a wind stress forcing near the first resonance frequency $\omega_{11}$. In I, $F_0/\pi = 2$, $\delta = 0.025$, $k = \pi$; in II $F_0/\pi = 2$, $\delta = 0.33$, $k = \pi$, in III $F_0/\pi = 1$, $\delta = 0.25$, $k = \pi$. In the region to the left of the dotted curve $A < 0$, to the right $A > 0$.

only depends on the fact that $\varphi_0$ is a normal mode and is not dependent on the particular basin shape.

Consider, for example, the first mode, $m = n = 1$. The steady circulation consists of two cells, one north of $y = 1/2$ the other south of $y = 1/2$. The circulation in the southern cell is counterclockwise, that in the northern clockwise. The senses of these circulations are unambiguous and do not depend on the parameters of the forcing aside from the frequency.

The shape of the response curve is shown in Fig. 1 for several parameter values in the case where $\Omega$ is near $\omega_{11}$. The form of the response curve is similar to those found for forcing at other normal mode frequencies. The solid curves in Fig. 1 give the amplitude of the response as a function of driving frequency for a fixed forcing amplitude. The response curves have a generally similar appearance to the response curves for a linear system except for the tilt of the entire response curves toward higher frequency. The amount of this tilt depends on the size of $N_{n,m}$, the measure of the non-linear interaction. In the absence of any forcing or dissipation $\{\mathfrak{F}_{nm} = \delta = 0\}$, the frequency of the free oscillation increases with amplitude, departing from the linear oscillation frequency of the normal mode.

The amplitude of the circulation is limited in two distinct ways. It is apparent from (3.15) that the maximum value of the amplitude, $A$ max, allowed by the theory is

$$A_{\max} = \frac{\sqrt{2}\, 8\mathfrak{F}_{nm}\omega_{mn}^2}{\delta}. \tag{4.2}$$

If $A$ is larger than $A_{\max}$ then $\Omega$ would be complex, indicating that the original assumption of a fixed amplitude oscillatory response would be inconsistent with the ensuing viscous dissipation. The limit (4.2) is therefore a viscous limit.

The value of $A_{\max}$ given in (4.2) yields the value of $A$ at the turn over point on the response curve (the dashed curve is also a locus of maximum amplitudes). In fact, if $\delta$ were zero the response curves would not be closed; the dashed lines in Fig. 1 would separate two distinct regimes of oscillation, and there would be no dissipative limit on the amplitude. However since it was originally assumed that $\Omega - \omega_{mn}$ was $O(\varepsilon^2)$, as long as $(\Omega - \omega_{mn})/\varepsilon^2 = 0(1)$ the relation (3.15) will yield finite amplitudes for all $\Omega$ even if $\delta = 0$. This can be seen from (3.15) by noting that for large amplitudes in the absence of viscosity ($\delta = 0$) the response curve asymptotically approaches the dashed curve in Fig. 1. This second limit for the amplitude, due to the tilt of the response curve, is purely inertial. This limit is insensitive to the value of the viscosity as long as that value is small. Another interesting consequence of (3.15) can be observed when the viscosity is small, as in the case of curve 1 in Fig. 1. The response curve is multivalued and in common with similar non-linear mechanical oscillators there is the possibility of interesting hysterisis effects with changing forcing frequency.

## 5. The possibility of boundary layers in the rectified flow

In section 3 it was seen that the solution for the steady part of $\varphi_1$, although it satisfied only a first order equation, was able to satisfy all the boundary conditions without the need of a boundary layer. This important result will now be proven for an arbitrarily shaped basin. The steady part of $\varphi_1$, called here $\varphi_{1s}$, satisfies the equation

$$-\frac{\partial \varphi_{1s}}{\partial x} = +J_s(\varphi_0, \nabla^2\varphi_0) = +(u^0\mathfrak{F}^0)_{sx} + (v^0\mathfrak{F}^0)_{sy}. \tag{5.1}$$

Here the subscript $s$ denotes the time independent part of the labelled quantity. The variable $\mathfrak{Z}^0$ is the vorticity ($\nabla^2\varphi_0$) of the zero order field, $u^0 = -\varphi_{0y}$, $v = +\varphi_{0x}$. Equation (5.1) is identical to the well known Sverdrup equation found in the theories of the steady ocean circulation (see for example STOMMEL, 1955). Here the steady part of the non-linear interactions takes the role of the steady wind stress in the above-mentioned theories. As is well known the crucial question is whether the $x$ integral of the right hand side of the Sverdrup equation across the basin is different from zero. If so, $\varphi_{1s}$ cannot satisfy zero boundary conditions as required on both sides of the basin and a boundary layer of some kind (in this case it would be inertial) would be necessary.

