SHORTER CONTRIBUTION

A note on internal gravity waves in a hydrostatic compressible fluid with vertical wind shear[1]

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Introduction

In a recent paper (Wiin-Nielsen, 1965), an investigation was made of the propagation of gravity waves in a hydrostatic, compressible fluid with a vertical wind shear. The perturbations were assumed to be independent of the coordinate (y) perpendicular to the direction of the basic flow. It was found that the speed of propagation in this case is determined entirely by the Richardson number (Ri) but is independent of the wave number.

The purpose of this note is to investigate the case in which the perturbations may depend on the y-coordinate, and where the fluid is bounded by vertical walls in the lateral direction. The basic state will still be characterized by a geopotential $\Phi = \Phi(p)$ and a flow $U = U(p)$.

The perturbation analysis

The linearized equations for our poroblem may be written as follows:

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + \omega\frac{dU}{dp} = -\frac{\partial \phi}{\partial x}, \qquad (1)$$

$$\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} = -\frac{\partial \phi}{\partial y}, \qquad (2)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial p}\right) + U\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial p}\right) + \bar{\sigma}\omega = 0, \qquad (3)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0, \qquad (4)$$

in which u, v and ω are the components of the perturbation velocity and ϕ the perturbation geopotential, while $\bar{\sigma} = -(\bar{\alpha}/\bar{\theta})(\partial\bar{\theta}/\partial p)$ is a parameter measuring the static stability in the basic state. In comparison with the earlier case (Wiin-Nielsen, 1965), we note the addition of equation (2) and the addition of the term $\partial v/\partial y$ in the continuity equation (4), while the other equations are unchanged.

We consider perturbations of the form:

$$a(x, y, p, t) = \hat{a}(y, p)e^{ik(x-ct)}, \qquad (5)$$

in which $k = (2\pi/L)$ is the wave number, L the wave length and c the phase speed.

When (5) is introduced in (1)–(4) we obtain four new equations from which we can eliminate all dependent variables except $\hat{\omega}(y, p)$. The final equation in $\hat{\omega}$ can be written in the form:

$$k^2E^2\frac{\partial^2\hat{\omega}}{\partial p^2} - \bar{\sigma}\frac{\partial^2\hat{\omega}}{\partial y^2} + k^2\left(\bar{\sigma} - E\frac{d^2E}{dp^2}\right)\hat{\omega} = 0, \qquad (6)$$

in which we have introduced the notation $E = U - c$.

The problem is now to solve the equation (6) under proper boundary conditions. We notice that (6) reduces to the earlier frequency equation if we neglect the y-dependence. The boundary condition on $\hat{\omega}$ is $\hat{\omega} = 0$ at $p = 0$ and $p = p_0 = 100$ cb when we restrict our attention to internal gravity waves. The lateral boundary condition is $v = 0$ at $y = \pm D$ assuming that the vertical walls are placed at the positions $y = \pm D$ while $y = 0$ is the middle of the channel. It is seen from (2) that $v = 0$ at $y = \pm D$ implies that $\partial\phi/\partial y = 0$ at the same positions. By differentiating (3) with respect to y, it is seen that the boundary condition $v = 0$ at $y = \pm D$ is equiva-

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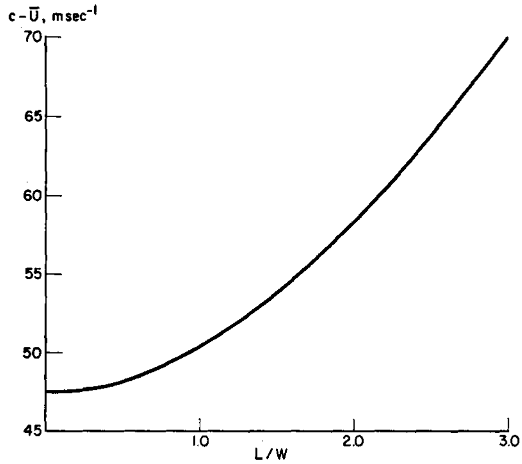

Fig. 1. $c - \bar{U}$ in m sec^{-1} as a function of L/W; L is the wave length and W the width of the channel. Parameters: $m = 1$, $\bar{\sigma} = 1.2$ and $s = 80$ m sec^{-1}.

