Inertial flow in the Gulf Stream

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ABSTRACT

This paper departs from previous studies of steady, laminar, purely inertial models of the Gulf Stream in that it considers two moving layers of different density and with different vertically uniform flow velocities instead of only one such layer. The flow takes place over a resting layer of denser water. Numerical methods are used where analytical results cannot be obtained. Instead of yielding a refined description that fits the observed Gulf Stream better than do the one-layer models, the two-layer models show anomalous features that seem physically unreasonable and that seem to prevent any correspondence with the observed Gulf Stream. Two types of anomalies occur. 1. If an overall density gradient is modelled by splitting the one-layer model into two equally thick layers where the lower layer has a density intermediate between that of the upper layer and the fluid at rest below, the inertial-dynamical solution is invalidated by discontinuities at latitudes near the middle of the range where the observed Gulf Stream seems steady and well-behaved, and where the one-layer models seem to give an adequate description of the flow. 2. If the interface between the lower layer and the fluid at rest below is made nearly horizontal in the interior of the ocean to the east of the Gulf Stream, so as to correspond to the observed 6°C isotherm, then no dynamical solution exists. This is because conservation of potential vorticity for transport of water over large ranges of latitude is incompatible with the small geostrophic velocities required in the lower layer by the nearly horizontal lower interface. Such difficulties lead to the tentative conclusion that steady, purely inertial models are inadequate to describe even the lower latitude growth region of the Gulf Stream.

1. Introduction

A theory of the growth region of the Gulf Stream as a steady, laminar, purely inertial boundary layer has been given by MORGAN (1956), and by CHARNEY (1955). Their results give as good agreement with observation as can be expected from their one-moving-layer models, and therefore, after Charney commented that his theory could be extended to a continuously stratified model, STOMMEL (1960) wrote, "At present such refinements do not seem worthwhile, however. One cannot help believing that a correct zero-order approximation has been achieved."

There is, however, one conspicuous failure of these inertial models. They do not give the Gulf Stream countercurrent which is found in association with the warm core, a surface lens of warm water in the swiftly moving portion of the stream. It seemed that to calculate this detail a second moving layer to represent the warm core would have to be added to the model.

This was the motivation for the present work which extends the analysis of Morgan and Charney from one to two moving layers. In the course of this investigation it became evident that there were problems of a more fundamental nature which must be discussed in terms of the two layer model, namely, do solutions even exist for purely inertial, continuously stratified models of the Gulf Stream? The indications of the present work are that they do not.

2. Formulation

The equations of motion are taken to be those for steady, adiabatic, inviscid flow on a beta-plane, the same conditions as in the papers of Morgan and Charney. These workers considered a model consisting of an upper layer of density, ϱ_1, thickness D_1, and of vertically uniform velocity (u_1, v_1), flowing over a resting layer of density ϱ_2, $\varrho_1 < \varrho_2$. This model reflects

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the fact that the velocity in the Gulf Stream decreases with depth. The present model is similar; the upper layer of Morgan and Charney's model is split into two layers, each with a vertically uniform, but different velocity. The equations of motion for this system are then

$$-fv_1' + \frac{1}{\varrho_1} p'_{1,x'} = 0, \tag{1}$$

$$u_1' v'_{1,x'} + v_1' v'_{1,y'} + fu_1' + \frac{1}{\varrho_1} p'_{1,y'} = 0, \tag{2}$$

$$(D_1' u_1')_{,x'} + (D_1' y_1')_{,y'} = 0, \tag{3}$$

$$-fv_2' + \frac{1}{\varrho_2} p'_{2,x'} = 0, \tag{4}$$

$$u_2' v'_{2,x'} + v_2' v'_{2,y'} + fu_2' + \frac{1}{\varrho_2} p'_{2,y'} = 0, \tag{5}$$

$$(D_2' u_2')_{,x'} + (D_2' v_2')_{,y'} = 0, \tag{6}$$

where commas indicate differentiation with respect to the variables following, and the subscripts (1, 2) indicate the upper and lower layers. Primes indicate variables with the usual physical dimensions as appropriate. The spatial variables (x,y) and velocity components (u, v) are positive eastward and northward respectively. p is the pressure, f is the coriolis parameter, $2\Omega \sin\theta_0(1 + (y'/R) \cot \theta_0)$, where θ_0 is the latitude of tangency of the beta-plane Ω is the earth's angular velocity, and R is the earth's radius.

