Stability of a non-divergent Ekman layer

By VICTOR BARCILON, *Div. of Engineering and Applied Physics, Harvard Univ. Cambridge, Mass.*

(Manuscript received June 2; revised version August 7, 1964)

ABSTRACT

The stability of a non-divergent Ekman layer generated by the horizontal motion of a rigid lid lying on the surface of a rotating fluid is investigated for large αR where α is the wave number of the perturbation and R the Reynolds number. On the assumption that whenever instability occurs rolls are formed at an angle β to the direction of motion of the lid, it is shown that the solutions of the sixth order differential equations governing the perturbation fields can be expressed in terms of the four solutions of the Orr-Sommerfeld equation for a two-dimensional plane parallel flow. The velocity profile of this flow is obtained by projecting the velocity vector of the Ekman flow onto a vertical plane perpendicular to the rolls.

As the angle β is varied, the nature of the instability (viz. inviscid or viscous) is discussed by examining the upper branch of the neutral stability curve for infinite Reynolds number.

The Coriolis force which affects the structure of the perturbation fields does not influence the nature of the neutral stability curve which is similar to that of a plane parallel flow having the same velocity profile as the component of the Ekman layer flow perpendicular to the direction of the rolls.

1. Introduction

In his study of the effects of wind stresses on ocean currents, EKMAN (1905) was the first to investigate a flow in which a balance between friction, Coriolis force and pressure gradient forces is achieved. Modeling the turbulent mixing by a constant eddy viscosity $\varkappa$, he has shown that the flow is confined to a narrow layer of depth $(\varkappa/f)^{\frac{1}{2}}$, where f is the component of the earth angular velocity along the vertical. Subsequent workers have refined his original analysis, in particular ELLISON (1956) making use of the advances in the theory of turbulent mixing reworked the problem with an Austausch coefficient varying linearly with depth. For laminar flows, the original theory is valid if the eddy viscosity $\varkappa$ is replaced by the kinematic viscosity ν.

FALLER (1963) has investigated experimentally the stability of a laminar Ekman layer produced on the bottom surface of a cylindrical rotating tank by forcing fluid in at the outer rim and withdrawing it at the inner one. For Reynolds numbers greater than 125, the flow in the boundary layer is unstable. A regular banded structure of vortex rolls spaced by a distance 11 times the boundary layer depth, and making an angle $\beta \simeq 16°$ with the interior geostrophic[1] current is observed. A theoretical investigation of the stability of the Ekman layer was made by STERN (1960). He was particularly interested in the interaction of the boundary layer flow with the interior geostrophic flow, and showed that there exist certain unstable modes which are not entirely confined to the boundary layer.

The stability of the simplest Ekman layer possible, i.e., one without divergence or underlying geostrophic current, has, however, not been considered. In the present paper we propose to study the stability of the "simple" Ekman layer flow generated by the motion of a horizontal rigid lid along the surface of an incompressible, infinitely deep, rotating fluid. We shall consider a fluid occupying the region $-\infty \leqslant x', y' \leqslant +\infty, z' < 0$, where x' and y' are the horizontal coordinates, and z' is the vertical one. The frame of reference, fixed with respect to the fluid, rotates about the vertical with an angular velocity Ω. A rigid lid, lying on the

[1] A geostrophic current exists whenever the Coriolis force balances the pressure gradient. In large regions of the atmosphere and oceans, inertia forces, viscous forces, etc. are negligible and a geostrophic balance occurs.

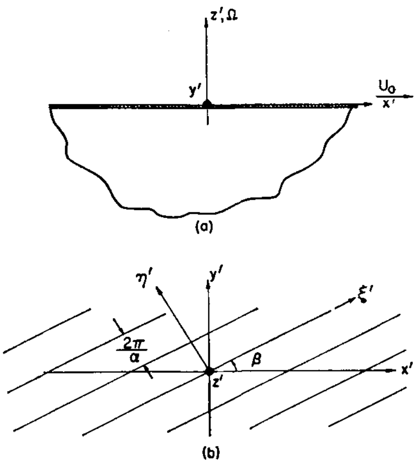

Fig. 1

surface $z'=0$ of the fluid, moves along the x'-direction with a velocity U_0 (see Fig. 1). The flow generated by the stress exerted at the surface is a function of z' only. The equations of motion are:

$$\left.\begin{aligned} \overline{W}'\frac{d\bar{U}'}{dz'} - 2\Omega\overline{V}' &= \nu\frac{d^2\bar{U}'}{dz'^2}, \\ \overline{W}'\frac{d\overline{V}'}{dz'} + 2\Omega\bar{U}' &= \nu\frac{d^2\overline{V}'}{dz'^2}, \\ \overline{W}'\frac{d\overline{W}'}{dz'} + \frac{1}{\varrho'}\frac{dP'}{dz'} &= \nu\frac{d^2\overline{W}'}{dz'^2}, \\ \frac{d\overline{W}'}{dz'} &= 0, \end{aligned}\right\} \quad (1.1)$$

where $\bar{U}'$, $\overline{V}'$, $\overline{W}'$ are the velocity components in the x', y', z' directions. Since $\overline{W}'=0$ at the surface $z'=0$, the continuity equation yields:

$$\overline{W}' \equiv 0. \quad (1.2)$$

Hence the flow is two-dimensional; solving for $\bar{U}'$ and $\overline{V}'$, we get:

$$\left.\begin{aligned} \bar{U}' &= U_0 \exp\left[\left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}} z'\right] \cos\left[\left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}} z'\right], \\ \overline{V}' &= V_0 \exp\left[\left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}} z'\right] \sin\left[\left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}} z'\right]. \end{aligned}\right\} \quad (1.3)$$

Equation (1.3) shows that the rotation has confined the effects of the surface stress to a shallow layer of depth δ, such that:

$$\delta = (\nu/\Omega)^{\frac{1}{2}}. \quad (1.4)$$

As z' decreases, the projection on a horizontal plane of the tip of the velocity vector follows the classical Ekman spiral.

We shall investigate the stability of this steady, horizontal flow by introducing small time-dependent perturbations in the velocity and pressure fields; the flow is unstable whenever these perturbations grow with time. Although the flow under consideration is not a parallel flow, the investigation of its stability is strikingly similar to the familiar analysis of plane parallel flows. In particular, we shall be able to reduce our problem to the solution of the classical Orr-Sommerfeld equation for an appropriate mean flow. The velocity profile of this flow will be the projection of the Ekman velocity field on the vertical plane perpendicular to the direction of the observed rolls.

2. Formulation of the eigenvalue problem

Relying on experimental evidence (Faller, 1963), we shall assume that rolls are formed whenever instability occurs. It will be more convenient to introduce a new coordinate system ξ', η', z' such that the ξ'-axis is along the direction of the rolls; the various perturbation fields will therefore depend only on η' and z'. The angle β between the ξ'-axis and the x'-axis is an unknown parameter of the problem. If U' and V' are the components of the mean flow in the ξ' and η' directions, we can write:

$$\left.\begin{aligned} U' &= \bar{U}'\cos\beta + \overline{V}'\sin\beta \\ &= U_0 \exp\left[\left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}} z'\right] \cos\left[\left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}} z' - \beta\right], \\ V' &= -\bar{U}'\sin\beta + \overline{V}'\cos\beta \\ &= U_0 \exp\left[\left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}} z'\right] \sin\left[\left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}} z' - \beta\right]. \end{aligned}\right\} \quad (2.1)$$

The equations satisfied by the perturbation fields can now be written as follows:

$$\left.\begin{aligned} &\frac{\partial u'}{\partial t'} + V'\frac{\partial u'}{\partial \eta'} + w'\frac{\partial U'}{\partial z'} - 2\Omega v' = \nu\nabla'^2 u', \\ &\frac{\partial v'}{\partial t'} + V'\frac{\partial \eta'}{\partial v'} + w'\frac{\partial V'}{\partial z'} + 2\Omega u' \\ &\qquad = -\frac{1}{\varrho}\frac{\partial p'}{\partial \eta} + \nu\nabla'^2 u', \\ &\frac{\partial w'}{\partial t'} + V'\frac{\partial w'}{\partial \eta'} = -\frac{1}{\varrho}\frac{\partial p'}{\partial z'} + \nu\nabla'^2 w', \\ &\partial v'/\partial \eta' = \partial w'/\partial z' = 0. \end{aligned}\right\} \quad (2.2)$$

The form of continuity equation enables us to introduce a stream function ψ' such that

$$\left.\begin{aligned} v' &= \partial\psi'/\partial z', \\ w' &= -\partial\psi'/\partial\eta'. \end{aligned}\right\} \quad (2.3)$$

Eliminating p' v' and w', we get

$$\left.\begin{aligned} &\frac{\partial u'}{\partial t'} + V'\frac{\partial u'}{\partial \eta'} - \frac{\partial U'}{\partial z'}\frac{\partial \psi'}{\partial \eta'} - 2\Omega\frac{\partial \psi'}{\partial z'} = \nu\nabla'^2 u', \\ &\frac{\partial}{\partial t'}\nabla'^2\psi' + V'\frac{\partial}{\partial \eta'}\nabla'^2\psi' - \frac{\partial^2 V'}{\partial z'^2}\frac{\partial \psi'}{\partial \eta'} \\ &\qquad + 2\Omega\frac{\partial u'}{\partial z'} = \nu\nabla'^4\psi'. \end{aligned}\right\} \quad (2.4)$$

At this point, a scaling of the various variables would be opportune. However, the existence of several characteristic units (of time, length, etc.) renders the choice of the appropriate scaling difficult. The correct scaling is arrived at by appealing to the following reasoning.

