A two-layer model of the Gulf Stream

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ABSTRACT

A theory is developed for a two-layer inertial model of the Gulf Stream. Both layers are in motion, but it is assumed that the ratio of the geostrophic drift in the lower layer to that of the upper layer is small. Approximate analytical solutions are obtained under this assumption. In addition, a criterion for the existence of inertial boundary currents is established. An important result is the prediction of deep and surface countercurrents to the east of the high velocity part of the Stream. These are due to the effect of bottom topography. Another important result is that the interface at the coast comes to the surface at a lower latitude if the deep water is in motion, and that the intersection of the interface and the sea surface extends out to sea in a north-easterly direction from the coast. The theory of the flow near the line of zero upper layer depth is as yet incomplete.

1. Introduction

This paper is about the Gulf Stream, or, more precisely, that portion of the Gulf Stream which lies along the North American continent. In the theory used here (cf. Charney, 1955 and Morgan, 1956) the boundary current is considered to be driven by advection of mass at its seaward edge rather than by the local wind stress. The boundary current can thus be studied as an isolated entity, provided that the mass flux into the current is known.

In Charney's and Morgan's papers a two-layer model was used, the lower layer being assumed motionless. Their results are in good agreement with observations except for an absence in the theoretical results of countercurrents east of the high velocity part of the Stream (see Stommel, 1965, p. 123).

More recent papers have attempted to extend the earlier work to take account of motion of the deep water. Robinson (1965) uses what is essentially a quasi-geostrophic theory for a stream with continuous density and velocity variation with depth. His paper is concerned more with setting up a theoretical framework than with obtaining detailed results. In another recent paper (Blandford, 1965), the earlier two-layer model was modified by considering three layers, the lower one being at rest. Blandford attempted to find a deep countercurrent, presumably due to advection of warm water from low latitudes which causes a zonal temperature contrast, but in this he was unsuccessful.

In the present work a two-layer model is employed with both layers in motion. Thus, by contrast to the other layer models, the effect of topography on the Stream can be included. An analytical solution is obtained under the assumption of small velocities in the lower layer. Some numerical results are also presented.

The effect of topography proves to be very important, for under the assumption that velocities in the lower layer are small, the relative vorticity of the lower layer is negligible except very near the coast; and consequently, the requirement of conservation of potential vorticity in a region of decreasing depth of the lower layer in the shoreward direction implies a deep countercurrent. This proves to be at approximately the same location and with the approximate magnitude of that observed (Stommel, p. 188). A surface countercurrent somewhat east of the deep countercurrent is also predicted, though this is somewhat weaker than observed.

In addition to the foregoing results, a criterion for the existence of inertial boundary

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currents is obtained. This generalizes earlier work by Greenspan (1963) but its conclusions are essentially the same, namely that a sharp northward variation of depth at the seaward edge of the Stream may be incompatible with the existence of an inertial boundary current.

2. Formulation

We consider a two-layer fluid on the β-plane, with Coriolis parameter $f = f_0 + \beta y$, and with coordinates (x, y, z) which measure respectively distance to the east, the north, and in the vertical. The flow is assumed to be steady, inviscid, and geostrophic in the x direction. The fluid is bounded below by bottom topography at $z = b$, above by a free surface at $z = H$, and to the east by a meridional wall at $x = 0$. No mass flux is allowed across the interface, $z = h$.

Let subscripts 1 and 2 denote quantities in the upper and lower layers, and let subscript k when it appears be either 1 or 2. Also, let D_1 and D_2 be the depths of the upper and lower layers at $x = \infty$, $y = 0$, and let Π_1 and Π_2 be defined by

$$H = D_1 + D_2 + \Pi_1 \Delta\varrho/\varrho_2, \tag{1a}$$

$$h = D_1 + \Pi_2 - \Pi_1 \varrho_1/\varrho_2, \tag{1b}$$

where $\Delta\varrho = \varrho_2 - \varrho_1$. Imposing the conditions that the pressure vanishes at the free surface and is continuous at the interface yields

$$p_1/\varrho_1 + gz = g(D_1 + D_2) + g'\Pi_1, \tag{2a}$$

$$p_2/\varrho_2 + gz = g(D_1\varrho_1/\varrho_2 + D_2) + g'\Pi_2, \tag{2b}$$

where $g' = g\Delta\varrho/\varrho_2$, while the definitions of Π_1 and Π_2 lead to

$$H - h = D_1 + \Pi_1 - \Pi_2, \tag{3a}$$

$$h - b = D_2 + \Pi_2 - \Pi_1\varrho_1/\varrho_2 - b$$
$$\approx D_2 + \Pi_2 - \Pi_1 - b. \tag{3b}$$

In the definition of the depth of the lower layer, (3b), we have assumed that $\Delta\varrho/\varrho_2 \ll 1$.

The equations of motion consistent with the above definitions and assumptions are

$$-fv_k + g'\Pi_{kx} = 0, \tag{4}$$

$$u_k v_{kx} + v_k v_{ky} + f u_k + g'\Pi_{ky} = 0, \tag{5}$$

$$[u_1(D_1 + \Pi_1 - \Pi_2)]_x + [v_1(D_1 + \Pi_1 - \Pi_2)]_y = 0, \tag{6}$$

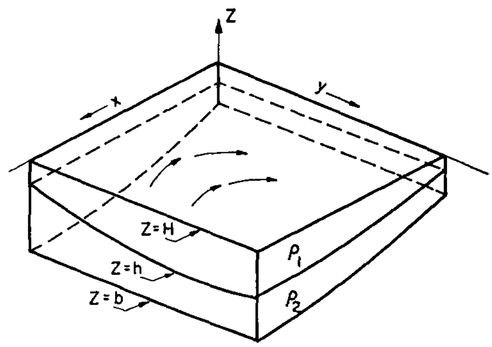

Fig. 1. Flow configuration.

and

$$[u_2(D_2 + \Pi_2 - \Pi_1 - b)]_x + [v_2(D_2 + \Pi_2 - \Pi_1 - b)]_y = 0 \tag{7}$$

Taking $\hat{u}_1(y)$ and $\hat{u}_2(y)$ to be known functions, we require that

$$u_k(\infty, y) = \hat{u}_k(y), \quad v_k(\infty, y) = 0, \tag{8a}$$

i.e., we consider the boundary current to be driven by a known zonal flow at its seaward edge. In addition, the kinematic condition at the coast implies

$$u_k(0, y) = 0. \tag{8b}$$

As a last condition, we require that there be no net transport between the coast and the point $x = \infty$, $y = 0$.