The first momentum equation in both layers has no nonlinear terms, and is therefore purely geostrophic, in accordance with the boundary layer nature of the flow, as discussed by Morgan and Charney.

Expressions for the pressure gradients in terms of the densities and the gradients of the layer thicknesses can be obtained by using the assumption of a level of no motion, and therefore vanishing pressure gradient, below the lower layer. We find,

$$p'_{1,x'} = \frac{\varrho_1}{\varrho_3} (\varrho_3 - \varrho_2)\, g[\gamma D'_{1,x} + D'_{2,x}], \tag{7}$$

$$p'_{2,x'} = g\left[\varrho_2 - \varrho_1 + \frac{\varrho_2}{\varrho_3}(\varrho_3 - \varrho_2)\right] D'_{1,x'} + g\frac{\varrho_2}{\varrho_3}(\varrho_3 - \varrho_2)\, D'_{2,x'}, \tag{8}$$

where $\gamma = (\varrho_3 - \varrho_1)/(\varrho_3 - \varrho_2)$. At this point, after the small density differences which cause the pressure gradients have been taken into account, the approximations $(\varrho_2/\varrho_3) = 1$, etc may be made. To nondimensionalize the equations we introduce as characteristic sizes of the variables: D_i, D_0; y', R; x', $\lambda = \sqrt{g'D_0}/2\Omega \sin\theta_0$; v', $\sqrt{g'D_0}$; u', $\sqrt{g'D_0}\, \lambda/R$; f, $2\Omega \sin\theta_0$; where D_0 is the sum of the layer thicknesses at $x = \infty$, $y = 0$ $g' = g(\varrho_3 - \varrho_2)/\varrho_3$; and g is the acceleration of gravity.

With the appropriate substitutions, the final nondimensional equations are

$$-\zeta v_1 + \gamma D_{1,x} + D_{2,x} = 0, \tag{9}$$

$$u_1 v_{1,x} + v_1 v_{1,y} + \zeta u_1 + \gamma D_{1,y} + D_{2,y}, \tag{10}$$

$$(D_1 u_1)_{,x} + (D_1 v_1)_{,y} = 0, \tag{11}$$

$$-\zeta v_2 + D_{1,x} + D_{2,y} = 0 \tag{12}$$

$$u_2 v_{2,x} + v_2 v_{2,y} + \zeta u_2 + D_{1,y} + D_{2,y} = 0 \tag{13}$$

$$(D_2 u_2)_{,x} + (D_2 v_2)_{,y} = 0 \tag{14}$$

The variable $\zeta = 1 + y \cot\theta_0$ has been introduced because in the subsequent development it plays a special role as the natural latitude coordinate for the problem.

Charney derived first integrals of the equations for his one-layer models. Exactly analogous procedures for the above equations yield:

$$\frac{v_{1,x} + \zeta}{D_1} = F_1(\psi_1), \quad \frac{v_{2,x} + \zeta}{D_2} = F_2(\psi_2), \tag{15}$$

$$\tfrac{1}{2} v_1^2 + \gamma D_1 + D_2 = G_1(\psi_1), \quad \tfrac{1}{2} v_2^2 + D_1 + D_2 = G_2(\psi_2), \tag{16}$$

where $F_i(\psi_i)$ and $G_i(\psi_i)$ are arbitrary functions of the stream functions, defined by $\psi_{i,x} = D_i v_i$, $\psi_{i,y} = -D_i u_i$.

The equations to be solved are found by substituting for v_i in (15), using (9, 12) to obtain

$$\gamma D_{1,xx} + D_{2,xx} - \zeta D_1 F_1(\psi_1) = -\zeta^2, \tag{17}$$

$$D_{1,xx} + D_{2,xx} - \zeta D_2 F_2(\psi_2) = -\zeta^2. \tag{18}$$

Fig. 1 is a schematic diagram of the model under consideration. Water flows toward the coast north of $\zeta = 1$ and turns north forming a boundary current. The thicknesses of the layers at the coast are h_i. The zero streamlines are along $\zeta = 1$ and the coast.