Let us introduce dimensionless variables as follows:

$$\left.\begin{aligned} t' &= \tau t, \\ \eta', z' &= L(\eta, z), \end{aligned}\right\} \quad (2.5)$$

where τ and L are characteristic time and length. It is also convenient to restrict the analysis to periodic disturbances of the form

$$\left.\begin{aligned} u &= u(z)e^{i\alpha(\eta - ct)}, \\ \psi &= \psi(z)e^{i\alpha(\eta - ct)}, \end{aligned}\right\} \quad (2.6)$$

where α is the wave number, and $c = c_r + ic_i$ is the complex phase speed. Substituting (2.5) and (2.6) in equations (2.4) we obtain

$$\left.\begin{aligned} &-\frac{i\alpha c}{\Theta R}u + i\alpha\lambda V u - i\alpha U_z\psi - \frac{2}{R}\psi_z \\ &\qquad = \frac{\lambda^2}{R}\{u_{zz} - \alpha^2 u\}, \\ &-\frac{i\alpha c}{\Theta R}\{\psi_{zz} - \alpha^2\psi\} + i\alpha\lambda V\{\psi_{zz} - \alpha^2\psi\} \\ &\qquad -\frac{i\alpha}{\lambda}V_{zz}\psi + \frac{2}{R}u_z \\ &\qquad = \frac{\lambda^2}{R}\{\psi_{zzzz} - 2\alpha^2\psi_{zz} + \alpha^4\psi\}, \end{aligned}\right\} \quad (2.7)$$

where $$R = \frac{U_0}{\delta\Omega} = \frac{U_0\delta}{\nu} \quad (2.8)$$

is the Reynolds number and

$$\left.\begin{aligned} \Theta &= \tau/\Omega^{-1}, \\ \lambda &= \delta/L. \end{aligned}\right\} \quad (2.8)$$

To determine Θ and λ we shall first require the phase speed of the modified Tollmien-Schlichting waves to be comparable to the speed of the mean flow, i.e.,

$$\frac{L}{\tau} = U_0 \quad \text{or} \quad (\lambda\Theta)^{-1} = R. \quad (2.10)$$

This assumption is plausible if we recall that it is satisfied by all known unstable plane parallel flows (e.g., for the Blasius flow $(c_r)_{\max} = .4$), and for the special case of an inviscid fluid, Rayleigh's theorem (LIN, 1955, p. 120) states that the value of the phase speed of neutral waves lies between the maximum and minimum values of the velocity of the mean flow. A second relation for Θ and λ is obtained by requiring that the transport of momentum of the mean flow by the vertical perturbation velocity be comparable to the rate of change of the momentum perturbation. From the first and third terms in the first equation (2.7) we deduce that

$$R\Theta = 1. \quad (2.11)$$

The equivalent argument in terms of the energy, would imply that the kinetic energy of the perturbations is increased by means of the work done by the Reynolds stress on the mean flow. From (2.10) and (2.11) we can easily deduce that

$$\left.\begin{aligned} L &= \delta, \\ \tau &= \delta/U_0. \end{aligned}\right\} \quad (2.12)$$

The Ekman layer depth δ is therefore the characteristic length, and as a consequence the Coriolis force is of the same order of magnitude as the viscous force even for the perturbation fields. However, the mere fact that these forces are of the same order of magnitude does not imply that they play comparable roles. Despite their smallness, the viscous terms are important in the equations because they are the most highly differentiated terms and their neglect would (1) prevent the boundary conditions from being satisfied and (2) introduce spurious singularities in the inviscid solutions. This is not the case for the Coriolis terms.

Using (2.12), equations (2.7) can now be written as:

$$\left.\begin{aligned} &(V-c)(D^2-\alpha^2)\psi - V''\psi \\ &\qquad = (i\alpha R)^{-1}[(D^2-\alpha^2)^2\psi - 2Du], \\ &(V-c)u - U''\psi = (i\alpha R)^{-1}. \\ &\qquad [(D^2-\alpha^2)u + 2D\psi], \end{aligned}\right\} \quad (2.13)$$

where either a prime or "D" represents a derivative with respect to z. The equations together with the following boundary conditions

$$\left.\begin{aligned} &\psi(0) = \psi'(0) = u(0) = 0, \\ &\psi,\ \psi' \text{ and } u \to 0 \text{ as } z \to -\infty \end{aligned}\right\} \quad (2.14)$$

constitute the eigenvalue problem for c which we propose to solve. The Ekman layer flow is stable, marginally stable or unstable according to whether c_i is positive, zero or negative.

The analogy between the first equation in (2.13) and the Orr-Sommerfeld equation suggests that we use a similar method of solution in which the eigenfunctions are expressed in the form of an asymptotic series in αR.

3. Asymptotic expansions for large αR

Expanding ψ and u in powers of $(\alpha R)^{-1}$, we can write

$$\left.\begin{aligned} \psi &= \psi^{(0)} + (\alpha R)^{-1}\psi^{(1)} + \dots, \\ u &= u^{(0)} + (\alpha R)^{-1}u^{(1)} + \dots. \end{aligned}\right\} \quad (3.1)$$

Substituting (3.1) in equations (2.13), we get

$$\left.\begin{aligned} (V-c)(D^2-\alpha^2)\psi^{(0)} - V''\psi^{(0)} &= 0, \\ (V-c)u^{(0)} - U'\psi^{(0)} &= 0, \end{aligned}\right\} \quad (3.2)$$

and for higher approximations

$$\left.\begin{aligned} &(V-c)(D^2-\alpha^2)\psi^{(n)} - V''\psi^{(n)} \\ &\qquad = \frac{1}{i}[(D^2-\alpha^2)^2\psi^{(n-1)} - 2Du^{(n-1)}], \\ &(V-c)u^{(n)} - U'\psi^{(n)} \\ &\qquad = \frac{1}{i}[(D^2-\alpha^2)u^{(n-1)} + 2D\psi^{(n-1)}]. \end{aligned}\right\} \quad (3.3)$$

In the present analysis we shall only consider the first terms of the various asymptotic expansions, and the superscripts, now superfluous, shall be dropped. Recognizing the first equation in (3.2) as the inviscid-Orr-Sommerfeld equation, we can avail ourselves of what is known about its two solutions ϕ_1 and ϕ_2 (LIN, 1955, Section 8.1). In particular, in the neighborhood of the critical point z_c, where

$$V(z_c) = c, \quad (3.4)$$

these two solutions can be written as

$$\left.\begin{aligned} \phi_1 &= (z-z_c)\left[1 + \frac{V_c''}{2V_c'}(z-z_c) + \dots\right], \\ \phi_2 &= 1 + \dots + \frac{V_c''}{V_c'}(z)\log(z-z_c), \end{aligned}\right\} \quad (3.5)$$

where V_c' and V_c'' stand for $V'(z_c)$ and $V''(z_c)$. Denoting the jth solution of (2.13) by $\{\psi_j, u_j\}$, where $j = 1, 2, \dots, 6$ we deduce from (3.2) that

$$\left.\begin{aligned}\{\psi_1, u_1\} &= \left\{\phi_1, \frac{U'\phi_1}{V-c}\right\}, \\ \{\psi_2, u_2\} &= \left\{\phi_2, \frac{U'\phi_2}{V-c}\right\}.\end{aligned}\right\} \quad (3.6)$$

By means of (3.5) we can see that u_1 is regular at $z = z_c$; however, u_2, rendered single-valued by an appropriate choice of the logarithmic function, is infinite at $z = z_c$ unless $U_c' = 0$.