It is convenient now to introduce non-dimensional variables. These are

$$x^* = (f_0/\sqrt{g'D_1})\,x, \quad y^* = (\beta/f_0)\,y,$$
$$u_k^* = (f_0^2/g'\beta D_1)\,u_k, \quad v_k^* = (1/\sqrt{g'D_1})\,v_k,$$
$$\Pi_k^* = (1/D_1)\,\Pi_k, \quad f^* = 1 + y^*, \quad D^* = 1 - b/D_2,$$

D^* being the non-dimensional depth of the lower layer in the absence of flow.

The non-dimensional version of the equations of motion is, with asterisks omitted,

$$-fv_k + \Pi_{kx} = 0, \tag{9}$$

$$u_k v_{kx} + v_k v_{ky} + f u_k + \Pi_{ky} = 0, \tag{10}$$

$$[u_1(1 + \Pi_1 - \Pi_2)]_x + [v_1(1 + \Pi_1 - \Pi_2)]_y = 0, \tag{11}$$

and

$$\{u_2[D + \gamma(\Pi_2 - \Pi_1)]\}_x + \{v_2[D + \gamma(\Pi_2 - \Pi_1)]\}_y = 0, \tag{12}$$

where $\gamma = D_1/D_2$.

This scaling is convenient, but overestimates somewhat the meridional extent of the current. It also overestimates, by a large amount, velocities in the lower layer.

We choose to work with first integrals of the equations of motion rather than with the equations themselves. To this end, we introduce transport stream functions ψ_1 and ψ_2 through

$$\psi_{1x} = v_1[1 + \Pi_1 - \Pi_2], \quad \psi_{1y} = -u_1[1 + \Pi_1 - \Pi_2], \tag{13}$$

$$\psi_{2x} = v_2[D + \gamma(\Pi_2 - \Pi_1)], \quad \psi_{2y} = -u_2[D + \gamma(\Pi_2 - \Pi_1)], \tag{14}$$

and define Bernoulli functions α_1 and α_2 by

$$\alpha_k = \Pi_k + \tfrac{1}{2} v_k^2. \tag{15}$$

It is easily shown that the potential vorticity and Bernoulli function of each layer are constant on streamlines, whence

$$\alpha_k = \alpha_k(\psi_k) \tag{16}$$

and
$$f + v_{1x} = [1 + \Pi_1 - \Pi_2]\,\alpha_1'(\psi_1), \tag{17}$$

$$f + v_{2x} = [D + \gamma(\Pi_2 - \Pi_1)]\,\alpha_2'(\psi_2). \tag{18}$$

Replacing Π_1 and Π_2 in these equations and in

$$\psi_{1x} = v_1[1 + \Pi_1 - \Pi_2], \tag{19}$$

$$\psi_{2x} = v_2[D + \gamma(\Pi_2 - \Pi_1)], \tag{20}$$

through the use of (15) provides a system of four equations in the four unknowns v_1, v_2, ψ_1, ψ_2 in place of the original system. As boundare conditions, we have

$$\psi_k(0, y) = 0 \tag{21a}$$

and also

$$\psi_1(\infty, y) = -\int_0^y \hat{u}_1[1 + \hat{\Pi}_1 - \hat{\Pi}_2]\,dy, \tag{21b}$$

$$\psi_2(\infty, y) = -\int_0^y \hat{u}_2[\hat{D}(y) + \gamma(\hat{\Pi}_2 - \hat{\Pi}_1)]\,dy, \tag{21c}$$

where $\hat{D}(y) = D(\infty, y)$ and where

$$\hat{\Pi}_k(y) = -\int_0^y f\hat{u}_k\,dy = \Pi_k(\infty, y). \tag{22}$$

An alternate integrated form of equations (9)–(12) which is useful for some purposes is

$$f + (1/f)\,\Pi_{1xx} = [1 + \Pi_1 - \Pi_2]\,F_1\left[\Pi_1 + \frac{1}{2f^2}\Pi_{1x}^2\right], \tag{23}$$

$$f + (1/f(\,\Pi_{2xx} = [D + \gamma(\Pi_2 - \Pi_1)]\,F_2\left[\Pi_2 + \frac{1}{2f^2}\Pi_{2x}^2\right]. \tag{24}$$

These follow from the fact that the potential vorticity and Bernoulli function of each layer are constant on streamlines and hence functionally related. Use of the x momentum equations to express v_1 and v_2 in terms of Π_1 and Π_2 leads immediately to the desired result, a set of two equations in the unknowns Π_1 and Π_2 in place of the original system. Since from the y momentum equations

$$\left[\Pi_k + \frac{1}{2f^2}\Pi_{kx}^2\right]_y = -u_k[f + (1/f)\,\Pi_{kxx}], \tag{25}$$

the boundary conditions for this last set of equations are

$$\Pi_k + \frac{1}{2f^2}\Pi_{kx}^2 = 0 \quad \text{at} \quad x = 0, \tag{26}$$

and (22).

3. Analytical solutions

Though the process of numerically integrating the equations is difficult and time consuming, some numerical solutions have been obtained and will be presented below. Here, however, we present an analytical solution. The method of obtaining this is based on the fact that velocities in the lower layer are much smaller than in the upper layer, so that to lowest order the flow in the upper layer and the position of the interface are decoupled from the flow in the lower layer. The flow in the lower layer and higher order approximations in the upper layer are then determined using the method of matched asymptotic expansions (Van Dyke, 1964). We will obtain solutions first for a particularly simple case and then generalize.

3A. *A quasi-linear case*

Suppose the zonal velocities at $x = \infty$ are

$$\hat{u}_1(y) = -1/f - \varepsilon\frac{1+\gamma}{(1-\gamma y)^3} \tag{27a}$$

$$\hat{u}_2(y) = -\varepsilon \frac{1+\gamma}{(1-\gamma y)^3}, \tag{27 b}$$

where ε is a constant, and suppose also that in the open sea the bottom is flat, so that $\hat{D}(y) = 1$. Then

$$\hat{\psi}_1(y) = y + \tfrac{1}{2}\varepsilon \left[\left(\frac{1+y}{1-\gamma y}\right)^2 - 1\right], \tag{28 a}$$

$$\hat{\psi}_2(y) = \varepsilon \left(\frac{1+\gamma}{1-\gamma y}\right) y, \tag{28 b}$$

$$\hat{\Pi}_1 = y + \tfrac{1}{2}\varepsilon \left[\left(\frac{1+y}{1-\gamma y}\right)^2 - 1\right], \tag{28 c}$$

and

$$\hat{\Pi}_2(y) = \tfrac{1}{2}\varepsilon \left[\left(\frac{1+y}{1-\gamma y}\right)^2 - 1\right]. \tag{28 d}$$

It follows that

$$\alpha_1(\psi_1) = \psi_1 \tag{29 a}$$

and

$$\alpha_2(\psi_2) = \tfrac{1}{2}\varepsilon[(1+\psi_2/\varepsilon)^2 - 1]. \tag{29 b}$$

For this flow, the potential vorticity of the upper layer is constant, in agreement with observations (Stommel, p. 111). We exploit this fact in using equation (23) to describe the flow in the upper layer, since (23) is linear in Π_1 for constant potential vorticity. The flow in the lower layer will be described by equations (18) and (20).