For the special case of constant potential vorticity in which $D_1(\infty, \zeta) = \delta\zeta/(1 + \delta)$, $D_2(\infty, \zeta) = \zeta/(1 + \delta)$, δ a constant, F_1 and F_2 are constant and the above equations become linear. Note that the sum of the thicknesses of the two layers is normalized to unity at $x = \infty$, $\zeta = 1$. Constant potential vorticity is of interest chiefly because, as shown by STOMMEL (1960), it is a good approximation to the actual Gulf Stream.

The particular solutions for constant potential vorticity are $\delta\zeta/(1+\delta)$ and $\zeta/(1+\delta)$ for the first and second equations respectively, and, assuming exponential solutions of the form $\exp(\alpha\zeta^{\frac{1}{2}}x)$, the characteristic equation for the general solution is

$$\alpha^2 = \{(1+\delta)/2(\gamma-1)\}\{\gamma + 1/\delta \pm [(\gamma - 1/\delta)^2 + 4/\delta]\}. \tag{19}$$

Only the negative roots are retained so that the north-south velocities will decay away from the boundary layer. The expressions for the layer thicknesses are therefore,

$$D_1 = a_1 n_1 e^{-\alpha_1\zeta^{\frac{1}{2}}x} + a_2 n_2 e^{-\alpha_2\zeta^{\frac{1}{2}}x} + \delta\zeta/(1+\delta), \tag{20}$$

$$D_2 = a_1 e^{-\alpha_1\zeta^{\frac{1}{2}}x} + a_2 e^{-\alpha_2\zeta^{\frac{1}{2}}x} + \zeta/(1+\delta), \tag{21}$$

where $n_i = [(1+\delta) - \alpha_i^2]/\alpha_i^2$. Since (D_1, D_2) must equal (h_1, h_2) at $x = 0$, it is possible to replace the variables (a_1, a_2) by (h_1, h_2). Then by the geostrophic equations (9) and (12) the velocities at the coast may be obtained in terms of (h_1, h_2). When these expressions are substituted in the two Bernoulli equations, (16), the result is two simultaneous quadratic equations for h_1 and h_2.

3. Constant potential vorticity solutions

A natural problem to consider is the effect of simply splitting the one-layer model in two. For this purpose we select the parameters $\gamma = 2$, $\delta = 1$, two layers with equal thicknesses and with equal density differentials across them.

Fig. 2 shows the results of the calculation in a plot of h_1 as a function of ζ. All real solutions to the pair of quadratics have been plotted in this figure. The symbols such as $(+\,+\,-\,+)$ give the signs of (h_1, h_2, v_1^0, v_2^0) where v_i^0 are the velocities at the coast at the closest computed point. The line of solutions extending north from $\zeta = 1.0$, $h_1 = 0.5$ is of greatest interest. It attains a maximum value of ζ at $h_1 = 0.112$, $\zeta = 1.464$, where $h_{1,\zeta} = \infty$.

In BLANDFORD (1964), hereafter referred to as I, analytical solutions are given for the case

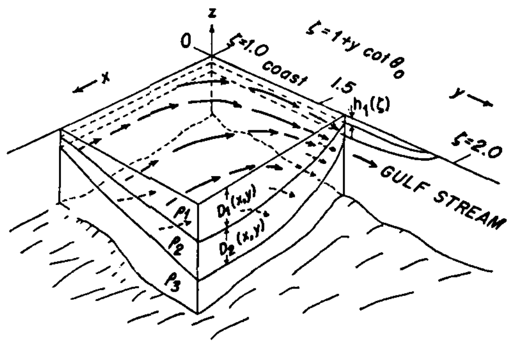

FIG. 1. The flow regime for Gulf Stream models in the present work. The water flows toward the coast in layers of thickness $D_i(x,y)$ and densities ϱ_i, over a resting layer of density ϱ_3 below which is the irregular ocean floor. The thickness of the layers at the (vertical) coast are $h_i(\zeta)$, where ζ is proportional to the coriolis parameter in the beta-plane approximation. The east-west coordinate is x, the north-south coordinate is y, and the vertical coordinate is z. The velocity vectors in the two layers (u_i, v_i), are indicated by solid lines in the upper layer and by dashed lines in the lower, and are not parallel in general. The streamlines along the coast move toward the coast at $y = 0$ or $\zeta = 1.0$. This latitude is drawn as a solid coast in the figure, and may be regarded as the north coast of South America, or simply as the $\psi = 0$ streamline.

in which the upper layer has moved away from the coast. This would happen at the point near $\zeta = 1.45$ in the first quadrant where $h_1 = 0$. These solutions are not presented here because they turn out to be of only incidental interest to the most important problems of the present work.