We still need four other fundamental solutions $\{\psi_3, u_3\}$, ..., $\{\psi_6, u_6\}$ in order to represent the general solution as

$$\{\psi, u\} = \sum_{i=1}^{6} A_i \{\psi_i, u_i\}, \quad (3.7)$$

where the A's are constants, determined by means of the boundary conditions. The character of equations (2.13) suggests the possibility of solutions of the form

$$\left.\begin{aligned}\psi &= \exp\,(\sqrt{\alpha R}\,Q)\{f_0 + (\alpha R)^{-\frac{1}{2}} f_i + \ldots\}, \\ u &= (\alpha R)^m \exp\,(\sqrt{\alpha R}\,Q) \\ &\quad \times \{g_0 + (\alpha R)^{-\frac{1}{2}} g_1 + \ldots\}.\end{aligned}\right\} \quad (3.8)$$

Substituting these expressions in (2.13), and comparing coefficients, we obtain

$$Q'^2 = i(V-c), \quad (3.9)$$

$$(\alpha R)^{\frac{1}{2}}\{5Q'^2 Q'' f_0 + 2Q'^3 f_0'\} - 2(\alpha R)^{m+\frac{1}{2}} Q' g_0 = 0, \quad (3.10)$$

$$i(\alpha R) U' f_0 + (\alpha R)^{m+\frac{1}{2}}\{2Q' g_0' + Q'' g_0\} = 0, \quad (3.11)$$

together with other equations similar to (3.10) and (3.11) governing the higher approximations.

An examination of (3.10) and (3.11) shows that m can only be equal to 1 or $\frac{1}{2}$.

When $m = \frac{1}{2}$, (3.10) becomes an equation for f_0 only, which can be readily solved, yielding:

$$f_0 = (V-c)^{-\frac{5}{4}} \quad (3.12)$$

whereas (3.11) is an inhomogeneous equation for g_0, whose solution is

$$g_0 = -\frac{i^{\frac{1}{2}}}{2}(V-c)^{\frac{1}{4}} \int_0^z \frac{U'dz}{(V-c)^{\frac{7}{4}}}. \quad (3.13)$$

Therefore, the third and fourth funamental solutions are

$$\left.\begin{aligned}\{\psi_3, u_3\} &= \left\{\phi_3,\ -\frac{(i\alpha R)^{\frac{1}{2}}}{2} (V-c)\phi_3 \int_0^z \frac{U'dz}{(V-c)^{\frac{7}{4}}}\right\}, \\ \{\psi_4, u_4\} &= \left\{\phi_4,\ -\frac{(i\alpha R)^{\frac{1}{2}}}{2} (V-c)\phi_4 \int_0^z \frac{U'dz}{(V-c)^{\frac{7}{4}}}\right\},\end{aligned}\right\} \quad (3.14)$$

where

$$\phi_3 = (V-c)^{-\frac{5}{4}} \exp\left(-\int_0^z [i\alpha R(V-c)]^{\frac{1}{2}} dz\right), \quad (3.15)$$

and

$$\phi_4 = (V-c)^{-\frac{5}{4}} \exp\left(+\int_0^z [i\alpha R(V-c)]^{\frac{1}{2}} dz\right) \quad (3.16)$$

are the third and fourth fundamental solutions of the Orr-Sommerfeld equation (LIN, 1955, p. 116).

When $m = 1$, the third term in equation (3.10), which represents the effects of the Coriolis force, is comparable to the first two which represent the effects of the inertia and viscous forces. Therefore, the remaining two fundamental solutions are the only ones which account for the Coriolis force. Solving for g_0 first and then for f_0, we get

$$\left.\begin{aligned}g_0 &= i(V-c)^{-\frac{1}{4}}, \\ f_0 &= z(V-c)^{-\frac{5}{4}},\end{aligned}\right\} \quad (3.17)$$

hence

$$\left.\begin{aligned}\{\psi_5, u_5\} &= \{z\phi_3,\ i\alpha R(V-c)\phi_3\}, \\ \{\psi_6, u_6\} &= \{z\phi_4,\ i\alpha R(V-c)\phi_4\}.\end{aligned}\right\} \quad (3.18)$$

We have therefore succeeded in expressing the six fundamental solutions $\{\psi_j, u_j\}$ in terms of the four fundamental solutions ϕ_1, ϕ_2, ϕ_3, ϕ_4 of the classical Orr-Sommerfeld equation encountered in the analysis of the stability of plane parallel flows. The various solutions have singularities at $z = z_c$; however, as in the case of plane parallel flows, z_c is an ordinary point of the exact equations (2.13) and the singularities are entirely due to the method of solution. In other words, the exact solutions of (2.13) are regular at $z = z_c$, but the representations (3.6), (3.14), (3.18) of these solutions are not valid at $z = z_c$. In the appendix we shall show that different representations of the solutions can be found in the neighborhood of z_c, and that the regions of validity of the two representations overlap. From a comparison of the two representations, the classical result will emerge, namely that the first representations are valid throughout the entire region if the various cuts at $z = z_c$, necessary to render the solutions single-valued are taken in the upper part of the complex z-plane whenever $V_c' > 0$.

As $z \to -\infty$, U and V decay exponentially and can be set equal to zero in order to obtain the behavior of the various fundamental solutions for large values of $|z|$. In particular, we can deduce from (3.15) and (3.16) the following asymptotic forms for ϕ_3 and ϕ_4, namely

$$\left.\begin{aligned} \phi_3 &\sim \exp\left\{-(-i\alpha Rc)^{\frac{1}{2}} z\right\}, \\ \phi_4 &\sim \exp\left\{+(-i\alpha Rc)^{\frac{1}{2}} z\right\}. \end{aligned}\right\} \tag{3.19}$$

If we choose the branch of the square root such that

$$\operatorname{Re}\,(-ic)^{\frac{1}{2}} > 0, \tag{3.20}$$

we can see that ϕ_3 becomes infinite as $z \to -\infty$. Since ψ and u are bounded at infinity, we must reject the fundamental solutions containing ϕ_3, namely ψ_3, u_3, ψ_5, and u_5, and write the general solution as

$$\left.\begin{aligned} \psi &= A\psi_1 + B\psi_2 + D\psi_4 + E\psi_6, \\ u &= Au_1 + Bu_1 + Du_4 + Eu_6, \end{aligned}\right\} \tag{3.21}$$

where A, B, D, and E are constants determined by the remaining boundary conditions. Since ϕ_4 decays exponentially as $z \to -\infty$ we expect ψ and u to satisfy the inviscid equations

$$\left.\begin{aligned} \psi'' - \alpha^2\psi &= 0, \\ u &= 0. \end{aligned}\right\} \tag{3.22}$$

If ψ is to be bounded, the solution e^{-az} should be rejected. Hence ψ, which is proportional to e^{az}, satisfies the following condition

$$\psi' - \alpha\psi = 0 \quad \text{for} \quad z \leqslant z_{\text{edge}}. \tag{3.23}$$

The edge of the boundary layer is rather arbitrarily taken to be at $z = -\pi$. Equation (3.23) together with the three boundary conditions $\psi(0) = \psi'(0) = u(0) = 0$, yield four linear homogeneous equations for the coefficients A, B, D, and E. The non-trivial solution is obtained if the following determinant vanishes, namely

$$\begin{vmatrix} \dfrac{U'(0)\phi_1(0)}{V(0)-c} & \dfrac{U'(0)\phi_2(0)}{V(0)-c} & 0 & i\alpha R\,[V(0)-c]\phi_4(0) \\ \phi_1(0) & \phi_2(0) & \phi_4(0) & 0 \\ \phi_1'(0) & \phi_2'(0) & \phi_4'(0) & \phi_4(0) \\ \phi_1'(-\pi)-\alpha\phi_1(-\pi) & \phi_2'(-\pi)-\alpha\phi_2(-\pi) & 0 & 0 \end{vmatrix} = 0. \tag{3.24}$$

If we define

$$\Delta_1 = \begin{vmatrix} \phi_0(0) & \phi_2(0) \\ \phi_1'(-\pi)-\alpha\phi_1(-\pi) & \phi_2'(-\pi)-\alpha\phi_2(-\pi) \end{vmatrix} \tag{3.25}$$

and

$$\Delta_2 = \begin{vmatrix} \phi_1'(0) & \phi_2'(0) \\ \phi_1'(-\pi)-\alpha\phi_1(-\pi) & \phi_2'(-\pi)-\alpha\phi_2(-\pi) \end{vmatrix} \tag{3.26}$$

the determinantal equation (3.24) can be written as

$$\frac{\Delta_1}{\Delta_2} = \frac{1}{\phi_4'(0)/\phi_4(0) - iU'(0)/\{V(0)-c\}^2\alpha R}, \tag{3.27}$$

In order to compare our results with those of plane parallel flows, it is convenient to introduce new quantities as follows

$$W(z;\,\beta) = V(z;\,\beta) - V(0),$$

i.e.,
$$W(z;\,\beta) = e^z \sin(z-\beta) + \sin\beta, \tag{3.28}$$

and
$$C = c - V(0).$$

Except for the second term in the denominator of the fraction on the right hand side,[1] equation (3.27) is identical to the determinantal equation that is obtained when the stability of a plane parallel flow with velocity profile $W(z)$ is investigated. Hence the U'_0-term contains the effect of the Coriolis force on the stability properties of the Ekman layer (viz. critical Reynolds number, growth rate, etc.). In particular, if this term were negligible the phase speed and growth rate of a given wave traveling at an angle β would be identical to the phase speed and growth rate of the Tollmien-Schlichting wave associated with the plane parallel flow with velocity profile $W(z; \beta)$. However, the detailed structure of these two waves would differ.