Another fact which will be exploited is that $\varepsilon < 1$ if the flow is to be representative of oceanic conditions, since the velocities in the lower layer are much smaller than those in the upper layer. Consequently, the interfacial position must be only weakly dependent on the flow in the lower layer, which thus behaves much like the flow of a one-lower fluid of given depth forced by a zonal velocity at infinity of magnitude ε. This motivates the introduction of a scaled stream function ψ and meridional velocity V through

$$\Psi_2 = \varepsilon\Psi, \quad v_2 = \sqrt{\varepsilon}\, V, \tag{30 a}$$

in which it is assumed that ψ and V are order unity. For notational reasons, we introduce two more new functions ζ and χ through

$$\zeta = \Pi_1, \quad \chi = \Pi_2. \tag{30 b}$$

The equations to be solved can now be written out. They are

$$y + (1/f)\,\zeta_{xx} = \zeta - \chi, \tag{31}$$

$$\sqrt{\varepsilon}\,\Psi_x = [D + \gamma(\chi - \zeta)]\, V, \tag{32}$$

$$\sqrt{\varepsilon}\, V_x = -f + [D + \gamma(\chi - \zeta)](1 + \Psi), \tag{33}$$

$$\chi = \tfrac{1}{2}\varepsilon[(1+\Psi)^2 - 1 - V^2], \tag{34}$$

which are to be solved subject to

$$\zeta + \frac{1}{2f^2}\zeta_x^2 = \Psi = 0 \quad \text{at} \quad x = 0, \tag{35 a}$$

and

$$\zeta = y + \tfrac{1}{2}\varepsilon\left[\left(\frac{1+y}{1-\gamma y}\right)^2, \quad \Psi = \left(\frac{1+\gamma}{1-\gamma y}\right) y, \tag{35 b}$$

at $x = \infty$. When these equations are solved, ψ_1 and v_1 can be computed from

$$v_1 = \zeta_x / f, \quad \psi_1 = \zeta + \tfrac{1}{2} v_1^2. \tag{36}$$

Also, the depth of the upper layer can be obtained; it is $1 + \zeta - \chi$.

The fact that the small constant $\sqrt{\varepsilon}$ multiplies the differentiated terms of (32) and (33) indicates that this is a singular perturbation problem. In what is obviously a boundary layer of thickness $\sqrt{\varepsilon}$ near the coast the variables must depend on a stretched coordinate $\xi = x/\sqrt{\varepsilon}$ in order that the differentiated terms enter into the balance. Away from the coast, the variables are smooth functions of x. In each region the equations may be solved by expanding in powers of $\varepsilon^{\frac{1}{2}}$; in the inner region near the coast the solutions are made to obey the boundary conditions at $x = \xi = 0$, and in the outer region the solutions are made to satisfy the boundary conditions at $x = \infty$. Any remaining ambiguity is resolved by requiring that the limit as $\xi \to \infty$ of an inner solution matches the limit as $x \to 0$ of the corresponding outer solution.

We turn first to the determination of ζ, and in this we let a subscript i denote the inner solution, a subscript o the outer solution, and a superscript the order of the term in the expansion. Thus, when x is order unity, x being the outer variable,

$$\zeta = \zeta_o(x) = \zeta_o^{(0)}(x) + \varepsilon^{\frac{1}{2}}\zeta_o^{(1)}(x) + \varepsilon\zeta_o^{(2)}(x) + \ldots, \tag{37 a}$$

and when x is small, of order $\sqrt{\varepsilon}$,

$$\zeta = \zeta_i(\xi) = \zeta_i^{(0)}(\xi) + \varepsilon^{\frac{1}{2}}\zeta_i^{(1)}(\xi) + \varepsilon\zeta_i^{(2)}(\xi) + \ldots, \tag{37 b}$$

where $\xi = x/\sqrt{\varepsilon}$ is the inner variable.

The process of substituting (37a) and (37b) into the equations, sorting out terms, and obtaining the solutions is straightforward (Jacobs, 1966). Here we present only the results, which give ζ with an error of order $\varepsilon^{\frac{3}{2}}$. These are

$$\zeta^{(0)} = y + a^{(0)} e^{-\sqrt{f}x}, \tag{38}$$

$$\zeta^{(1)} = 0, \tag{39}$$

$$\zeta^{(2)} = \int_0^\infty G^{(2)}(x, x')\, \chi_0^{(2)}(x, x')\, dx', \tag{40}$$

where $\quad a^{(0)} = (1 - y^2)^{\frac{1}{2}} - f,$

and the Green's function $G^{(2)}(x, x')$ is

$$G^{(2)}(x, x') = \frac{f}{\sqrt{1-y}} e^{-\sqrt{f}x_>} \left[\sinh \sqrt{f}x_< + \frac{a^{(0)}}{f} \cosh \sqrt{f}x_>\right], \tag{41}$$

where $x_>$ is the greater and $x_<$ the lesser of (x, x'). These expressions are uniformly valid in x; to this order, the flow in the upper layer does not exhibit boundary layer phenomena.

From the foregoing results, the meridional velocity and transport stream function are computed to be

$$v_1 = -\frac{a^{(0)}}{\sqrt{f}} e^{-\sqrt{f}} + (\varepsilon/f)\, \zeta_x^{(2)} + 0(\varepsilon^{\frac{3}{2}}), \tag{42}$$

and

$$\psi_1 = y + a^{(0)} e^{-\sqrt{f}x} + [(a^{(0)})^2/2f]\, e^{-2\sqrt{f}x} + \varepsilon\{\zeta^{(2)} - (a^{(0)}/f^{\frac{3}{2}})\, \zeta_x^{(2)} e^{-\sqrt{f}x}\} + 0(\varepsilon^{\frac{3}{2}}). \tag{43}$$

Also, the depth at the coast of the upper layer is

$$[1 + \zeta^{(0)} + \varepsilon(\zeta^{(2)} - \chi^{(2)})]_{x=0} = \sqrt{1-y^2} + \varepsilon[\zeta^{(2)}(0) - \chi^{(2)}(0)].$$

We now turn to solution of the lower layer. With an error of order ε, these equations are

$$\sqrt{\varepsilon}\, \Psi_x = [D - \gamma\zeta^{(0)}]\, v \tag{44 a}$$

$$\sqrt{\varepsilon}\, V_x = -f + [D - \gamma\zeta^{(0)}](1 + \Psi). \tag{44 b}$$