Integrate 5.1 from the left hand boundary, given by $x_L = x_L(y)$ to the right hand boundary $x_R = x_R(y)$ of the basin.

$$-\int_{x_L}^{x_R} \varphi_{1sx}\,dx = (u^{(0)}(x_R)\,\mathfrak{Z}^0(x_R))_s - (u^{(0)}(x_L)\,\mathfrak{Z}^{(0)}(x_L))_s + \frac{\partial}{\partial y}\int_{x_L(y)}^{x_R(y)} (v^0\mathfrak{Z}^0)_s\,dx - \left[(v^0(x_R)\,\mathfrak{Z}^0(x_R))_s\frac{dx_R}{dy} - (v^0(x_L)\,\mathfrak{Z}^0(x_L))_s\frac{dx_L}{dy}\right]. \quad (5.2)$$

Since the zero order field by hypothesis satisfies the condition of zero normal flow

$$u^{(0)} - v^{(0)}\frac{dx}{dy} = 0 \quad \text{at} \quad x = x_R \quad \text{and} \quad x_L. \quad (5.3)$$

So that

$$-\int_{x_L}^{x_R} \varphi_{1sx}\,dx = \frac{\partial}{\partial y}\int_{x_L}^{x_R} (v^0\mathfrak{Z}_1^0)_s\,dx. \quad (5.4)$$

Further, if the zero order motion satisfies the equation for a normal mode

$$\mathfrak{Z}_t^0 + v^0 = 0. \quad (5.5)$$

Therefore

$$(v^0\mathfrak{Z}^0)_s = -\tfrac{1}{2}\left(\frac{\partial \mathfrak{Z}_0^2}{\partial t}\right)_s = 0 \quad (5.6)$$

since the amplitude of the mode is time independent. Thus the inhomogeneity in (5.1) has a zero $x$ average and $\varphi_{1s}$ can satisfy all boundary conditions. Therefore no boundary layer is required if $\varphi_0$ has the structure of the Rossby wave normal modes.

On the other hand if $\varphi_0$ is not a normal mode, and therefore does not satisfy (5.5), the integral of $J_s$ need not be zero. Of course such solutions $\varphi_0$ must be forced solutions at frequencies other than the resonant frequencies, and therefore will generally have smaller amplitudes. They may however give rise to *steady circulations with boundary layers* yielding large velocities where the normal modes yield none, in the boundary region. We may note from (3.10) that $\varphi_{1s}$ has zero tangential as well as normal velocity at the eastern and western boundaries, precisely where a boundary layer has its maximum (tangential) velocity.

An example of such non-resonant forcing yielding a boundary layer is not difficult to discover. Since the forcing is non-resonant we may use the scaling of (I) (i.e. $\varepsilon = \tau_0/\varrho_0 H\,\beta^2 L^3$) to find

$$\nabla^2\varphi_{0t} + \varphi_{0x} = T, \quad (5.7\text{a})$$

$$\nabla^2\varphi_{1t} + \varphi_{1x} = -J(\varphi_0, \nabla^2\varphi_0). \quad (5.7\text{b})$$

Consider the forcing

$$T = F_{mn}[m\pi \cos \omega t \cos m\pi x + (m^2+n^2)\pi^2\omega \sin \omega t \sin m\pi x] \sin n\pi y + F_{lj}[j\pi \cos \omega t \cos j\pi x + (j^2+l^2)\pi^2\omega \sin \omega t \cos j\pi x] \sin l\pi y. \quad (5.8)$$