If we define $u+iv$ as follows:

$$u+iv=\left(1+\frac{W'_0}{C}\frac{\Delta_1}{\Delta_2}\right)^{-1}, \tag{3.29}$$

the determinatal eqation (3.27) can be written as

$$u+iv=G\equiv\frac{C\phi'_4(0)/W'_0\phi_4(0)-iU'_0/\alpha RW'_0C}{1+C\phi'_4(0)/W'_0\phi_4(0)-iU'_0/\alpha RW'_0C}, \tag{3.30}$$

where W'_0 and U'_0 stand for $W'(0)$ and $U'(0)$. The advantage of writing the determinantal equations in this manner will become evident shortly.

In order to express $u+iv$ in terms of αR, C, α, β, we must evaluate $\phi_1(z)$ and $\phi_2(z)$. Expanding ϕ_1 and ϕ_2 in power series of α^2, and substituting in the inviscid Orr-Sommerfeld equation, we obtain (cf. Lin, 1955, p. 34):

$$\left.\begin{aligned}\phi_1&=(W-C)\left[1+\alpha^2\int_0^z(W-C)^{-2}dz\times\int_0^z(W-C)^2dz+\dots\right],\\ \phi_2&=(W-C)\left[\int_0^z(W-C)^{-2}dz+\alpha^2\times\int_0^z(W-C)^{-2}dz\int_0^z(W-C)^2dz\times\int_0^z(W-C)^{-2}dz+\dots\right].\end{aligned}\right\} \tag{3.31}$$

[1] Henceforth, this term will be referred to as the U'_0-term.

To simplify the present analysis we shall neglect terms of order α^2 or higher, thus implicitly assuming that α is small. We shall see that this assumption is not always valid and that it can lead to erroneous conclusions. The determinants Δ_1 and Δ_2 can now be easily obtained, viz.

$$\left.\begin{aligned}\Delta_1&=-C[-(V_0+C)^{-1}+\{(V_0+C)+W'(-\pi\}K],\\ \Delta_2&=-\frac{W'_0}{C}\Delta_1+\frac{\alpha(V_0+C)+W'(-\pi)}{C},\end{aligned}\right\} \tag{3.32}$$

where $$K_1=\int_0^{-\pi}(W-C)^{-2}dz, \tag{3.33}$$

and where $W(-\pi)$, which is equal to $(1+e^{-\pi})\sin\beta$, has been approximated by $\sin\beta(=-V_0)$. The definition of K_1 given in (3.33) has no meaning unless we further specify how to indent the path of integration. This problem is discussed in Appendix A where it is shown that the result obtained for plane parallel flows is also valid in this case, namely at each critical point z_c, the path is indented below the real axis if W'_c is positive and vice versa. Using the definition (3.29) of $u+iv$ and the expressions for Δ_1 and Δ_2 obtained in (3.32) we deduce that:

$$u+iv\equiv1+CW'_0\times\left[K_1-\frac{1}{[\alpha(V_0+C)+W'(-\pi)][V_0+C]}\right]. \tag{3.34}$$

It should be noted at that point that since

$$v\equiv CW'_0\ \mathrm{Im}\ (K_1), \tag{3.35}$$

v is a function of C and β *only*, whereas u is a function of C, β and α. The right hand side of the determinantal equation (3.30), on the other hand, is a function of αRC^3, C, and β. This can easily be seen by evaluating $C\phi'_4(0)/W'_0\phi_4(0)$ by means of (3.16), namely:

$$\frac{C}{W'_0}\frac{\phi'_0(0)}{\phi_4(0)}=\frac{5}{4}+\frac{(i\alpha R)^{\frac{1}{2}}C(-C)^{\frac{1}{2}}}{W'_0}. \tag{3.36}$$

Hence, we can write formally the imaginary and real parts of the determinantal equation as follows:

$$v(C;\ \beta) = G_i(\alpha RC^3,\ C;\ \beta), \tag{3.37}$$

$$u(C,\ \alpha;\ \beta) = G_r(\alpha RC^3,\ C;\ \beta). \tag{3.38}$$

The method for solving these equations in order to obtain the neutral stability curve in the α–R plane is the following:

(i) Consider a specific value of β.

(ii) Select a value of C and evaluate v.

(iii) Solve (3.37) for $\mu = \alpha RC^3$. If (3.37) has no solution, the flow is stable for this value of C. As in the case of plane parallel flows, the nature of the curves G_i vs. μ is such that they can be intersected by a horizontal line with ordinate $v(c)$ in two points with abscissae μ_1 and μ_2 $(\mu_1 > \mu_2)$.

(iv) Evaluate $G_r(\alpha RC^3, C; \beta)$.

(v) Solve (3.38) for α. Knowing α and αR, the neutral stability curve (N.S.C.) $\alpha = \alpha(R; \beta)$ can easily be obtained. Also in this case, the flow is stable unless (3.38) has a solution. For each value of C such that both (3.37) and (3.38) have a solution, we obtain two points (α_1, R_1) and (α_2, R_2) of the N.S.C. The branch $\alpha_1 = \alpha_1(R; \beta)$ is usually referred to as the first or upper branch.

The procedure outlined above is identical to the one followed to solve the determinantal equation for plane parallel flows. There are, however, certain important differences. For instance, in our case, the right hand side $G = G_r + iG_i$ of the determinantal equation is not only a function of $\mu = \alpha RC^3$, but also of C. Furthermore, because of the parameter β, we are forced to investigate several velocity profiles each of which is typical of certain ranges of β. For certain values of C, some of these profiles can be intersected in *two points*. In the following paragraph, we shall derive expressions for u, v, G_i, and G_r valid in different ranges of β and deduce the shape of the N.S.C.

Equation (3.36) expressing $C\phi_4'(0)/W_0'\phi_4(0)$ in terms of αR and C is not valid when

$$\frac{1}{C} < (\alpha R)^{\frac{1}{3}} C^{\frac{1}{3}},$$

i.e.,
$$\mu < 1. \tag{3.39}$$

When this inequality is satisfied, the representation of ϕ_4 given in the Appendix should be used, and hence $\phi_4'(0)/\phi_4(0)$ should be expressed in terms of Airy-functions. This complicates the calculations a great deal, and we shall not extend our analysis to the case in which $\mu < 1$. As a consequence, we shall not be able to obtain the second branch of the N.S.C.

4. Neutral stability curve

We shall try to draw some conclusions about the whole N.S.C. from an analysis of its upper branch. To obtain an expression of the form $\alpha = \alpha(R; \beta)$ valid for large R and large αRC^3, we shall solve simultaneously the real and imaginary parts of the determinantal equation. The procedure we shall follow was outlined at the end of the preceding paragraph. Before we can use it, however, we must obtain expressions for $\operatorname{Re}\{K_1\}$, $\operatorname{Im}\{K_1\}$, G_r and G_i.