In solving (44 a) and (44 b) it is expedient *not* to use the method of matched asymptotic expansions. Instead, let

$$t = \int_0^x (D - \gamma\zeta^{(0)})\, dx;$$

then

$$\sqrt{\varepsilon}\, \Psi_t = V, \tag{45}$$

$$\sqrt{\varepsilon}\, V_t = 1 + \Psi - f/(D - \gamma\zeta^{(0)}), \tag{46}$$

which yield

$$\varepsilon \Psi_{tt} - \Psi = 1 - f/(D - \gamma\varepsilon^{(0)}), \tag{47}$$

which in turn has solution

$$\begin{aligned} \Psi = & -1 + e^{-t/\sqrt{\varepsilon}} \\ & + (f/2\sqrt{\varepsilon}) \left\{ e^{-t/\sqrt{\varepsilon}} \int_0^t \frac{e^{t'/\sqrt{\varepsilon}}}{(D - \gamma\zeta^{(0)})}\, dt' \right. \\ & + e^{t/\sqrt{\varepsilon}} \int_t^\infty \frac{e^{t'-/\sqrt{\varepsilon}}}{(D - \gamma\zeta^{(0)})}\, dt' \\ & \left. - e^{-t/\sqrt{\varepsilon}} \int_0^\infty \frac{e^{-t'/\sqrt{\varepsilon}}}{(D - \gamma\zeta^{(0)})}\, dt'. \right. \end{aligned} \tag{48}$$

Repeated integration by parts yields a series in ascending powers of $\sqrt{\varepsilon}$, of which we keep only the first two terms. The result is

$$\Psi = \left[\frac{f}{D - \gamma\zeta^{(0)}} - 1\right] - \left(\frac{f - \Gamma}{\Gamma}\right) e^{-t/\sqrt{\varepsilon}} + 0(\varepsilon), \tag{49}$$

where Γ is $(D - \gamma\zeta^{(0)})$ evaluated at $x = 0$, i.e., it is the approximate (with an error of order ε) depth of the lower layer at the coast.

Recalling the definitions of t, Ψ, and V, we obtain from the above the uniformly valid solutions

$$\psi_2 = \varepsilon \left\{ \left[\frac{f}{D - \gamma\zeta^{(0)}} - 1\right] - \left(\frac{f - \Gamma}{\Gamma}\right) \times \exp\left[-\frac{1}{\sqrt{\varepsilon}} \int_0^x (D - \gamma\zeta^{(0)})\, dx\right] \right\} + 0(\varepsilon^2), \tag{50}$$

and

$$v_2 = \sqrt{\varepsilon} \left(\frac{f - \Gamma}{\Gamma}\right) \exp\left[-\frac{1}{\sqrt{\varepsilon}} \int_0^x (D - \gamma\zeta^{(0)})\, dx\right] - \varepsilon f \frac{D_x - \gamma\zeta_x^{(0)}}{(D - \gamma\zeta^{(0)})^3} + 0(\varepsilon^{\frac{3}{2}}). \tag{51}$$

Note that the stretched boundary layer variable is not $x/\sqrt{\varepsilon}$ as assumed earlier, but rather

$$(1/\sqrt{\varepsilon}) \int_0^x (D - \gamma\zeta^{(0)})\, dx.$$

Note also that away from the coast

$$\psi_2 = \varepsilon\left[\frac{f}{D-\gamma\zeta^{(0)}} - 1\right] + 0(\varepsilon^2),$$

$$v_2 = 0 + 0(\varepsilon),$$

so the term $\chi_0^{(2)}$ which appears in (51) is

$$\chi_0^{(2)} = \frac{1}{2}\left[\frac{f^2}{(D-\gamma\zeta^{(0)})^2} - 1\right].$$

It is in order now to discuss these results. As noted earlier, the flow in the upper layer and the position of the interface behave to lowest order as if the fluid in the lower layer were at rest. The flow in the lower layer is as if it were the flow of a one-layer fluid of given depth forced by a zonal velocity at infinity of magnitude ε. The meridional velocity in the lower layer is of magnitude ε except in a layer of thickness $\sqrt{\varepsilon}$ near the coast where it is of order $\sqrt{\varepsilon}$. Consequently, the relative vorticity of the lower layer is smaller than the planetary vorticity by a factor of ε except very near the coast, wherè it is comparable to the planetary vorticity.

This has very important consequences, for it means that the direction of the deep meridional velocity is greatly influenced by the topography. Away from the coast the potential vorticity is essentially equal to the planetary vorticity divided by the depth of the layer, as shown by the first term in equation (50), in which, it will be remembered $(D-\gamma\zeta^{(1)})$ is the approximate depth of the lower layer. Consequently, if the depth of this layer decreases in the shoreward direction, the streamlines must deviate to the south in order to conserve potential vorticity. Since in the ocean there are regions in which the depth of the lower layer does decrease in the shoreward direction and in which there is a deep countercurrent (see Stommel, pp. 188–190), the present theory provides a possible explanation for the deep countercurrent.

Near the coast the relative vorticity of the lower layer is not small, and in this region the deep current according to this theory is in the same direction as the surface current.

Turning again to the flow in the upper layer, we note that the correction due to the motion of the lower layer must be important at least near $y = 1$, as seen from (41). Near $y = 1$,

$$G^{(2)}(x, x') \approx -\frac{2}{\sqrt{1-y}}\, e^{-\sqrt{2}(x_> + x_<)},$$

and the non-dimensional depth of the upper layer becomes

$$1+\zeta-\chi \approx 2 + [\sqrt{1-y^2} - 2]\, e^{-\sqrt{2}x} - \frac{2\varepsilon}{\sqrt{1-y}}\, e^{-\sqrt{2}x} \times \int_0^\infty e^{-\sqrt{2}x'} \chi_0^{(2)}(x')\, dx',$$

which vanishes on the line

$$x = \frac{1}{\sqrt{2}} \log\left\{\left[1 - \sqrt{\frac{1-y}{2}}\right] + \frac{\varepsilon}{\sqrt{1-y}} \times \int_0^\infty e^{-\sqrt{2}x'} \chi_0^{(2)}(x')\, dx'\right\}.$$

This intersects the coast at a value of y somewhat smaller than 1,

$$y \approx 1 - \sqrt{2}\,\varepsilon \int_0^\infty e^{-\sqrt{2}x'} \chi_0^{(2)}(x')\, dx',$$

and extends in a north-easterly direction from the point of intersection with the coast. At $y = 1$ the method of solution is invalid, because the correction term due to motion in the lower layer is no longer small and because the non-linear terms in the x momentum equation, which were neglected at the outset, also become large. In fact, the values of u_1 and u_2 become infinite like $+1/\sqrt{1-y}$ as $y \to 1$.

We note also that even for $y < 1$ the present theory is invalid unless $\sqrt{\varepsilon} < \Gamma$, for otherwise the basic approximation that $\zeta < \chi$ does not hold true.

