SHORTER CONTRIBUTION

Equatorial planetary waves

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The mass transport stream function

$$\Phi = r^k \sin^k\theta e^{i(k\varphi - \sigma t)} \tag{1}$$

will be shown to be a non-hydrostatic exact solution of the inviscid equations of motion in a rotating spherical shell of fluid. In (1) r, θ, and φ are spherical polar coordinates, k is the zonal wave number, and σ is the frequency of oscillation. The "shellular" modes Φ are a subset of the linear free oscillations in a rotating homogeneous fluid spheroid. The full set has been studied by Bryan (1888), Cartan (1922), Stewartson & Roberts (1963), and recently by Greenspan (1964, 1965). The isolation of the subset Φ as solutions of the shell problem, and in a radially stratified fluid as well, was discussed in a recent paper on hydromagnetic planetary waves by Malkus (1967). A rederivation of the spheroidal free oscillations here would seem to have small merit for meteorological or oceanographic studies. However, an exhibit of the character of the shellular modes may lend support to the approximate two dimensional solutions which have been proposed in recent literature (e.g. Longuet-Higgins, 1966, 1967).

The components of the mass transport vector are

$$V_r = 0, \quad V_\theta = \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\Phi, \quad V_\varphi = -\frac{\partial}{\partial\theta}\Phi, \tag{2}$$

hence $$\nabla\cdot\mathbf{V} = 0. \tag{3}$$

Since the divergence of $\mathbf{V}$ vanishes, the density associated with equation (1) cannot be a function of time. We restrict attention to a density field which is a function of r only. The equations of motion then may be written

$$\frac{\partial\mathbf{V}}{\partial t} + 2\hat{z}\times\mathbf{V} - \nabla P = \tfrac{1}{2}[\mathbf{V}\times\nabla\times(\mathbf{V}/\varrho) + (\mathbf{V}/\varrho)\times\nabla\times\mathbf{V}], \tag{4}$$

where a non-dimensionalization has been made in terms of a characteristic radius R, a characteristic density $\varrho(R)$ and the angular rotation rate Ω:

$$\mathbf{x}^1 = R\mathbf{x}, \quad \varrho\mathbf{v} = [\varrho(R)\,\Omega R]\,\mathbf{V}, \quad t^1 = \Omega^{-1}t,$$

$$P^1 = \frac{P}{\varrho(R)\,\Omega^2 R}, \tag{5}$$

where $\mathbf{v}$ is the vector velocity of the flow, $\hat{z}$ is a unit vector along the axis of rotation, and P incorporates the (radial) gravitational potential and the dynamic terms.

The components of (4) are written in terms of Φ and P, using (2), as

$$2\sin\theta\frac{\partial}{\partial\theta}\Phi = \frac{\partial P}{\partial r} + N_1, \tag{6}$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\frac{\partial\Phi}{\partial t} + 2\cos\theta\frac{\partial}{\partial\theta}\Phi = \frac{1}{r}\frac{\partial P}{\partial\varphi} + N_2, \tag{7}$$

$$-\frac{\partial}{\partial\theta}\frac{\partial\Phi}{\partial t} + 2\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\Phi = \frac{1}{r\sin\theta}\frac{\partial P}{\partial\varphi} + N_3, \tag{8}$$

where

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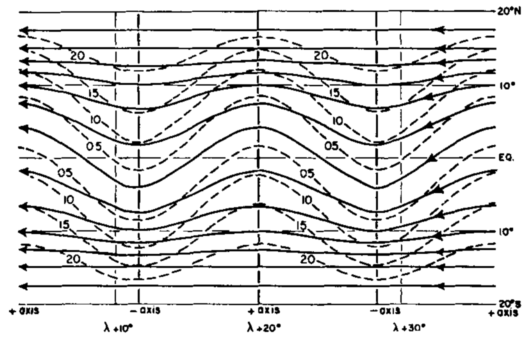

Fig. 1. Basic current disturbed by an equatorial wave (C. E. Palmer, 1952). Solid lines are stream lines. Dashed lines are isotach (knots).