1. *Evaluation of* $\operatorname{Im}\{K_1\}$. Since the evaluation of $\operatorname{Im}\{K_1\}$ is relatively easy, we shall consider it first. This evaluation is best achieved by regarding W as the independent variable. In other words, we shall map the z-plane onto the W-plane. The integral K_1 can now be written as

$$K_1 = \int_\Gamma \frac{dW}{W'(z)\{W - C\}^2}, \tag{4.1}$$

where Γ is the appropriate path of integration in the W-plane. Since $(z_M) = 0$, where

$$z_M = \begin{cases} -\dfrac{5\pi}{4} + \beta & \text{for} \quad \dfrac{\pi}{4} < \beta < \dfrac{\pi}{2}, \\[2ex] -\dfrac{\pi}{4} + \beta & \text{for} \quad \dfrac{\pi}{2} < \beta < \dfrac{\pi}{4}, \end{cases} \tag{4.2}$$

the mapping
$$W = W(z) \tag{4.3}$$

is one to one only if we introduce a cut in the W-plane at $W_M = W(z_M)$. In the neighborhood of the branch point W_M, we can easily deduce that

$$W' \simeq (2W_M'')^{\frac{1}{2}}(W - W_M)^{\frac{1}{2}}. \tag{4.4}$$

Therefore, the integrand in (4.1) is integrable in the neigborhood of W_M. Furhermore, except at W_M, the mapping (4.3) is conformal. We shall use this property to find the path of integration Γ, and in particular to deduce the correct indentations at the critical points. We recall that in the z-plane the path of integration

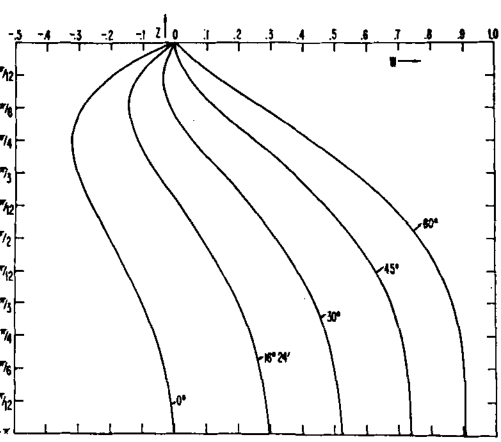

Fig. 2

for K_1 must be indented at the critical point $z = z_c$ below the real axis if W_c' is positive, and vice versa.

The imaginary part of K_1 results entirely from the integration around the critical points. It is, therefore, permissible to expand the integrand in a power series about the critical points. This procedure could be used regardless of whether W or z is the independent variable. The advantage of using W lies in that the critical point(s) is (are) known to be at $W = C$. Expanding W' in powers of $W - C$, we get

$$W' = W_c' \left[1 + \frac{W_c'^2}{W_c''}(W - C) + \ldots \right]. \quad (4.5)$$

As an example, let us evaluate $\mathrm{Im}\{K_1\}$ for the case in which C is negative and is such that there are two critical points z_1 and z_2 $(z_1 > z_2)$. From Fig. 2, we can see that W_1' is positive whereas W_2' is negative. Hence, the path of integration is indented in the z-plane as shown in Fig. 3a. Using the conformal property of the mapping, the path of integration Γ in the W-plane can easily be obtained, and is shown in Fig. 3b. If γ_1 and γ_2 are the two small half circles around $W = C$, we can write, using (4.5)

$$\int_{\gamma_j} \frac{dW}{W'(W-C)^2} = \frac{1}{W_j'} \int_{\gamma_j} \frac{dW}{(W-C)^2} - \frac{W_c''}{W_j'^2} \int_{\gamma_j} \frac{dW}{W-C} + \ldots, \quad (4.6)$$

for $j = 1, 2$. Integrating term by term, we note that only the second term on the right hand side, which is proportional to $[\log (W - C)]_{\gamma_j}$, has an imaginary part. On account of the branch line of the logarithmic function at $W = C$ shown on Fig. 3b, we can write

$$\mathrm{Im}\,\{K_1\} = \pi \left[\frac{W_1''}{W_1'^3} - \frac{W_2''}{W_2'^3} \right]. \quad (4.7)$$

For the case in which C is positive and such that there are two critical points, an identical procedure would yield

$$\mathrm{Im}\,\{K_1\} = -\pi \left[\frac{W_1''}{W_1'^3} - \frac{W_2''}{W_2'^3} \right], \quad (4.8)$$

whereas
$$\mathrm{Im}\,\{K_1\} = \mp \pi \frac{W_c''}{W_c'^3}, \quad (4.9)$$

when C is respectively positive or negative and is such that there is only one critical point.

2. *Behavior of G for large αRC^3:* To separate G into its real and imaginary parts, we must first select the correct determination of $(-C)^{\frac{1}{2}}$ [cf. (3.30) and (3.36)]. This is done by recalling that $(-C)^{\frac{1}{2}}$ is the limit of $(W - C)^{\frac{1}{2}}$ as W approaches zero. Since the branch of the square-root function at $W = C$ is always in the upper half of the W-plane, we can write

$$(-C)^{\frac{1}{2}} = \begin{cases} -iC^{\frac{1}{2}} & \text{for} \quad C > 0, \\ |C|^{\frac{1}{2}} & \text{for} \quad C < 0. \end{cases} \quad (4.10)$$

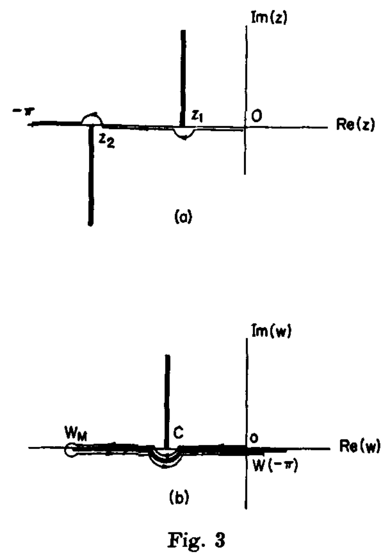

Fig. 3

For large values of $\alpha R|C|^3$, we can, therefore, deduce from (3.30) and (3.36),[1] the following expressions for G_r and G_i

$$G_r \simeq 1 \mp \frac{1}{2Y} \quad (\text{for } C \gtrless 0), \tag{4.11}$$

and

$$G_i \simeq -\frac{1}{2Y}, \tag{4.12}$$

where Y is defined as follows

$$Y = \frac{1}{W_0'}\left\{\frac{\alpha R|C|^3}{2}\right\}^{\frac{1}{3}}. \tag{4.13}$$

Since Y is large for the upper branch, the real and imaginary parts of the determinantal equation can be written as

$$u \equiv CW_0'\left[\mathrm{Re}\,\{K_1\} \frac{1}{\{\alpha(V_0+C)+W'(-\pi)\}(V_0+C)}\right] \simeq \mp\frac{1}{24} \tag{4.14}$$

and

$$v \equiv CW_0'\,\mathrm{Im}\,\{K_1\} \simeq -\frac{1}{2Y}. \tag{4.15}$$

We shall first consider equation (4.15). Since we are only considering large $|Y|$'s, the right hand side of (4.15) is very small, and for this equation to have a solution we should select values of C for which $v(C)$ is also small. We are, therefore, led to search for all the values of C such that v is zero. These values depend upon the angle β.

If there is one critical point, v is proportional to CW_c'' [cf. (3.35) and (4.9)], and hence is zero either when $C = 0$, or when $C = C_I$, where

$$C_I = W(z_I) = -\exp\left(-\frac{\pi}{2}+\beta\right) + \sin\beta, \tag{4.16}$$

and $z_I = -\pi/2 + \beta$ is the abscissa of the inflexion point.

If there are two critical points, v is proportion to $C\{W_1''/W_1'^3 - W_2''/W_2'^3\}$ [cf. (4.7) and (4.8)]. Again v is zero for $C = 0$. But, since there exists a value $C_*(\beta)$ of C for which

$$\left.\begin{aligned} &\frac{W''(z_3^*)}{[W'(z_1^*)]^3} - \frac{W''(z_2^*)}{[W'(z_2^*)]^3} = 0,\\ \text{and}\quad &W(z_1^*) = W(z_2^*) = C_*, \end{aligned}\right\} \tag{4.17}$$

v is also zero for $C = C_*$. Omitting certain tedious but straightforward numerical calculations we can obtain the following expressions for C_*, z_*^1, and z_2^*:

$$\left.\begin{aligned} C_* &= -.1741\exp(\beta) + \sin\beta,\\ z_1^* &= -.2182 + \beta,\\ z_2^* &= -1.734 + \beta. \end{aligned}\right\} \tag{4.18}$$

These expressions are meaningful only when $-\pi < z_1^* < z_2^* < 0$. We can see from (4.18) that C_* exists only in a restricted range of β, namely

$$\beta_* > \beta > \beta_{**}, \tag{4.19}$$

$$\left.\begin{aligned} \text{where}\quad &\beta_* = .2182(\simeq 12^\circ\!.5),\\ \text{and}\quad &\beta_{**} = -\pi + 1.734(\simeq -80^\circ\!.5. \end{aligned}\right\} \tag{4.20}$$

Hence, v is small whenever C is close to either one or more of the characteristic values obtained above, namely 0^-, 0^+, C_I and C_*. However, for a given β, these values are not all relevant; in particular, if $C = C_I$ intersects the velocity profile at a second point distinct from the inflexion point, expressions (4.7) or (4.8) should be used to calculate v, and it is easy to see that $v(C_I) \neq 0$. In fact C_I is a relevant characteristic value only if it is non-negative. Or, using its definition given in (4.16), we can say that C_I is a relevant characteristic value only for $\beta \geqslant \beta_0$, where β_0 is the root of the equation:

$$C_I(\beta_0) = 0 \tag{4.21}$$

which can be solved numerically, yielding $\beta_0 \simeq 16°24'$.