In order to obtain numerical results we must assign numerical values to the constants, pick a representation for the bottom topography, and carry out the integration in (40). We take $y = 0$ to coincide with 15° latitude, and chose $D_1 = 500$ m, $D_2 = 4700$ m, $f_0 = 6.14 \times 10^{-5}$ m^{-1}, $\beta = 2.07 \times 10^{-11}$ m^{-1} sec^{-1}, $\Delta\varrho/\varrho_2 = 2 \times 10^{-3}$, $\varepsilon = 0.04$. Thus x is measured in units of 51 km, y in units of 2970 km, u_1 and u_2 in units of 5.4 cm/sec, and v_1 and v_2 in units of 313 cm/sec. The transports T_1 and T_2 for the two layers are

$$T_1 = (79.8 \times 10^6 \text{ m}^3/\text{sec})\,\psi_1,$$

$$T_2 = (750 \times 10^6 \text{ m}^3/\text{sec})\,\psi_2,$$

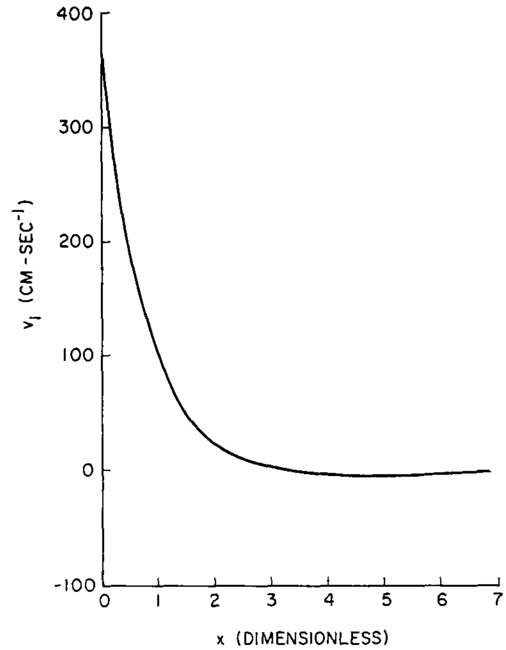

Fig. 2. Meridional velocity of upper layer vs. dimensionless distance from coast at $y = 0.75$.

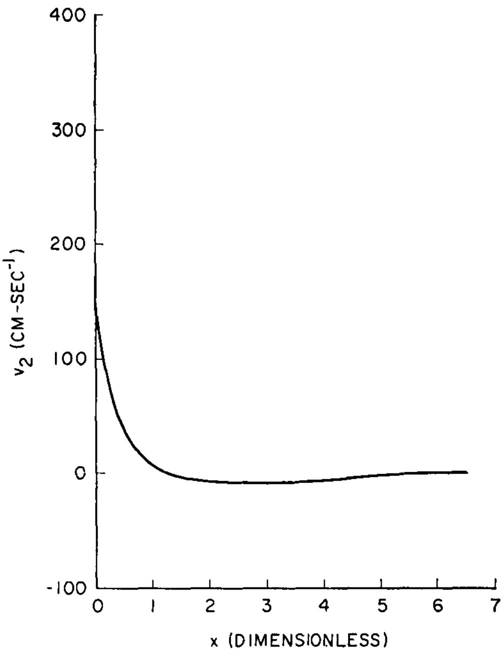

Fig. 3. Meridional velocity of lower layer vs. dimensionless distance from coast at $y = 0.75$.

and with $\varepsilon = 0.04$ this yields

$$T_1 = 63.8 \times 10^6 \text{ m}^3/\text{sec},$$

$$T_2 = 27.0 \times 10^6 \text{ m}^3/\text{sec}$$

at $y = 0.75$, which is close to 35° latitude. These values are at least representative of those for the Gulf Stream.

For the topography, we use a smoothed representation by taking

$$D = 0.5 + x/12, \quad 0 \leqslant x \leqslant 6,$$

$$D = 1, \quad x \geqslant 6.$$

This was chosen more for convenience than for accuracy, though apart from omission of the steep part of the continental shelf it is not unreasonable.

For evaluating the integral in (40), we make use of the fact that $\gamma = D_1/D_2$ is small, of the order of 0.1. Indeed, a theory could be made (and has been, by the author, though it is not presented here) based entirely on the fact that the lower layer is much thicker than the upper. For small γ, we make the approximation

$$\frac{1}{(D-\gamma\zeta^{(0)})^2} = \frac{1}{(D-\gamma y-\gamma a^{(0)}e^{-\sqrt{f}x})^2} \approx \frac{1}{(D-\gamma y)^2} + \frac{2\gamma a^{(0)}}{(D-\gamma y)^3}e^{-\sqrt{f}x}$$

in the integrand of (40), and then find that for a constant slope bottom $\zeta^{(2)}$ can be expressed in terms of elementary functions and various types of exponential integrals, which are tabulated. The expression for $\zeta^{(2)}$ is lengthy and uninformative, and is not presented here.

In Figs. 2 and 3 the velocities v_1 and v_2 are plotted. The maximum value of the deep countercurrent is 7 cm/sec, and it occurs some distance from the coast. There is also a surface countercurrent, but this is weaker than the deep countercurrent, its maximum velocity being 3 cm/sec. The location of the countercurrents appears to be in reasonable agreement with observations, but their magnitudes are too small. These could be increased by using a larger value of ε or a greater amplitude for the bottom topography, but then the method of solution used here becomes invalid.

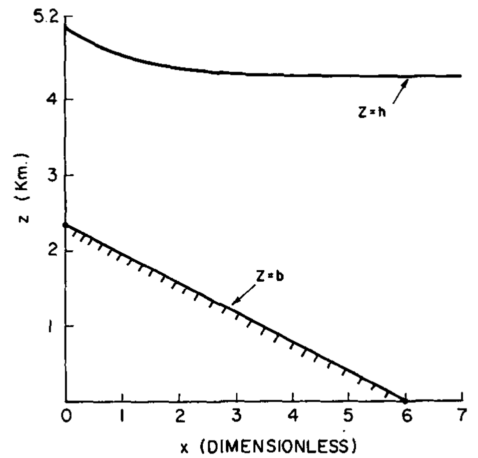

Fig. 4. Position of interface at $y = 0.75$.

The position of the interface is shown in Fig. 4. As can be seen, there is no indication of a warm core, which would be characterized by a decrease of the depth of the upper layer in the seaward direction.

3.B. *General theory*

We turn now to the task of generalizing the previous results, and in this we assume only that the magnitude of $\hat{u}_1$ is of order unity, while that of $\hat{u}_2$ is of order ε. It follows that $\alpha_1(\psi_1)$ will be of the form

$$\alpha_1(\psi_1) = A(\psi_1) + \varepsilon B(\psi_1) + 0(\varepsilon^2),$$

and that

$$\alpha_2(\psi_2) = \varepsilon C(\psi_2/\varepsilon) + 0(\varepsilon^2);$$

here A, B, and C are functions with amplitude of order unity.