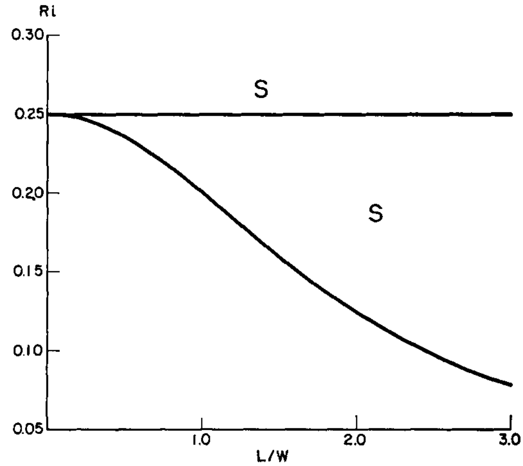

Fig. 2. Regions of stability in a diagram with L/W as abscissa and the Richardson number as ordinate.

lent to the condition $\partial\hat{\omega}/\partial y = 0$ at $y = \pm D$. (6) must therefore be solved under the conditions:

$$\left.\begin{aligned} &\hat{\omega} = 0 \quad \text{at } p = 0 \text{ and } p = p_0, \\ &\frac{\partial\hat{\omega}}{\partial y} = 0 \quad \text{at } y = +D \text{ and } y = -D. \end{aligned}\right\} \qquad (7)$$

The latter boundary condition is automatically satisfied if we assume $\hat{\omega}(y, p)$ to have the form

$$\hat{\omega}(y,p) = \Omega(p) \sin\left(\frac{\pi}{2}\frac{y}{D}\right). \qquad (8)$$

An expression containing $\sin(n\pi y/2D)$ where n is an integer can naturally also be used but the various values of n are equivalent to variations in the width of the channel and can therefore be disregarded.

Introducing (8) in (6) we obtain the following equation for $\Omega(p)$:

$$E^2 \frac{d^2\Omega}{dp_*^2} + \bar{\sigma}p_0^2\left(1 + \frac{\pi^2}{4k^2D^2}\right)\Omega = 0, \qquad (9)$$

in which we have changed the independent variable to the nondimensional value $p_* = p/p_0$, and where we have made the assumption that U is a linear function of p, i.e. $d^2E/dp^2 = 0$. Comparing (9) with the previous frequency equation we note that the only change is that $\bar{\sigma}p_0^2$ in the last term has been replaced by $\bar{\sigma}p_0^2(1 + \pi^2/(4k^2D^2))$. We can therefore adopt the previous solution when we make the replacement mentioned above assuming as before that the basic current has the form

$$U = \bar{U} + s(1/2 - p_*). \qquad (10)$$

The solution for the phase speed may be written:

$$c = \bar{U} + \frac{1}{2}s \coth\left(\frac{m\pi}{2\lambda}\right) \quad (m = \pm 1, \pm 2, \ldots), \qquad (11)$$

where $\lambda = (q^2 - 1/4)^{1/2}$ and $q^2 = (\bar{\sigma}p_0^2/s^2)(1 + \pi^2/(4k^2D^2))$. We note in particular that $\mathrm{Ri} = (\bar{\sigma}p_0^2)/s^2$ as shown by Wiin-Nielsen (1965).

Discussion of the solution

The only difference between the present and previous solution is in the definition of q^2 which in the present case contains an additional factor: $1 + \frac{1}{4}L^2/W^2$ where L is the wave length and $W = 2D$ is the width of the channel. The introduction of the y-dependence of the perturbations thus makes the phase speed dependent on the wave length while this was not the case in the previous investigation. In view of the fact that a detailed investigation was made of the dependence of c on the parameters Ri and m in the previous paper (Wiin-Nielsen, 1965), we shall here restrict ourselves to illustrate the wave length dependence. Selecting a value of $s = 80$ m sec^{-1} which corresponds to a value of the windshear of approximately 4 m sec^{-1} km^{-1}, setting $m = 1$ and $\sigma_0 = 1.2$ MTS-units we have tabulated $c - \bar{U}$ as a function of the ratio L/W. The results are shown in Fig. 1 which shows that the speed of internal gravity increases when L/W increases. The previous solution applies for $L/W = 0$. We notice that the value of $c - \bar{U}$ increases by a factor of 1.5, when L/W goes from 0 to 3. The previous estimates of $c - \bar{U}$ are therefore too small by a significant factor when applied to atmospheric motions.

In the previous investigation we found for a general wind profile $U = U(p)$ that a sufficient criterion for stability is that $\mathrm{Ri} \geqslant 1/4$. By applying the same methods we get in this case that a sufficient criterion for stability is

$$\mathrm{Ri} \geqslant \frac{1}{4(1 + \frac{1}{4}(L/W)^2)}. \quad (12)$$

This relation is illustrated in Fig. 2 where the horizontal line $\mathrm{Ri} = 1/4$ is the critical curve for the previous case above which the disturbances are stable. The second curve corresponds to the right hand side of (12). The area above this curve represents a region of stability in the present investigation. We find therefore that the introduction of the y-dependence of the disturbances expands the region of stability.

REFERENCES

Wiin-Nielsen, A. 1965. On the propagation of gravity waves in a hydrostatic compressible fluid with vertical windshear. *Tellus*. Vol. 17, No. 3, pp. 306–320.