In addition to the angles β_{**}, β_*, and β_0 already encountered with reference to C_* and C_I, we can easily see that $\beta = 0$ and $\beta = \pi/4$ will arise in connection with $C = 0^-$ and $C = 0^+$, subdividing still further the range of β.

3. *Imaginary part of the determinantal equation.* The first column of Table 1 shows the various subintervals of the range of β, the corresponding relevant characteristic values of C being shown in the second column. Although the selection of a value of C close to a relevant

[1] Since the U_0'-term in (3.30) is negligible, the Coriolis force does not enter in the calculations which follow.

TABLE 1

β	Charcteristic values of C	Imiginary part of determinantal equation	Real part of determinantal equation	Type of instability	No. of loops
$\pi/2$					
	C_I	$C \to C_I^+$	$\mathrm{Re}\{K_1(C_I)\} > 0$	Inv. (1)	1
	0^+	Not acceptable	—	—	
$\beta_1 \simeq 55°$					
	C_I	$C \to C_I^+$	$\mathrm{Re}\{K_1(C_I)\} < 0$	Stable	0
	0^+	Not acceptable	—	—	
$\pi/4$					
	C_I	$C \to C_I^+$	$\mathrm{Re}\{K_1(C_I)\} < 0$	—	
	0^+	Not acceptable	—	—	
	0^-	$C \to 0^-$	$\mathrm{Re}\{K_1(C \to 0^-)\} > 0$	Vis. (2)	1
$\beta_0 \simeq 16°$					
	0^+	$C \to 0^+$	$\mathrm{Re}\{K_1(C \to 0^+)\} > 0$	Vis. (1)	2
	0^-	$C \to 0^-$	$\mathrm{Re}\{K_1(C \to 0^+)\} > 0$	Vis. (2)	
$\beta_* \simeq 12°$					
	0^+	$C \to 0^+$	$\mathrm{Re}\{K_1(C \to 0^+)\} > 0$	Vis. (1)	
	0^-	Not acceptable	—	—	2
	C_*	$C \to C_*^+$	$\mathrm{Re}\{K_1(C_*)\} > 0$	Inv. (2)	
0					
	0^-	$C \to 0^-$	$\mathrm{Re}\{K_1(C \to 0^-)\} > 0$	Vis. (1)	2
	C_*	$C \to C_*^+$	$\mathrm{Re}\{K_1(C_*)\} > 0$	Inv. (2)	
$\beta_2 \simeq -40°$					
	0^-	$C \to 0^-$	$\mathrm{Re}\{K_1(C \to 0^-)\} > 0$	Vis. (1)	1
	C_*	$C \to C_*^+$	$\mathrm{Re}\{K_1(C_*)\} < 0$	—	
$\beta_{**} \simeq -80°$					
	0^-	$C \to 0^-$	$\mathrm{Re}\{K_1(C \to 0^-)\} > 0$	Vis. (1)	
$-(\pi/2)$					

characteristic value will insure that $v(c)$ is small, it does not guarantee that $v(C)$ is of the sign of G_i, namely negative for $-\pi/2 < \beta < \pi/4$ and positive for $\pi/4 < \beta < \pi/2$. The third column of Table 1 shows how C should be selected for v to be both small and of the sign of G_i, i.e., for the imaginary part of the determinantal equation to have a solution.

4. *Real part of the determinantal equation.* To complete the description of the upper branch of the N.S.C. we shall now solve (4.14). The evaluation of $\mathrm{Re}\{K_1\}$ necessary to obtain α as a function of C is carried out in Appendix B. Since we are only considering the "tail" of the upper branch, this evaluation is made for C approaching the characteristic values 0^+, 0^-, C_I and C_*.

Values of C for which (4.14) yields negative values of α should be rejected. We shall see that the sign of α depends upon that of $\mathrm{Re}\{K_1\}$.

In the case in which C approaches a finite characteristic value, namely C_I or C_*, α also approaches a finite value α_I or α_* such that

$$\alpha_I = -\frac{W'(-\pi)}{V_0 + C_I} + \frac{1}{(V_0 + C_I)^2 \,\mathrm{Re}\{K_1(C_I)\}}; \qquad (4.22)$$

α_* is given by an identical expression in which C_I is replaced by C_*. Since we are mainly concerned at this stage by the over all properties of the N.S.C. we shall neglect the first term in equation (4.22) and simply say that $\mathrm{Re}\{K_1(C_I)\}$ and $\mathrm{Re}\{K_1(C_*)\}$ must be positive for α_I and α_* to be positive. Fig. 4, which represents the variations of $\mathrm{Re}\{K_1(C_I)\}$ and $\mathrm{Re}\{K_1(C_*)\}$, shows that this requirement is fulfilled for

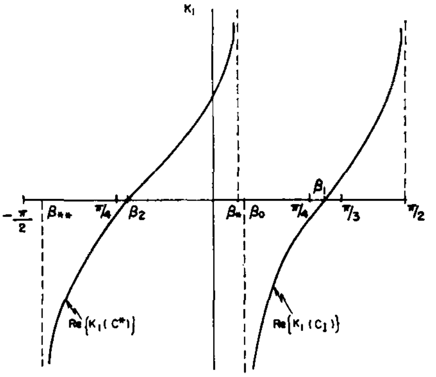

Fig. 4

$\beta_1 < \beta < \pi/2$ and $\beta_2 < \beta < \beta_*$, respectively, where $\beta_1 \simeq 55°$ and $\beta_2 \simeq -40°$.

When C approaches zero, we can write, using (B.3) and (B.6)

$$\text{Re}\ [K_1(C \to 0)] = -\frac{1}{W_c' C} + 0\ (\log |C|)$$

where W_c' is the value of W' at the critical point closest to the origin. Substituting this expression in (4.14), we deduce that α tends to zero as C, namely

$$\alpha \sim -\left[\frac{W_c'}{V_0^2}\right] C. \qquad (4.23)$$

Therefore, α is positive if $-W_c' C$ and hence $\text{Re}\,\{K_1(C \to 0)\}$ are positive. An investigation of the sign of W_c' both when C approaches 0^- and 0^+ shows that $\text{Re}[K_1(C \to 0)]$ is always positive.

The fourth column of Table 1, which shows the sign of $\text{Re}\,\{K_1(C)\}$ for the various relevant, acceptable characteristic values of C, indicates whether the real part of the determinantal equation has a solution. The fifth and sixth columns are discussed in the next paragraph.

5. Conjectures about the N. S. C.

Investigations of plane parallel flows showed that whenever they are unstable, the instability is either of viscous or inviscid character. An instability which occurs on account of the presence of an inflexion point in the velocity profile is called inviscid. In this case the upper branch of the N.S.C. has an asymptote α_I for large R, for which it can be shown that (LIN, 1945, pp. 287)

$$R \sim (\alpha - \alpha_I)^{-2}. \qquad (5.1)$$

The other type of instability is called viscous because viscosity plays a destabilizing role. In this case the upper branch of the N.S.C. is such that as R tends to infinity, α tends to zero in the following manner:

$$\alpha \sim R^{-1/6}. \qquad (5.2)$$

Therefore, a knowledge of the "tail" of the upper branch of the N.S.C. is sufficient to distinguish between the two kinds of instabilities.

In the present analysis what seems to be a third kind of instability was found to occur for $\beta_* > \beta > \beta_2$. By means of calculations similar to the ones necessary to obtain (5.1) and (5.2) we would get the following relation for the upper branch of the N.S.C. associated with this instability

$$R \sim (\alpha - \alpha_*)^{-2} \qquad (5.3)$$

On account of the analogy between (5.3) and (5.1), we conjecture that this instability is of an inviscid nature and that it differs from the usual one because of the existence of two viscous layers. In particular, for infinite Reynolds number, the phase speed of the neutral wave is C_* rather than C_I.

From Table 1, we can see that two of the above mentioned types of instability can occur for the same β. In this case the N.S.C. has two loops. A study of the lower branches of both loops should be done to determine whether they are stacked one on top of the other or whether one loop is inside the other. We have indicated in the fifth column of Table 1 the kind of instability (i.e., inviscid or viscous) and the number of viscous layers or regions. Finally, the sixth column shows the number of loops of the N.S.C. for each subinterval of the range of β.