It may easily be verified that the solution of (5.7a) is

$$\varphi_0 = F_{mn} \cos \omega t \sin m\pi x \sin n\pi y + F_{lj} \cos \omega t \sin j\pi x \sin l\pi y \quad (5.9)$$

while the integral

$$-\int_0^1 J_s(\varphi_0, \nabla^2\varphi_0) = \frac{F_{mn}F_{lj}}{2}[(-1)^{j+m}-1]\left\{\frac{jm(l+n)}{j^2-m^2}\sin(l+n)\pi y + \frac{jm(l-n)}{j^2-m^2}\sin(l-n)\pi y\right\} \quad (5.10)$$

which is not in general zero. This would require, as discussed above, a boundary layer current whose total transport is equal to the right hand side of (5.10). Thus a purely oscillatory wind forcing can produce a steady boundary current by purely non-linear effects.[1]

## 6. Applications of the theory

Consider an ocean for which $L = 5000$ km, $\beta = 10^{-13}$, $H = 5$ km. If, along with VERONIS & STOMMEL (1956) we consider a travelling stress wave with an amplitude $\tau/\varrho = 1$ cm²/sec², $\varepsilon = 10^{-2}$. If $\tau$ represented the scaling for a steady wind stress, this value of $\tau$ would yield values of approximately 0.05 cm/sec for the steady mid-ocean velocities computed from the Sverdrup relation

$$\beta v = \frac{k \cdot \operatorname{curl} \tau}{\varrho_0 H} \tag{6.1}$$

It is of interest therefore to compare this with the values for the circulations produced by a fluctuating stress field of the same amplitude. If we choose a value of $A$ of 0.1, which is seen to be a reasonable value from Fig. 1, this leads to a fluctuating response of approximately 75 cm/sec and a steady circulation of approximately 3 cm/sec to a stress wave with a period of about 5 days. Note that the rectified steady circulation is greater, by two orders of magnitude, than the velocities produced by a steady wind stress of the same strength.

In order that the response be inertially dominated, we required $R \leqslant \varepsilon^2 \beta L$. Actually the slightly weaker condition, verifiable *a posteriori* by the form of the solution, $R \leqslant \pi^2 \varepsilon^2 \beta L$ is really needed. If, from the Ekman theory $R = \sqrt{\nu f/2}\, H^{-1}$ (where $f$ is the Coriolis parameter and $\nu$ the vertical Austausch co-efficient) this latter condition requires an Austauch coefficient less than 10 cm²/sec which is not unreasonable. There is a suggestion therefore, that the effect of an oscillatory wind stress may be important in producing significant steady, as well as fluctuating oceanic velocities.

[1] This result has also been found by Dennis Moore (Private Communication).

A weakness in the theory is of course the assumption that the stress wave, which is meant to represent the presence of migratory cyclone waves, can be represented by an oscillation at a single frequency. This is a very highly simplified picture. In order to deal with a realistic forcing field, the more difficult problem of the response due to a statistically steady continuous spectrum must be obtained. The very high values of the response obtained in this simple model would no doubt be reduced in the more realistic model, but even a reduction of two orders of magnitude will not affect the significance of the rectified circulation. The results, however, of this simple model should be considered as only suggestive.

The model dynamics formulated in this paper are, however, certainly relevant to a physically realizable laboratory experiment, in which fluid contained in a rotating cylinder with a slanted bottom (in which Rossby wave modes, similar to those discussed here, are possible) is driven by an oscillating upper surface. The dimensions of the cylinder, and the degree of the forcing can be adjusted to satisfy the scaling requirements of section 2.

Thus the analysis presented here, while perhaps not *directly* applicable to the oceanic situation, is relevant to the interesting problem of the dynamical behavior of Rossby waves in enclosed basins.

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## ФЛУКТУИРУЮЩИЕ ВЕТРЫ И ОКЕАНИЧЕСКАЯ ЦИРКУЛЯЦИЯ

Вычисляется флуктуирующая и установившаяся реакция простой модели ограниченного океана на поверхностное напряжение ветра, осциллирующее с частотой, равной или близкой к резонансной частоте одной из волн Россби для данного бассейна. Вычисления проведены в предположении, что эта реакция ограничена эффектами конечной амплитуды.

Показано, что колебания с частотами нормальных мод не могут вызывать интенсивных течений вблизи границ, до движения, порождаемые при частотах, отличных от резонансных, способны вызывать установившиеся пограничные слои.