The portion of the solution between $\zeta = 1.0$ and 1.464 is plotted in more detail in Fig. 3. The dashed lines show (D_1, D_2) in the interior; the solid lines show (h_1, h_2) at the coast. To the left is a representation of the density structure. In the present case ϱ_2 is midway between ϱ_1 and ϱ_3, $\gamma = 2$. We see that $h_{2,\zeta}$ diverges near $\zeta =$ 1.464 also. By the geostrophic equation, these divergences imply that the east-west velocities also diverge, and therefore that the non-linear terms become important in the first momentum equation, and therefore that the entire theory fails. We can see that in any event the solution could not have extended farther than $\zeta = 2.0$, by equating the total flux across a given latitude to the total flux toward the coast below that latitude. The flux north is given by

$$\int_0^\infty (D_1 v_1 + D_2 v_2)\,dx.$$

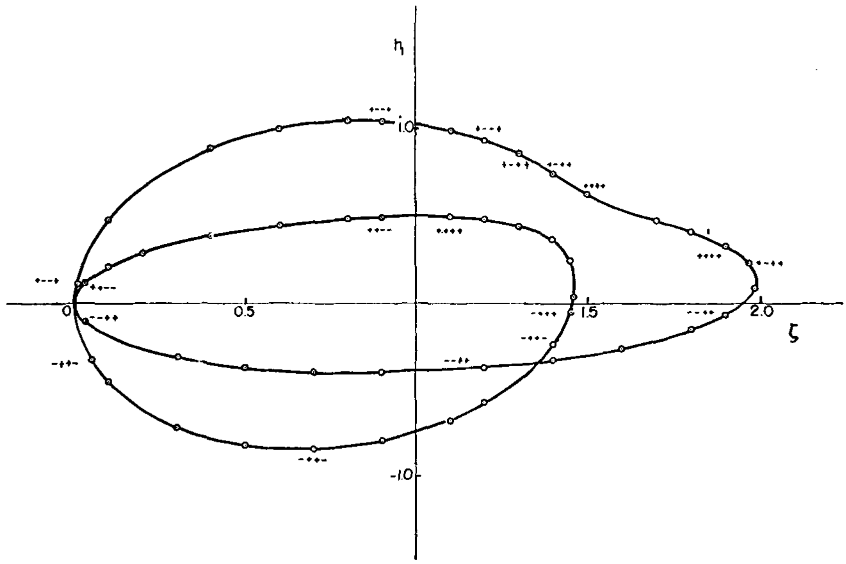

FIG. 2. The solution for a constant potential vorticity model with $\gamma = 2.0$, $\delta = 1.0$. Solution plotted in the (h_1, ζ) plane, where h_1 is the thickness of the upper layer at the coast and ζ is the latitude coordinate. The symbols like $(+ + - +)$ give the signs of (h_1, h_2, v_1^0, v_2^0) at the nearest calculated point. The velocities at the coast are v_1^0 and v_2^0. The correct solution has $h_1 = 0.5$ at $\zeta = 1.0$, and extends north.

By substituting for v_i from the geostrophic equations the integral may be performed. If the result is equated to the flux toward the coast north of $\zeta = 1$ we find,

$$\gamma h_1^2 + 2h_1 h_2 + h_2^2 = \frac{\zeta(2-\zeta)}{(1+\delta)^2}(\gamma\delta^2 + 2\delta + 1). \quad (22)$$

This shows that if h_1 and h_2 are positive there can be no solution north of $\zeta = 2$. The same result is obtained for one layer, a special case of two layers for which $h_1 = \delta = 0$. However, in that case the inertial solution does extend smoothly from $\zeta = 1$ to 2. It is therefore clear that simply splitting the one-layer model in two

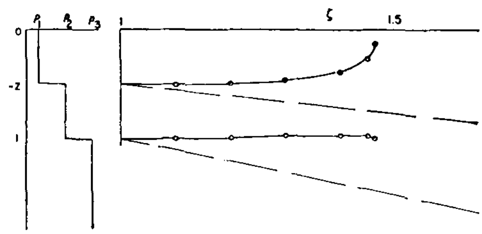

FIG. 3. The correct solution for $\delta = 1.0$, $\gamma = 2$. The dashed lines show the layer thicknesses in the interior, the solid lines show the layer thicknesses at the coast. The solution is plotted only to the point where $h,_\zeta = \infty$. To the left is a representation of the density structure.

has greatly restricted the range of validity of inertial models. This is the most important result in the present work and there is no known physical explanation for it.