Defining Ψ, V, ζ and χ as before, and again letting superscripts denote the order of a term in expansions in powers of $\varepsilon^{1/2}$, we have

$$\Psi_{1x} = v_1(1 + \zeta - \chi), \tag{52}$$

$$v_{1x} = -f + (1 + \zeta - \chi)(A'(\psi_1) + \varepsilon B'(\psi_1) + \dots), \tag{53}$$

$$\sqrt{\varepsilon}\,\Psi_x = V(D - \gamma\zeta + \gamma\chi), \tag{54}$$

$$\sqrt{\varepsilon}\,V_x = -f + (D - \gamma\zeta + \gamma\chi)(C'(\Psi) + \dots), \tag{55}$$

where $$\zeta = A(\psi_1) + \varepsilon B(\psi_1) + \dots - \tfrac{1}{2} v_1^2, \tag{56}$$

$$\chi = \varepsilon[C(\Psi) + \dots - \tfrac{1}{2} V^2], \tag{57}$$

to be solved subject to

$$\left.\begin{aligned} \psi_1(0, y) &= \Psi(0, y) = 0, \\ \psi_1(\infty, y) &= \hat{\psi}_1(y), \\ \Psi(\infty, y) &= \hat{\Psi}_1(y). \end{aligned}\right\} \tag{58}$$

In discussing the top layer, we use the results found previously, that with an error of order $\varepsilon^{1/2}$ the variables in the top layer do not exhibit boundary layer character, and furthermore have no term proportional to $\varepsilon^{1/2}$ in their expansion. Also, we use

$$f v_1 = \zeta_x,$$

which comes from the x momentum equation.

Now $$\psi_{1x} = \zeta_x(1 + \zeta)/f - v\chi, \tag{59}$$

so $$\psi_1 = (\zeta + \tfrac{1}{2}\zeta^2)/f + \int_x^\infty v\chi\,dx + b. \tag{60}$$

Here b is a function of y which is determined by the values of ψ_1 and ζ at $x = \infty$. In general, it is of the form

$$b = b^{(0)} + \varepsilon b^{(2)} + \dots.$$

Inserting perturbation series and sorting out terms, we arrive at

$$\psi_1^{(0)} = \zeta^{(0)}(1 + \tfrac{1}{2}\zeta^{(0)})/f + b^{(0)}, \tag{61}$$

$$\psi_1^{(2)} = [(1 + \zeta^{(0)})/f]\,\zeta^{(2)} + \int_x^\infty v_1^{(0)}\chi_0^{(2)}\,dx + b^{(2)}. \tag{62}$$

Since

$$A(\psi_1) + \varepsilon B(\psi_1) = A(\psi_1^{(0)}) + \varepsilon[\psi_1^{(2)} A'(\psi_1^{(0)}) + B(\psi_1^{(0)})] + 0(\varepsilon^2), \tag{63}$$

and since

$$f + v_{1x}^{(0)} = (1 + \zeta^{(0)})\,A'(\psi_1^{(0)}), \tag{64}$$

we have from (66)

$$\zeta^{(0)} = A(\psi_1^{(0)}) - (\zeta_x^{(0)})^2/2f^2 \tag{65}$$

and

$$\zeta^{(1)} = \psi_1^{(2)}(f + V_{1x}^{(0)})/(1 + \zeta^{(0)}) + B(\psi_1^{(0)}) - v_1^{(0)}\zeta_x^{(2)}/f. \tag{66}$$

Consequently, from (61) and (65), we obtain

$$\zeta^{(0)} + (\zeta_x^{(0)})^2/2f^2 = A[\zeta^{(0)}(1 + \tfrac{1}{2}\zeta^{(0)})/f + b^{(0)}], \tag{67}$$

a differential equation which can be solved by quadrature, while the equation for $\zeta^{(2)}$ becomes

$$(\zeta^{(2)}/v_1^{(0)})_x = [f/(v_1^{(0)})^2] \times \left\{ B(\psi_1^{(0)}) + A'(\psi_1^{(0)}) \left[\int_x^\infty v_1^{(0)} \chi_0^{(2)} dx + b^{(2)} \right] \right\}, \tag{68}$$

which can also be solved by quadrature. This completes the solution for the flow in the upper layer, since from the solutions for $\zeta^{(0)}$ from (67) and for $\zeta^{(2)}$ from (68) the other variables of interest can be calculated.

If we let

$$t = \int_0^x (D - \gamma\zeta^{(0)})\, dx$$

as before and neglect terms of order ε, the equations describing the flow in the lower layer become

$$\sqrt{\varepsilon}\Psi_t = V, \tag{69}$$

$$\sqrt{\varepsilon}\, V_t = D'(\Psi) - f/G(t), \tag{70}$$

where $G(t) = D - \gamma\zeta^{(0)}$.

These will be solved by the method of matched asymptotic expansions, the inner variable being $\tau = t/\sqrt{\varepsilon}$. Thus when t is order unity,

$$\Psi = \Psi_0^{(0)}(t) + \sqrt{\varepsilon}\Psi_0^{(1)}(t) + \dots,$$

and when t is small, of order $\sqrt{\varepsilon}$,

$$\Psi = \Psi_i^{(0)}(\tau) + \sqrt{\varepsilon}\Psi_i^{(1)}(\tau) + \dots,$$

and V is treated in the same way.

In the outer region,

$$V_0^{(0)} = 0, \quad C'(\Psi_0^{(0)}) = f/G(t), \tag{71}$$

and
$$V_0^{(1)} = \Psi_{0t}^{(0)}, \quad \Psi_0^{(1)} = 0. \tag{72}$$

Bearing in mind the definition of $G(t)$, this indicates that as in the case treated in 3.A the velocity in the lower layer is found by requiring conservation of f divided by the depth of the layer.