As $\beta \to \beta_1$, the ordinate $\alpha_1(\beta)$ of the asymptote of the upper branch increases indefinitely. This is easy to see from equation (4.22) if we recall that $\text{Re}\,\{K_1(C_I)\}$ approaches zero as β approaches β_1 (cf. Fig. 1B). In other words the instability is greater as $\beta \to \beta_1$. This result can be understood heuristically by noting that for $-\pi < z < 0$, the velocity profiles $W(z, \beta)$ are similar to velocity profiles of boundary layer type with an adverse pressure gradient which increases as β decreases. It is well known that the existence of an adverse pressure gradient destabilizes plane parallel flows (SCHLICHTING, 1955, p. 339; cf. also Fig. 17.1). On account of the increase of α_I, the approximation which was made in order to evaluate $\phi_1(-\pi)$, $\phi_1'(-\pi)$, $\phi_2(-\pi)$ and $\phi_2'(-\pi)$ namely $\alpha < 1$, is no longer justified. We must, therefore, regard the results obtained in the neighborhood of β_1 (and by the same token β_2) as suspicions. In particular, it would be otherwise difficult to reconcile the conclusion that $W(z, \beta)$ is stable for $\beta_1 > \beta > \pi/4$ with the well-known result that all two-dimensional flows of boundary layer type are unstable for sufficiently large R. Therefore, the angles β_1 and β_2 have no real physical significance and stem in the present analysis from the mathematics.

The angle $\bar{\beta}$ that one would actually observe in an experiment is determined by minimizing $R_{\min}(\beta)$ with respect to β, namely

$$\left\{\frac{dR_{\min}(\beta)}{d\beta}\right\}_{\beta=\bar{\beta}} = 0. \tag{5.4}$$

We were not able to obtain an analytic expression for $R_{\min}(\beta)$, and the determination of $\bar{\beta}$ is now being attempted by numerical means. If $\bar{\beta}$ is equal to $\beta_0 (\beta_0 \simeq 16°24')$ as suggested by the experiments (FALLER, 1963), we can see from Table 1 that the instability associated with this angle is of viscous rather than inviscid character. Such a result is surprising, unless another loop associated with an inviscid instability is missing on account of our approximation. Actually since β_1 and β_2 are spurious subdivisions of the range of β, we suspect that a more refined analysis in which the assumption $\alpha < 1$ is relaxed, would show that such an inviscid instability does occur in the intervals $\beta_0 < \beta < \beta_1$ and $\beta_{**} < \beta < \beta_2$.

Finally, we should recall that all our results concerning the upper branch of the N.S.C. were obtained without the U_0'-term ever entering in the calculations. This term is negligible in comparison with $C\phi_4'(0)/W_0'\phi_4(0)$ when the latter is expressed in terms of Airy functions even for the lower branch.[1] Therefore the Coriolis force does not seem to affect the shape of the N.S.C. which is similar to that of a plane parallel flow with velocity $W(Z:\beta)$.

[1] The author is indebted to Prof. C. C. Lin for pointing this out to him.

6. Conclusion

We have investigated the stability of a non-divergent Ekman layer for large αR by the method of small perturbations in the form of traveling waves. For each value of the angle β between the direction of motion of the lid and that of the waves, we were led to investigate the stability of a plane parallel flow whose velocity profile is identical to the component of the Ekman velocity along the direction of propagation of the waves. In particular we have shown that the perturbation fields can be expressed in terms of the four fundamental solutions of the Orr-Sommerfeld equation associated with this plane parallel flow. On account of the Coriolis force, these traveling waves differ from the Tollmien-Schlichting waves corresponding to this equivalent plane parallel flow. For large Reynolds numbers, however, the Coriolis force does not affect the stability properties of the Ekman layer (viz. phase speed and growth rate of the waves), and the N.S.C. is similar to the one that would be obtained by investigating a two-dimensional plane parallel flow with velocity profile $W(z; \beta)$. In this respect, we can say that the Coriolis force does not play a significant role in the stability analysis. Although the viscous force is comparable in magnitude to the former, it plays a different role. As in the classical theory of plane parallel flow, it is important both in the neighborhood of the rigid wall where it enables the no-slip condition to be satisfied, and at the critical points where it determines the phase shift. Finally, we must recall that these two forces also influence the stability properties in an indirect way in so far as they determine the profile of the mean flow.

We have found that for infinite Reynolds number the phase speed of the neutral wave differs from C_I whenever there are two critical points, and is equal to C_* such that

$$C_* = W(z_1^*) = W(z_2^*),$$

and
$$\frac{W''(z_1^*)}{[W'(z_1^*)]^3} - \frac{W''(z_2^*)}{[W'(z_2^*)]^3} = 0,$$

where z_1^* and z_2^* are the abscissae of the two critical points. If we agree to denote by C_* the phase speed of the neutral wave for any number of critical points, we generalize the above formulae, which were obtained in § 4 under the assumption $\alpha < 1$, to the case of n critical points. In order to find their abscissae $z_1^*, z_2^*, \dots, z_n^*$ and C_* we simply require (1) that these critical points lie along the same vertical line, an unknown distance C_* away from the z-axis and (2) that $\text{Im}\{K_1\}$ equal zero. Recalling that each critical point contributes to $\text{Im}\{K_1\}$, we can write:

$$C_* = W(z_1^*) = \dots = W(z_n^*), \qquad (6.1)$$

and

$$\sum_{k=1}^{n} (-1)^k \frac{W''(z_k^*)}{[W'(z_k^*)]^3} = 0. \qquad (6.2)$$

We can immediately see that for $n = 1$, this system of equations yields the classical results: $C_* = C_I$ and $z_1^* = z_I$.

Acknowledgments

I would like to thank Professor L. N. Howard for many valuable discussions during the course of this investigation and for carrying out an extensive machine computation of the integral K_1. The financial support provided by the National Science Foundation is gratefully acknowledged.

Appendix A. Representation of solutions valid at a critical point

We have seen that the representation of the solutions obtained in Section 3 is not valid in the neighborhood of a critical point z_c. As in the case of the classical Orr-Sommerfeld equation this feature is entirely due to the method of solution. To obtain a representation valid in the neighborhood of z_c we follow the approach first used by Heisenberg (Lin, 1955, pp. 39 and 126). Introducing a generalization of the usual change of variables, we define:

$$\left.\begin{aligned} \zeta &= (\alpha R)^{\frac{1}{3}}(z - z_c), \\ \chi(\zeta) &= \psi(z), \\ \omega(\zeta) &= (\alpha R)^{-n} u(z). \end{aligned}\right\} \qquad (A.1)$$

Substituting (A.1) in (2.13) and expanding the non-constant coefficients in powers of ζ we get

$$\frac{d^4\chi}{d\zeta^4} - iV_c'\zeta\frac{d^2\chi}{d\zeta^2} - 2(\alpha R)^{n-1}\frac{d\omega}{d\zeta} + 0(\alpha R)^{-\frac{1}{3}} = 0, \qquad (A.2)$$

$$(\alpha R)^{n-\frac{1}{3}}\left\{\frac{d^2\omega}{d\zeta^2} - iV_c'\zeta\omega\right\} + U_c'\chi + 0(\alpha R)^{-\frac{2}{3}} = 0, \qquad (A.3)$$

where U_c' and V_c' stand for dU/dz and dV/dz evaluated at $z = z_c$. We can easily see that six independent solutions are obtained when n is successively equal to $\frac{1}{3}$ and 1. We shall examine the solutions of (A.2) and (A.3) for each value of n separately. In analogy with plane parallel flows certain solutions are expressed in terms of Hankel functions of order one-third, and it is convenient to define

$$\left.\begin{aligned} \varrho_1 &= \zeta^{\frac{1}{2}} H_{\frac{1}{3}}^{(1)}[\tfrac{2}{3}(iV_c'^{\frac{1}{3}}\zeta)^{\frac{3}{2}}], \\ \varrho_2 &= \zeta^{\frac{1}{2}} H_{\frac{1}{3}}^{(2)}[\tfrac{2}{3}(iV_c'^{\frac{1}{3}}\zeta^{\frac{3}{2}})]. \end{aligned}\right\} \qquad (A.4)$$

When $n = \frac{1}{3}$, equations (A.2) and (A.3) can be written as

$$\left.\begin{aligned} &\frac{d^4\chi}{d\zeta^4} - i\zeta V_c'\frac{d^2\chi}{d\zeta^2} = 0, \\ &\frac{d^2\omega}{d\zeta^2} - i\zeta V_c'\omega = -U_c'\chi. \end{aligned}\right\} \qquad (A.5)$$

The first equation in (A.5), which is identical to the one obtained in the theory of plane parallel flows, admits the following solutions (Lin, 1955, pp. 127):

$$\left.\begin{aligned} \chi_1 &= \zeta, \\ \chi_2 &= 1, \\ \chi_3 &= \int_{+\infty}^{\zeta} d\zeta \int_{+\infty}^{\zeta} \varrho_1 d\zeta, \\ \chi_4 &= \int_{-\infty}^{\zeta} d\zeta \int_{-\infty}^{\zeta} \varrho_2 d\zeta. \end{aligned}\right\} \qquad (A.6)$$

Using the method of variation of constants, the corresponding expressions for ω are easily obtained, namely

$$\omega_k = \frac{\pi U_c'}{6i}\left[\varrho_2 \int_{+\infty}^{\zeta} \varrho_1 \chi_k d\zeta - \varrho_1 \int_{-\infty}^{\zeta} \varrho_2 \chi_k d\zeta\right], \tag{A.7}$$

where $k = 1, 2, 3, 4$.