Three limits are of particular interest:

1. $\gamma \to \infty$. In this case the "maximum" in Fig. 2 tends to $\zeta = 2$, the upper layer becoming like a one-layer model.

2. $\gamma \to 1$. The maximum tends to $\zeta = 1$, thus severely restricting the range of the solution. However, the second solution in the first quadrant of Fig. 2 tends to the one-layer solution except near $\zeta = 1$ where there is a divergence and h_1 tends to infinity.

3. $\delta \to 0$. This has the same behavior as 2 above. Fig. 4 shows h_1 as a function of ζ for the parameters $\gamma = 1.1$, $\delta = 0.1$.

The last limit is of particular interest because it indicates that there may be no continuously stratified solution. This is so because the parameters for the third limit have been selected to model a constant "gradient" of density in an infinitesimally thick layer. To see this, note that to have equal "gradients" in both layers, $\theta = [(\varrho_2 - \varrho_1)/\delta]/[(\varrho_3 - \varrho_2)/1] = 1$. But $\gamma = 1 + \theta\delta$, so if $\theta = 1$ and $\delta = 0.1$, $\gamma = 1.1$, which is the set of parameters in the third limit above.

Perhaps the discontinuity in Fig. 4 could be eliminated by a continuously stratified model, but this remains to be shown. Otherwise the suggestion is strong that no solution can be found for a purely inertial, continuously stratified model of the Gulf Stream.

4. The warm core countercurrent

A model for the countercurrent must include layers with non-constant potential vorticity which requires that equations (17) and (18) be solved numerically. This is done by integrating a system of six first-order differential equations, four of which come from (17, 18) and two of which are $\psi_{i,x} = D_i v_i$. If a value is chosen for h_1, then an equation analogous to (22) gives h_2. Then the Bernoulli equations will give v_1 and v_2 at the coast. Thus all the initial conditions are determined as a function of h_1. For large x the integration diverges, corresponding to the growing exponentials discarded from the constant potential vorticity solutions. As h_1 passes through its proper value in a sequence of trials, the asymptotic behavior of all the fields changes sign. By watching this behavior, the correct solution may be successively approximated. This computational method was suggested by N. P. Fofonoff. The numerical techniques are presented in more detail in I.

The countercurrent is thought to exist because warm water carried north creates a pressure gradient opposite to that of the main stream, and by geostrophy this implies a countercurrent.

Fig. 5 shows a density section in the North Atlantic, approximately along the 66°W longitude circle. One possible model for a countercurrent is a horizontal interface between ϱ_1 and ϱ_2 at a depth of 160 meters (corresponding to $\sigma_t = 26.1$), and a lower layer bounded below by $\sigma_t = 27.0$ and going from 400 meters at 15°N which is taken as $\zeta = 1.0$, to a thickness of 750 meters at 30°N which is approximately $\zeta = 2.0$. (Another possibility is certainly one in which the upper layer decreases in thickness with latitude. Such models have been computed and give essentially the same results as this one.) Characteristic σ_t values are 25.0, 26.5, and 27.5 for the upper, lower, and resting layers respectively.

In the numerical calculations, the interior layer thicknesses are assumed of the form: $D_1(\infty,\zeta) = a + b\zeta$, $D_2(\infty,\zeta) = c + (1 - a - b - c)\zeta$.

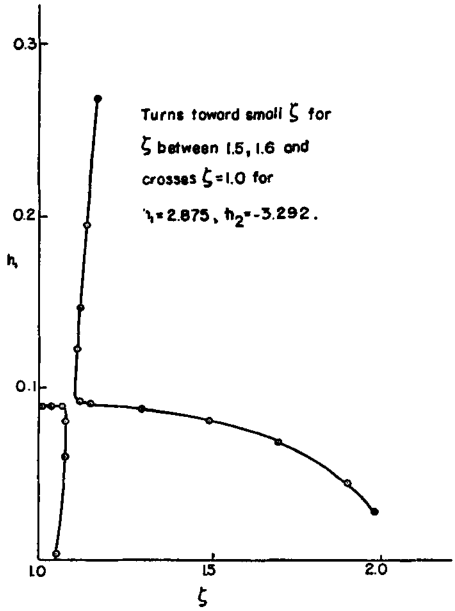

FIG. 4. Thickness of the upper layer, h_1, at the coast as a function of the latitude variable, ζ, for the case $\delta = 0.1$, $\theta = 1.0$, $(\gamma = 1.1)$.