For matching the outer solution to the inner solution, we need the results that when t is order $\sqrt{\varepsilon}$,

$$\Psi_0 \approx \Psi_0^{(0)}(0) + \sqrt{\varepsilon}\tau\Psi_{0t}^{(0)}(0),$$

$$\frac{1}{G(t)} \approx \frac{1}{G(0)} - \sqrt{\varepsilon}G'(0)\tau/G^2(0),$$

so
$$C'[\Psi_0^{(0)}(0)] = f/G(0),$$

$$C''[\Psi_0^{(0)}(0)] = -fG'(0)/[G^2(0)\,\Psi_{0t}^{(0)}(0)].$$

In the inner region,

$$\Psi_{i\tau} = V_i, \tag{73}$$

$$V_{i\tau} = C'(\Psi_i) - f/G(\sqrt{\varepsilon}\tau), \tag{74}$$

which combine to

$$\Psi_{i\tau\tau} = C'(\Psi_i) - f/G(\sqrt{\varepsilon}\tau). \tag{75}$$

Consequently,

$$\Psi_{i\tau\tau}^{(0)} = C'(\Psi_i^{(0)}) - f/G(0), \tag{65}$$

$$\Psi_{i\tau\tau}^{(1)} = \Psi_i^{(1)}C''(\Psi_i^{(0)}) + fG'(0)\,\tau/G^2(0). \tag{77}$$

Multiplying (76) by $\Psi_{i\tau}^{(0)}$ and integrating, we obtain

$$\tfrac{1}{2}(\Psi_{i\tau}^{(0)})^2 = C(\Psi_i^{(0)}) - f\Psi_i^{(0)}/G(0) + d^{(0)}. \tag{78}$$

Here $d^{(0)}$ is a constant of integration which is found by using

$$\Psi_i^{(0)} \to \Psi_0^{(0)}(0) \quad \text{as} \quad \tau\to\infty.$$

Using the value of $d^{(0)}$ obtained in this way, we can then solve (78) by quadrature, using the boundary condition at $\tau = 0$.

Before solving (77), we note that as $\tau\to\infty$, $\Psi_i^{(1)} \to \tau\Psi_{0\tau}^{(0)}(0)$, hence

$$\Psi_i^{(1)}C''(\Psi_i^{(0)}) \to -fG'(0)\,\tau/G^2(0),$$

as it should. Now

$$C''(\Psi_i^{(0)}) = \frac{1}{v_i^{(0)}}[C'(\Psi_i^{(0)})]_\tau = V_{i\tau\tau}^{(0)}/V_i^{(0)},$$

hence (77) becomes

$$[V_i^{(0)}\Psi_{i\tau}^{(1)} - V_{i\tau}^{(0)}\Psi_i^{(1)}]_\tau = fG'(0)\,\tau V_i^{(0)}/G^2(0), \tag{79}$$

and integration together with application of the matching condition yields

$$[\Psi_i^{(1)}/V_i^{(0)}]_\tau = -[fG'(0)/(G(0)\,V_i^{(0)})^2] \int_\tau^\infty \tau V_i^{(0)}\, d\tau, \tag{80}$$

so the solution of this equation also is obtained by quadrature. Hence the equations of the lower layer are solved, and a uniformly valid solution can be constructed.

It is important to note that as in 3.A the relative vorticity of the lower layer is negligible except very near the coast. Consequently, away from the coast, a decrease of depth of the lower layer in the shoreward direction implies southward motion of the deep water.

4. Numerical results

In order to check the analysis an attempt has been made to obtain numerical solutions for the case discussed in 3.A. The method consists of guessing values of v_1 and v_2 at $x = 0$ and then integrating. A Runge-Kutta scheme with spatial steps of 0.005 was used.

In general the initial guesses will be incorrect and the integration must be repeated with different values of $v_1(0)$ and $v_2(0)$ until the solutions appear to obey the boundary conditions at $x = \infty$. The integration is easier if topography is ignored, for then it can be shown that

$$\Lambda = f(\psi_1 + \psi_2/\gamma) - \Pi_1 - \Pi_2/\gamma - \tfrac{1}{2}(\Pi_1 - \Pi_2)^2$$

is independent of x. Since Λ is a function only of y it can be computed from the known conditions at $x = \infty$. This provides a relation between v_1 and v_2 so that (say) only $v_2(0)$ need be guessed. No such simplification was found for the topographic case.

It is difficult to say whether the numerical integrations represent true solutions because in many cases the numerical solutions diverge with x. It is felt that this is due to incorrect values of $v_1(0)$ and $v_2(0)$, since the solutions are extremely sensitive to the initial conditions. The results of a typical calculation are shown in Fig. 5, which represents the best result that could be obtained with a reasonable amount of effort. It is apparent that for x greater than 2.5 the numerical solution is inaccurate. In Table 1 the values of $v_1(0)$ and $v_2(0)$ as found analytically and numerically are compared for different values of y. The agreement is much better for the non-topographic case, in which both the analytical and numerical solutions are more accurate.

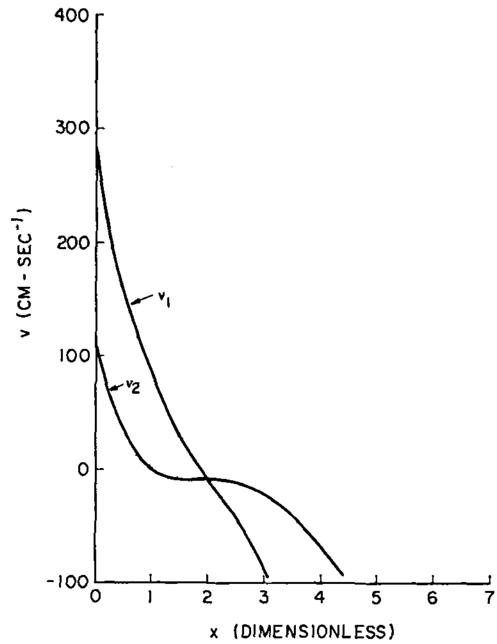

Fig. 5. Numerical solution for meridional velocities at $y = 0.75$.

Table 1. *Comparison of analytical and numerical results*

Velocities are given in cm-sec^{-1}

y	Analytic $v_1(0)$	Analytic $v_2(0)$	Numerical $v_1(0)$	Numerical $v_2(0)$
Non-topographical case				
.25	83	16	82	16
.50	174	32	171	31
.75	285	46	265	45
Topographical case				
.25	105	82	79	82
.50	215	114	178	111
.75	364	152	284	119

In view of the great difficulty in obtaining numerical solutions the principle conclusions of this paper must rest on the analytical work of the previous section. The numerical results serve to check some of the qualitative features of the analysis, however, and for this reason have been presented.

5. Concluding remarks

An interesting result is the separation of the boundary current from the coast at a slightly smaller value of y than that predicted by the Charney-Morgan theory and the existence of large positive zonal velocities in both layers near the separation latitude. The flow in the region near this latitude is not accurately described by the present theory. In order to obtain such a description and thus to treat the portion of the Gulf Stream northward and eastward of Cape Hatteras a much more extensive theory is necessary.

The present work is applicable south of the separation latitude and appears to account for a number of observed features of the Stream. However, frictional effects, which have been ignored here, are undoubtedly important at least very near the coast. An extension of the present theory by inclusion of frictional effects would serve to put the work on a firmer basis.

Acknowledgments

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APPENDIX

An existence criterion

A necessary condition for the existence of solutions can be obtained by linearization about the flow at infinity. The solutions of the linearized equations are then examined to see if they decay with x, as they should.