When $n = 1$, equations (A.2) and (A.3) become

$$\left.\begin{aligned} &\frac{d^4\chi}{d\zeta^4} - iV_c'\zeta\frac{d^2\chi}{d\zeta^2} = 2\frac{d\omega}{d\zeta}, \\ &\frac{d^2\omega}{d\zeta^2} - iV_c'\zeta\omega = 0. \end{aligned}\right\} \tag{A.8}$$

Following the reverse procedure, we solve for ω first and then for χ. The solutions are:

$$\left.\begin{aligned} &\omega_5 = \varrho_1, \\ &\omega_6 = \varrho_2 \end{aligned}\right\} \tag{A.9}$$

and

$$\begin{aligned} \chi_5 = \frac{\pi}{3i}\Bigg\{ &-\zeta \int_{+\infty}^{\zeta} \varrho_1' [\varrho_2\chi_3' - \varrho_1\chi_4'] d\zeta \\ &+ \chi_3 \int_0^{\zeta} \varrho_1'\varrho_2 d\zeta - \chi_4 \int_{+\infty} \varrho_1'\varrho_1 d\zeta \\ &+ \int_{+\infty}^{\zeta} \varrho_1' [\zeta(\varrho_2\chi_3' - \varrho_1\chi_4') - (\varrho_2\chi_3 - \varrho_1\chi_4)] d\zeta \Bigg\}, \\ \chi_6 = \frac{\pi}{3i}\Bigg\{ &-\zeta \int_{-\infty}^{\zeta} \varrho_2' [\varrho_2\chi_3' - \varrho_1\chi_4'] d\zeta \\ &+ \chi_3 \int_{-\infty}^{\zeta} \varrho_2'\varrho_2 d\zeta - \chi_4 \int_0^{\zeta} \varrho_2'\varrho_1 d\zeta \\ &+ \int_{-\infty}^{\zeta} \varrho_2' [\zeta(\varrho_2\chi_3' - \varrho_1\chi_4') - (\varrho_2\chi_3 - \varrho_1\chi_4)] d\zeta \Bigg\}. \end{aligned} \tag{A.10}$$

These representations are clearly valid for fixed ζ and $\alpha R \to \infty$, i.e., for $z \to z_c$. If the argument of ζ satisfies the familiar inequality

$$-\frac{7\pi}{6} < \arg\,(V_c'^{\frac{1}{3}}\zeta) < \frac{\pi}{6}, \tag{A.11}$$

we can show by using the appropriate asymptotic expansions for the Hankel functions entering in the definition of ϱ_1 and ϱ_2 that the two representations are identical in a region of overlapping validity. Furthermore, since inequality (A.11) is identical to the one obtained in the theory of plane parallel flows, the same criterion for the indentation of the path of integration holds, namely the indentation is below the real axis for $V_c' > 0$ and above for $V_c' < 0$.

Appendix B. Evaluation of Re $\{K_1(C)\}$

The difficulty encountered in the evaluation of $K_1(C)$ resides in that the integrand has one or two double poles. We can circumvent this difficulty by defining a new integral $L(C, \beta)$ closely related to K_1, but such that its integrand has no singularities in the range of interest.[1] For values of C for which there exists one critical point, we shall define the integral $L(C, \beta)$ as follows:

$$L(C,\beta) = \int_0^{-\pi} \frac{1 - W'/W_c' + (W_c''/W_c'^3)W'(W-C)}{(W-C)^2}\,dz. \tag{B.1}$$

We can see that this integrand in (B.1) is finite and well defined everywhere since its numerator has a double zero at $W = C$. By first evaluating L numerically and then integrating each of the three terms of its integrand separately we get the following formula for K_1

$$K_1(C,\beta) = L_{\text{num.}} - \frac{1}{W_c'}\left[\frac{1}{W-C}\right]_0^{-\pi} - \frac{W_c''}{W_c'^3}\left[\log\,(W-C)\right]_0^{-\pi}, \tag{B.2}$$

where L_{num} stands for the numerical value of L. We have used formula (B.2) to evaluate Re $\{K_1(C_I)\}$, Re $\{K_1(C \to 0^+)\}$ and Re $\{K_1(C \to 0^-)\}$ for $\beta_0 < \beta < \pi/2$, $0 < \beta < \pi/4$ and $\beta < 0$, respectively. Taking advantage of the smallness of C, we can obtain a single analytical expression for both Re $\{K_1(C \to 0^+)\}$ and Re $\{K_1(C \to 0^-)\}$, namely

$$\text{Re}\,\{K_1(C \to 0\} = -\frac{1}{W_c' C} + 0\,(\log |C|). \tag{B.3}$$

[1] The author is indebted to Prof. L. N. Howard for suggesting this method as well as the form of the integral L.

Since no such simplification is possible for $\mathrm{Re}\{K_1(C_I)\}$ we have plotted the latter on Fig. 4.

The calculation procedure is only slightly modified for the case in which there are two critical points; in particular the definition of L becomes

$$L = \int_0^{z_M} \frac{1 - W'/W_1' + (W_1''/W_1'^3)W'(W-C)}{(W-C)^2}\, dz + \int_{z_M}^{-\pi} \frac{1 - W'/W_2' + (W_2''/W_2'^3)W'(W-C)}{(W-C)^2}\, dz, \quad \text{(B.4)}$$

where W_1', W_1'', W_2', W_2'' stand respectively for W', W'' evaluated at the critical points z_1 and z_2; z_M is defined in (4.2). Once L is evaluated numerically, we use the following formula to compute K_1

$$K_1 = L_{\text{num.}} - \frac{1}{W_1'}\left[\frac{1}{W-C}\right]_0^{z_M} - \frac{1}{W_2'}\left[\frac{1}{W-C}\right]_{z_M}^{-\pi} - \frac{W_1''}{W_1'^3}[\log\,(W-C)]_0^{z_M} - \frac{W_2''}{W_2'^3}[\log\,(W-C)]_{z_M}^{-\pi}. \quad \text{(B.2)}$$

We should use (B.5) to evaluate $\mathrm{Re}\,\{K_1(C_*)\}$ and $\mathrm{Re}\,\{K_1(C\to 0^-)\}$ for $\beta_{**} < \beta < \beta_*$ and $0 < \beta < \pi/4$ respectively. As previously, when $C \to 0^-$ (and $0 < \beta < \pi/4$) we can deduce an analytic expression for $\mathrm{Re}\,\{K_1(C\to 0^-)\}$ namely,

$$\mathrm{Re}\,\{K_1(C\to 0^-)\} = \frac{-1}{W_1'C} + 0\,(\log |C|)$$

or since z_1 is close to the origin:

$$\mathrm{Re}\,\{K_1(C\to 0^-)\} = \frac{-1}{W_0'C} + 0\,(\log |C|). \quad \text{(B.6)}$$

The graph of $\mathrm{Re}\ K_1(C_*)\}$ vs. β is shown on Fig. 4.

REFERENCES

Ekman, W. W., 1905, On the influence of the earth's rotation on ocean currents. *Ark. Mat. Astr. Fys.*, 2.

Ellison, T. H., 1956, Atmospheric Turbulence. Survey in Mechanics (ed. G. K. Batchelor and R. M. Davies), p. 414, Camb. Univ. Press.

Faller, A. J., 1963, An experimental study of the instability of the laminar Ekman boundary layer. *J. Fluid Mech.*, **15**, pp. 560–576.

Lin, C.-C., 1945, On the stability of two dimensional parallel flows. Parts I and III. *Quart. J. Appl. Math.*, **3**, pp. 117–142 and pp. 277–301.

Lin, C.-C., 1955, *The Theory of Hydrodynamic Stability*. Camb. Univ. Press.

Schlichting, H., 1955, *Boundary Layer Theory*, p. 339. McGraw-Hill, New York.

Stern, M. E., 1960, Instability of Ekman flow at large Taylor number. *Tellus*, **12**, pp. 399–417.