The last coefficient gives the normalization at $\zeta = 1.0$. For the above parameters we have $\gamma = 2.7$, $a = 0.4$, $b = 0.0$, $c = -0.15$. The solution is shown in Fig. 6. The upper illustration has the same format as Fig. 3. We see that h_1 increases along the coast until $h_{1,\zeta} = \infty$ near $\zeta = 1.7$. The lower illustration is a cross-section at $\zeta = 1.6$. The divergence at large x from the asymptotic solution given by the dashed lines reflects the divergence of the approximating numerical solution. There is no countercurrent in this model or in any of a large number of other models computed, because the slope of the lower interface is always great enough to force a positive velocity in both layers. Instead of the upper layer being thickest away from the coast, creating a warm core, it is thickest at the coast, in disagreement with observation.

This result might have been expected because of the following argument. A solution can be computed analytically for a layer of constant potential vorticity between two resting layers. In this solution h_1 increases along the coast. Now the boundary conditions imposed in the present attempt to find a countercurrent cause the upper layer velocity to be smaller than the lower layer velocity, which means that the upper

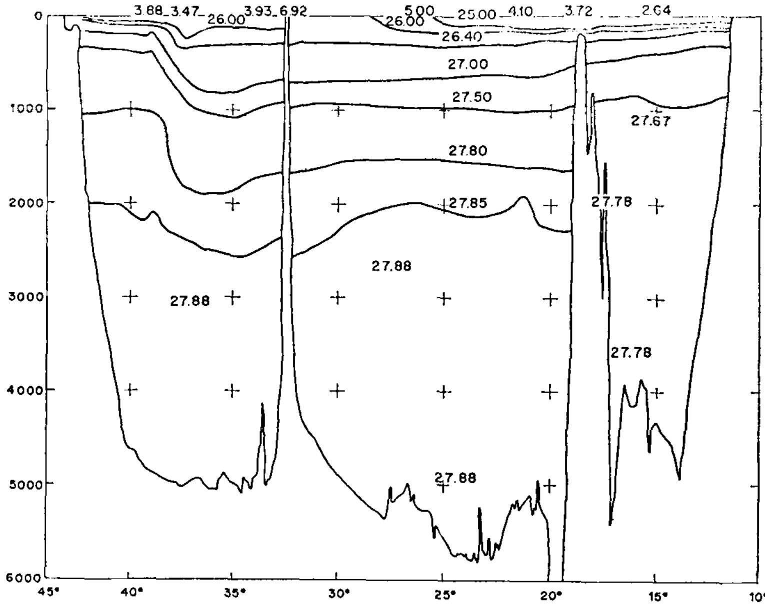

FIG. 5. North-south density section in the Atlantic near 66°W. Stations plotted are: Atlantis 5176–5202, 5232–5235, 5237–5263, Crawford 312–328. The numbers plotted are values of σ_t which is 1000 $(\varrho - 1)$, where ϱ is the density.

