A note on geostrophic scale analysis of planetary waves

By ROBERT E. DICKINSON, *Massachusetts Institute of Technology*

(Manuscript received September 26, 1967)

Introduction and formulation

In the troposphere one generally distinguishes two types of geostrophic motion: class I, discovered by Charney (1947), which pertains to motions with a horizontal distance scale small compared to the radius of the earth; and class II, planetary scale motions, discovered by Burger (1958). These two kinds of motion have been reviewed by Phillips (1963). We shall show here that, for some combination of sufficiently strong zonal winds and high static stability, a third type of scaling may be applied to planetary scale geostrophic motions. The resulting prognostic equation is similar to that derived by class I scaling, so the analysis of this paper may be considered a variation on Charney's scaling applicable to class I motions of planetary scale.

It is convenient in discussing planetary scale motions to select a nondimensional stability parameter that does not vary with the velocity or distance scale of motion. For this purpose we define the planetary static stability S by

$$S = \frac{R}{\Omega^2 a^2}\left(\frac{\partial T}{\partial z} + \varkappa T\right), \qquad (1)$$

where $z = -\log p$, with p as pressure in units of the mean surface pressure, R = gas constant, $\varkappa = R/C_p$ where C_p is the specific heat at constant pressure, T = temperature, Ω = earth's rotational frequency, and a = radius f the earth. Let us take ε to be a constant which is a characteristic value of S. We take

$$\varepsilon = 1/10, \qquad (2)$$

which may be regarded as a typical value of S in the upper stratosphere ($S = 1/10$ for a 265° isothermal atmosphere.) We shall use ε as a small parameter for the scaling of this paper. Let x, y denote eastward and northward distance respectively, $u = dx/dt$, $v = dy/dt$, $w = dz/dt$, where t is time, d/dt is a substantial derivative, and finally let h denote geopotential height.

For the remainder of this paper, we shall use an asterisk (*) to denote a dimensional variable. We assume that all nondimensional variables are $O(1)$ and are related to dimensional variables as follows:

$$\left.\begin{aligned} x &= x^*/a, \\ y &= \varepsilon^{-\frac{1}{2}} y^*/a, \\ t &= \varepsilon \Omega t^*, \\ u &= \varepsilon^{-1}(\Omega a)^{-1} u^*, \\ v &= \varepsilon^{-\frac{3}{2}}(\Omega a)^{-1} v^*, \\ w &= \varepsilon^{-\frac{3}{2}}\Omega^{-1} w^*, \\ T &= \varepsilon^{-\frac{3}{2}}(\Omega a)^{-2} R T^*, \\ h &= \varepsilon^{-\frac{3}{2}}(\Omega a)^{-2} g h^*. \end{aligned}\right\} \qquad (3)$$

Assuming a β-plane geometry and neglecting possible external forcing, we obtain the following nondimensional horizontal equations of motion

$$\varepsilon^{\frac{1}{2}}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \varepsilon^{\frac{1}{2}} w \frac{\partial u}{\partial z}\right) - (f_0 + \varepsilon^{\frac{1}{2}}\beta y)\, v + \frac{\partial h}{\partial x} = 0, \qquad (4)$$

$$\varepsilon^{\frac{3}{2}}\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \varepsilon^{\frac{1}{2}} w \frac{\partial v}{\partial z}\right) + (f_0 + \varepsilon^{\frac{1}{2}}\beta y)\, u + \frac{\partial h}{\partial y} = 0, \qquad (5)$$

where if φ_0 = some mean latitude, the constants

f_0 and β are $f_0 = 2 \sin \varphi_0$, $\beta = 2 \cos \varphi_0$. It is necessary to assume f_0 is $O(1)$, so the present analysis does not apply within a distance of $O(\varepsilon^{\frac{1}{2}})$ ($\sim 20°$ latitude) of the equator.

As continuity equation we obtain

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \varepsilon^{\frac{1}{2}}\left(\frac{\partial w}{\partial z} - w\right) = 0; \qquad (6)$$

while the hydrostatic equation has the form

$$\frac{\partial h}{\partial z} = T; \qquad (7)$$

and the thermodynamic equation may be written as

$$\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) + wS/\varepsilon = 0. \qquad (8)$$

We shall assume that errors of $O(\varepsilon^{\frac{1}{2}})$ are allowable. Then from (4) and (5) we obtain the geostrophic approximation.

$$\left.\begin{aligned} f_0 v &= \frac{\partial h}{\partial x} + O(\varepsilon^{\frac{1}{2}}), \\ f_0 u &= -\frac{\partial h}{\partial y} + O(\varepsilon^{\frac{1}{2}}). \end{aligned}\right\} \qquad (9)$$

By cross differentiation of (4) and (5) and using (6) we obtain the vorticity equation

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\frac{\partial u}{\partial y} - \beta v + f_0\left(\frac{\partial w}{\partial z} - w\right) = O(\varepsilon^{\frac{1}{2}}).$$

All terms in (7), (8), (9), and (10) are the same order of magnitude. Elimination of u, v, T in terms of h by (9) and (7), and w by (6), then reduces (8) and (10) to a single geostrophic potential vorticity equation which, defining

$$q = \left(f_0^2 e^z \frac{\partial}{\partial z}\frac{e^{-z}}{S/\varepsilon}\frac{\partial h}{\partial z} + \frac{\partial^2 h}{\partial y^2}\right),$$

then is written

$$\frac{\partial q}{\partial t} + J(h/f_0, q) + \beta\frac{\partial h}{\partial x} = O(\varepsilon^{\frac{1}{2}}), \qquad (11)$$

where J is a Jacobian.