$$N_1 = \frac{1}{\varrho}\left[\frac{1}{r\sin^2\theta}\left(\frac{\partial\Phi}{\partial\varphi}\right)\left(\frac{\partial}{\partial r}r\frac{\partial\Phi}{\partial\varphi}\right) + \frac{1}{r}\left(\frac{\partial\Phi}{\partial\theta}\right)\left(\frac{\partial}{\partial r}r\frac{\partial\Phi}{\partial\theta} - \frac{1}{2}\left\{\frac{1}{\sin^2\theta}\left(\frac{\partial\Phi}{\partial\varphi}\right)^2 + \left(\frac{\partial\Phi}{\partial\theta}\right)^2\right\}\frac{\partial\ln\varrho}{\partial r}\right]\right.,$$

$$N_2 = \frac{1}{\varrho}\left[\frac{1}{r\sin\theta}\left(\frac{\partial\Phi}{\partial\theta}\right)\left(\frac{\partial\Phi}{\partial\theta}\sin\theta\frac{\partial\Phi}{\partial\theta} + \frac{\sin\theta}{1}\frac{\partial^2\Phi}{\partial\varphi^2}\right)\right],$$

$$N_3 = \frac{1}{\varrho}\left[\frac{1}{r\sin^2\theta}\left(\frac{\partial\Phi}{\partial\varphi}\right)\left(\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\Phi}{\partial\theta} + \frac{\sin\theta}{1}\frac{\partial^2\Phi}{\partial\varphi^2}\right)\right].$$

It is seen that Φ, eq. (1), is a complete solution of (6, 7, 8) if

$$P = \frac{2k}{k+1}r\cos\theta\,\Phi + \frac{k(k+1)}{2\varrho}\Phi^2 \tag{9}$$

and if
$$\sigma = -\frac{2}{k+1} \tag{10}$$

for the case of constant density. For arbitrary radial stratification of an incompressible fluid, Φ is no longer a complete solution, but remains a linear solution.

The phase velocity of these shellular modes from (10) is

$$C_p = -\frac{2}{k(k+1)}. \tag{11}$$

Hence they move toward the west, as do the planetary waves of the approximate hydrostatic solutions in the recent literature (e.g., S. Rosenthal, 1960, 1965; T. Matsuno, 1966). M. S. Longuet-Higgins (1966, 1967) has studied that large class of planetary and gravity waves which are eigenfunctions of Laplace's tidal equations for a spherical shell. His linear hydrostatic solutions have many of the properties of the non-hydrostatic shellular modes and are identical in their latitudinal amplitude dependence and phase velocity near the equator. In a private communication Longuet-Higgins has suggested a mechanical interpretation of the shellular modes. He noted that eq. (1) represents the spherical oscillation of lines of fluid particles which, on the average, are bisected by the equatorial plane and parallel to the axis of rotation.

The atmosphere of the earth near the equator usually has a negligible latitudinal gradient of density. Hence it meets the model conditions of $\varrho = \varrho(r)$ very well. However, the typical easterly wind field varies from 250 cm/sec at the equator to 1000 cm/sec at 15° north and south of the equator. This shearing flow probably is the energy source for the equatorial easterly waves that are observed (C. E. Palmer, 1952), but the shear is also a significant departure from the uniform motion required for shellular modes.

Observed waves range about 15° of longitude in wavelength and have phase velocities between 500 cm/sec and 750 cm/sec relative to the ground. An idealization, drawn from the data, of the basic equatorial current disturbed by an equatorial wave is reproduced in Figure 1 from the work of C. E. Palmer (1952). These waves are seen to be swiftly damped north and south of 10° latitude. Similar waves are also seen on satellite photographs and offer promise of vast amounts of new data.

The dimensional phase velocity for $k = 24$ from (11) is approximately 200 cm/sec plus the velocity of any uniform zonal flow. An average zonal flow between the equator and 10° latitude appears to be approximately 500 cm/sec from Palmer's idealized synthesis of the data. Hence the phase velocity relative to the ground of a shellular mode of $k = 24$ in such a zonal flow would be approximately 700 cm/sec, which is within the range of the observed speeds. The 10° latitude average is chosen here because a $k = 24$ wave has reduced to e^{-1} of its equatorial amplitude at 10° north and south. The general latitudinal and vertical amplitude reduction of a shellular mode of high k from (1) is

$$\frac{\Phi(\theta=\pi/2-\gamma, r=1-\Delta)}{\Phi(\theta=\pi/2, r=1)} = e^{-k\Delta} e^{-k\gamma^2/2} \qquad (12)$$

for small values of γ and Δ.

It may be that the easterly waves seen in the baroclinic regions between 10° and 20° latitude are kinematically related to the shellular equatorial modes. The horizontal density gradients could both be the energy source of the waves and also create, in a kinematic sense, a pseudo-equator at the latitude of maximum gradient. However, the shellular modes are at best a useful framework for the rationalization of the observations. An adequate dynamic theory must account for both the instability and the distortion of shellular modes due to either horizontal shears at the equator or the horizontal density gradients further north and south.

In conclusion it should be noted that the shellular mode is equally applicable to oceanic planetary waves. Coupling the atmospheric and oceanic waves would be difficult except in regions where the zonal flow was quite small. However, the oceanic equatorial current could exchange energy with the shellular modes. It may be of value to look for them.

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