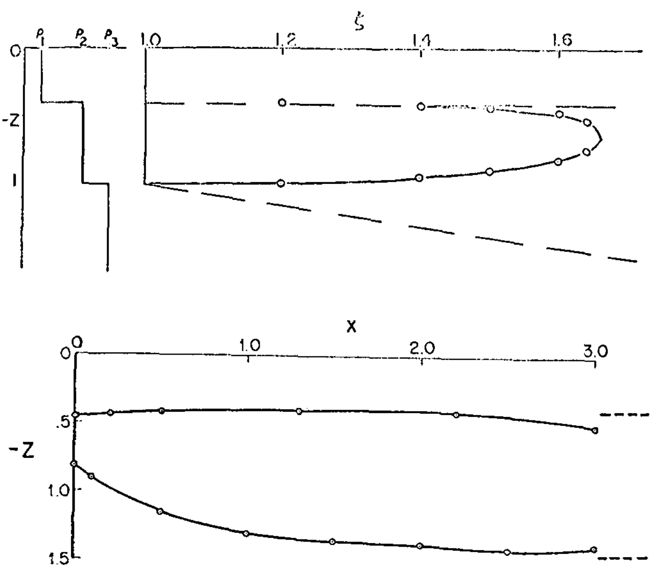

FIG. 6. Results of calculations for a two-layer Gulf Stream using boundary conditions from observed data which it was thought would give an off-shore countercurrent; $a = 0.4$, $b = 0$, $c = -\cdot 15$, and $\gamma = 2.7$. In the upper figure the dashed lines indicate the layer thickness in the interior, and the solid lines show the layer thicknesses at the coast. To the left is a representation of the density structure. The solution ends when $h_2 = 0$. The lower diagram gives a cross-section at $\zeta = 1.6$. The failure of the solution to go to the dashed lines which give the proper asymptotic values shows the divergence of the numerical integration for large x.

layer is an approximation to a resting layer, so we could expect h_1 to increase along the coast, as it does.

Although no countercurrent has yet been computed, I have found a warm core in some solutions which are analogous to the "second solution" in the first quadrant of Fig. 1. These solutions are given in I.

These results indicate that a two-layer model cannot simulate an inertial countercurrent. However, because of computational difficulties I have not been able to compute cross-sections north of where $D_1(\infty,\zeta)=0$, because in the interior $D_1(\infty,\zeta)=0$ implies infinite potential vorticity. This is an artifact of the finite number of layers, and perhaps continuously stratified models can give an inertial countercurrent. The problem of an inertial countercurrent is therefore not settled. There is more detail on the problem in I.

5. A second failure of purely inertial flow

In § 2 we saw a failure of purely inertial flow for which there exists no known physical explanation. However, the failure of inertial flow discussed in the present section does have a physical explanation. Consider a two-layer model in which the lower layer is almost at rest. Then by (12) the lower interface must be almost horizontal, and there must be a small flux toward the coast. But the upper layer is approaching a one-layer model, so its layer thickness is determined. Therefore the lower layer thickness is determined. But if water flows through this lower layer, however slowly, it must conserve potential vorticity. So we must expect $v_{2,x}$ to be large. But it cannot be large and still have v_2 and the transport small. So the lower layer cannot be purely inertial.

The above is not a rigorous argument, but it gives the physical idea. This failure of inertial flow is observed in the most realistic two-layer model I have computed of the Gulf Stream. The layer above $\sigma_t=27.0$ extends from 400 meters at $\zeta=1$ to 750 meters at $\zeta=2$ and the lower interface, $\sigma_t=27.5$, varies between 900 and 1000 meters depth, almost horizontal. The mean densities of the upper, lower, and resting layers are 26.0, 27.25, and 27.75 respectively.

These parameters imply $a=0.111$, $b=0.333$, $c=0.778$, $\gamma=3.5$. When calculations are performed for these parameters it is found that there is no value of h_1 on either side of which all the fields of the problem change sign for large x. That is, the numerical criteria for the existence of a solution are not satisfied, and therefore there is no solution. This point is discussed in more detail in I.

The suggestion from this computation is that in the actual Gulf Stream the water in the lower layers moves too slowly for the flow to be purely inertial. If we were to imagine some ideal, model flow in which the above boundary conditions were imposed, it would presumably become unsteady and turbulent, reflecting the impossibility of a steady, laminar solution. Perhaps analogous instabilities occur in the actual Gulf Stream. It is not clear at present whether continuously stratified inertial models would or would not exhibit the same difficulties.

6. Summary and conclusions

The net result of the present work seems only to add more confusion to the problem of the Gulf Stream than existed previously. The simple one-layer models of Morgan and Charney, which gave good agreement with observation, when split into two layers, break up at mid-latitudes. So the task now is to explain why the one-layer models are so successful. It also seems that the inertial theory cannot explain the warm core countercurrent, and that the deeper waters of the stream cannot be purely inertial. Perhaps the next steps toward the solution of these puzzling problems will include viscous models and existence proofs for inviscid models.

This may be the place to note that I also considers two moving layer models flowing over a rigid bottom. All the problems of the present models are found in these models also. The result of greatest interest is the demonstration that as $\varrho_2-\varrho_1$ increases from zero, the solution is restricted to lower and lower latitudes.

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