Discussion

From (3) we see that the above model assumes the following characteristic scales (i.e. $1/2\pi$ of the time period or wavelength):

Time scale $= (\varepsilon\Omega)^{-1} \simeq 1$ day.
Latitude scale $= \varepsilon^{\frac{1}{2}} a \simeq 20°$ of latitude.
Longitude scale $= a \simeq 90°$ of longitude at 50° N.
Vertical scale = one scale height (7 km).

Characteristic velocities are assumed to be:

Zonal velocity $= U = \varepsilon\Omega a \sim 50$ m/sec.
Meridional velocity $= \frac{1}{3}U = \varepsilon^{\frac{3}{2}}\Omega a \sim 15$ m/sec.

The advective time scale proportional to a/u^* is a lower limit; the model is applicable to motions of longer period as well. Reference to published mean monthly stratospheric maps suggests that the assumed longitudinal and latitudinal scales apply to observed dominant winter time disturbances in the stratosphere. While the longitudinal scale of disturbances is observed to be roughly 60° of longitude, the latitudinal scale is only 15°–20° of latitude. Meteorologists who have studied the structure of planetary waves have commented on the typical "sausage" shape of the patterns on a constant pressure surface, with a much smaller latitude than longitude scale, Muench (1967).

From (3) we see that the parameter ε may be interpreted as being a Rossby number for the zonal motion. An essential feature of the above scaling is that the north-south motions are less by a factor of $\varepsilon^{\frac{1}{2}}$ than east-west motions, rather than the same magnitude, as assumed by Burger. Consequently, when we assume planetary scale motions, (4) gives geostrophic balance only to $O(\varepsilon^{\frac{1}{2}})$ rather than $O(\varepsilon)$, and for geostrophic balance h^* must follow v^* in being $O(\varepsilon^{\frac{3}{2}})$. Then for a balance in the thermodynamic equation to be achieved, w^* must also be taken proportional to $\varepsilon^{\frac{3}{2}}$. Also, for geostrophic balance to hold in (5), y^* must be $O(\varepsilon^{\frac{1}{2}})$. This being assumed, $\partial u/\partial x + \partial v/\partial y = O(\varepsilon^{\frac{1}{2}})$, showing that the motion is nondivergent to lowest order. Vorticity is given by $\varepsilon\, \partial v/\partial x - \partial u/\partial y$; we obtain a potential vorticity equation which is the same as for class I scaling, except to lowest order the vorticity is simplified by omission of a $\partial v/\partial x$ term.

A minimum value of S in the upper troposphere of 0.02 would give $\varepsilon = 2 \times 10^{-2}$, $u^* \simeq 10$ m/sec,

$v^* \simeq 1$ m/sec. In this case the meridional velocity would be restricted to very small amplitudes such as might be assumed in perturbation theories about a mean zonal wind. Also, disturbances would be limited to those with a latitudinal scale of less than 10°, which probably can no longer be regarded as planetary scale.

For simplicity we have derived the above theory in Cartesian coordinates. For the scaling assumed above, the neglect of spherical geometry introduces errors which are $O(\varepsilon^{\frac{1}{2}})$ less than terms retained, and so, in principle, are no larger than other terms omitted; in practice, however, these errors may be unacceptable. In another work the author (Dickinson, 1968) uses the above scaling to derive a spherical geometry model with variable Coriolis parameter for a perturbation study of planetary wave propagation. When latitudinal propagation of disturbances may occur, neglect of the spherical geometry and neglect of the variation of the Coriolis parameter cannot be justified.

We note that Murakami (1963) has discussed yet another kind of planetary wave where the longitudinal scale is less than the latitudinal scale. He has shown how the different scalings may be used to analyze orders of magnitude of various energetic terms.

In conclusion, we have given an alternate scaling to that of Burger for planetary scale geostrophic wave motions. Because many interacting scales of motion actually occur in the atmosphere, it is not possible to establish rigorously the applicability of our analysis. The assumed scaling does suggest, however, that the polar night stratospheric planetary waves are quasi-nondivergent to a first approximation, and will therefore be better described by the class I than the class II model for geostrophic motions.

Acknowledgements

The author is grateful to Dr. Jack Geisler and Dr. Norman Gaut for reading the manuscript. The author was supported in this work by a National Science Foundation Grant (GA-400).

REFERENCES

Burger, A. P. 1958, Scale considerations of planetary motions of the atmosphere. *Tellus, 10,* 195–205.
Charney, J. G., 1948, On the scale of atmospheric motion. *Geof. Publ. 17,* 2, 17–30.
Dickinson, R. E., 1968, On the exact and approximate linear theory of vertically propagating planetary Rossby waves forced at a spherical lower boundary. *Mon. Wea. Rev. 96.* In press.
Muench, S., 1967, Planetary waves in winter. Paper presented at the 1967 Stanstead Seminar.
Murakami, T., 1963, Analysis of various large-scale disturbances and the associated zonal mean motions in the atmosphere. *Geofisica pura e applicata, 54,* 119–165.
Phillips, N. A., 1963, Geostrophic motion. *Reviews of Geophysics, 1,* 123–176.