We will use the dimensional form of (23) and (24) for this purpose,

$$\frac{f+(g'/f)\Pi_{1xx}}{D_1+\Pi_1-\Pi_2} = F_1\left[\Pi_1+\frac{g'}{2f^2}\Pi_{1x}^2\right], \quad (A.1)$$

$$\frac{f+(g'/f)\Pi_{2xx}}{D_2+\Pi_2-\Pi_1-b} = F_2\left[\Pi_2+\frac{g'}{2f^2}\Pi_{2x}^2\right], \quad (A.2)$$

and linearize taking $\varphi_k=\Pi_k-\hat{\Pi}_k$ to be small. We will also take $b=\hat{b}(y)$, thus assuming that there is no appreciable dependence of the topography on x far from the coast.

In what follows, we let

$$\left.\begin{aligned} P_1 &= f/[g'(D_1+\hat{\Pi}_1+\hat{\Pi}_2)], \\ P_2 &= f/[g'(D_2^2+\hat{\Pi}-\hat{\Pi}_1-\hat{b})], \end{aligned}\right\} \quad (A.3)$$

and remember that

$$f\hat{u}_k+g'\hat{\Pi}_{ky}=0. \quad (A.4)$$

Then

$$\begin{aligned} F_k[\Pi_k+g'/2f^2\Pi_{kx}^2] &\approx F_k[\Pi_k+\varphi_k] \\ &\approx F_k(\hat{\Pi}_k)+\varphi_k F_k'(\hat{\Pi}_k) \\ &= F_k(\hat{\Pi}_k)+(\varphi_k/\hat{\Pi}_{ky})\,\partial/\partial y\, F_k(\hat{\Pi}_k) \\ &= g'[P_k-(g'\varphi_k/g\hat{u}_k)P_{ky}]. \quad (A.5) \end{aligned}$$

We now let

$$Q_k=(1/\hat{u}_k)\,\partial/\partial y \log P_k \quad (A.6)$$

and obtain, after linearization of the left sides of (A.1) and (A.2),

$$(1/f)\varphi_{1xx}+(Q_1-P_1)\varphi_1+P_2\varphi_2=0, \quad (A.7)$$

$$(1/f)\varphi_{2xx}+(Q_2-P_2)\varphi_2+P_1\varphi_1=0. \quad (A.8)$$

Assuming solutions of the form $\exp[f^{\frac{1}{2}}\lambda x]$ leads to the characteristic equation

$$\lambda^4-[(P_1-Q_1)+(P_2-Q_2)]\lambda^2 + [(P_1-Q_1)(P_2-Q_2)-P_1P_2]=0, \quad (A.9)$$

which in turn leads to

$$2\lambda^2=(P_1-Q_1)+(P_2-Q_2)\pm\{[(P_1-Q_1) + (P_2-Q_2)]^2-4[(P_1-Q_1)\times(P_2-Q_2)-P_1P_2]\}^{\frac{1}{2}}. \quad (A.10)$$

The expression inside the curly bracket is easily shown to be positive, and hence the eigenvalues λ are either pure real or pure imaginary. In order that they be real, so that the solutions decay exponentially with x,

$$\begin{aligned} (P_1-Q_1)+(P_2-Q_2) &> 0, \\ (P_1-Q_1)(P_2-Q_2) &> P_1P_2, \end{aligned} \quad (A.11)$$

which is the desired criterion.

Now let

$$\delta_1 = D_1 + \hat{\Pi}_1 - \hat{\Pi}_2,$$

$$\delta_2 = D_2 + \hat{\Pi}_2 - \hat{\Pi}_1 - \hat{b};$$

these are the depths of the upper and lower layers at $x = \infty$. Using (A.4) and the fact that each of the terms $(P_1 - Q_1)$, $(P_2 - Q_2)$ must be separately positive, we arrive at

$$\hat{u}_1 \hat{u}_2 > g' \hat{u}_1 \beta \delta_1 / f^2, \quad \text{(A.13a)}$$

$$\hat{u}_1 \hat{u}_2 > g' \hat{u}_2 (\beta \delta_2 + f \hat{b}_y)/f^2, \quad \text{(A.13b)}$$

and

$$\hat{u}_1 \hat{u}_2 [\beta(\hat{u}_1 \delta_1 + \hat{u}_2 \delta_2) + f \hat{u}_2 \hat{b}_y] < \hat{u}_1 \hat{u}_2 g' [(\beta \delta_1)(\beta \delta_2 + f \hat{b}_y)]/f^2. \quad \text{(A.13c)}$$

It is now a matter of working through the inequalities to find that:

(1) for $\beta \delta_2 + f \hat{b}_y \geqslant 0$

$$\hat{u}_1 < 0, \quad u_2 < 0 \quad \text{(A.14)}$$

and

(2) for $\beta \delta_2 + f \hat{b}_y < 0$,

$$\beta(\hat{u}_1 \delta_1 + \hat{u}_2 \delta_2) + f u_2 \hat{b}_y < g'(\beta \delta_1)(\beta \delta_2 + f \hat{b}_y)/f^2 \quad (u_1 u_2 > 0); \quad \text{(A.15)}$$

and

$$g'(\beta \delta_1)(\beta \delta_2 + f \hat{b}_y)/f^2 < \beta(\hat{u}_1 \delta_1 + \hat{u}_2 \delta_2) + f u_2 \hat{b}_y < 0 \quad (\hat{u}_1 \hat{u}_2 < 0). \quad \text{(A.16)}$$

These conditions are subsumed by

$$\beta(\hat{u}_1 \delta_1 + \hat{u}_2 \delta_2) + f \hat{u}_2 \hat{b}_y < 0, \quad \text{(A.17)}$$

which is a special case of a result proved by Pedloskey (1965) for a baroclinic fluid and which reduces to Greenspan's criterion if $\hat{u}_1 = \hat{u}_2$. It should be noted that a rapid variation of topography with y can be highly important even though $\hat{u}_2$ is in general quite small.

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ДВУХСЛОЙНАЯ МОДЕЛЬ ГОЛЬФСТРИМА

Развивается теория для двухслойной инерционной модели Гольфстрима. Оба слоя находятся в движении, однако, предполагается, что отношение геострофического смещения в нижнем слое к аналогичному смещению в верхнем слое мало. При этих условиях получаются приближенные аналитические решения. Дополнительно устанавливается критерий существования инерционных граничных течений. Важным результатом является предсказание глубокого и поверхонстного противотечений к востоку от той области течения, в которой наблюдаются большие скорости. Они представляют собою эффект придонной топографии. Другой важный результат состоит в том, что поверхность раздела выходит у берега на поверхность в низких широтах, если глубинная вода находится в движении, и линия пересечения поверхности раздела с поверхностью моря уходит в море в северо-восточном направлении от берега. Теория течения вблизи линии нулевой глубины верхнего слоя все еще не полна.