Planetary waves in a symmetrical polar basin

By PAUL H. LeBLOND, *Institut für Meereskunde der Universität Kiel*

(Manuscript received July 8; revised version September 9, 1964)

ABSTRACT

The problem of low frequency zonal vorticity waves in a symmetric polar basin is formulated in a plane projection tangent to the sphere at the pole. Eigensolutions are found when only first order curvature effects are retained in the plane of projection. Comparison with the results of Longuet-Higgins (1964*b*) shows that such an approximation provides a reliable analogue, applicable to the polar regions, of the mid-latitude beta-plane.

1. Introduction

The oceanographic exploration of the Arctic has progressed rapidly in the last two decades, and enough information has been collected to provide a satisfactory picture of the main hydrographic and bathymetric features (Coachman, 1961; Gordienko, 1961). The next logical step in Arctic oceanographic research should be a study of the dynamics of the Arctic Ocean. There are too few time series to investigate the situation from an observational point of view, but it may prove useful to have a preliminary estimate of the time scales of the motions to be encountered. It is the purpose of this paper to find theoretical estimates which can be useful as guides in future observational programmes, and also to give a firm basis to some analytical techniques applicable to the study of oceanographic phenomena in Arctic regions.

Particular attention will be given to the long period planetary (or Rossby) waves which, because of their long period and large scale will be representative of Arctic conditions. The nature and properties of these waves will depend on the main bathymetric features of the Arctic basin; in this paper, however, we will consider only a basin in which the depth contours and the boundary are symmetrical about the pole.

2. The model

The Arctic Ocean is represented by a basin centered on the pole of rotation. The depth is allowed to vary with latitude, but not with longitude, and the southern boundary is taken at a distance of 1500 km from the pole. It will be realised that this is a very gross model of the Arctic topography, and that some of the large scale asymmetries, like the Lomonosov Ridge, and the opening to the Atlantic Ocean will actually play an important role. How much of a role they do play can be later ascertained by comparing observations with the results of theoretical works such as this one. It is preferable in any case not to overburden a preliminary investigation with too many details. The slight vertical stratification associated with the seasonal surface phenomena (melting of the ice cover) is not taken into account, and the Arctic waters are considered homogeneous in density.

The motions studied will be assumed to have time and distance scales large enough to ignore the effects of viscosity and to regard the pressure field as purely hydrostatic, as is usual with long waves. The constant total energy of the water contained in the basin is assumed to be small enough to allow linearization of the momentum equations; the Rossby number $U/L\Omega$, formed from the angular frequency of rotation of the Earth (Ω) and from the horizontal velocity (U) and length (L) scales, will then be small. A small Rossby number also implies that the local vertical vorticity component of the fluid, of order U/L, will be small compared with the planetary vorticity.

Under the above approximations the momentum equations for an incompressible fluid take the form

$$\frac{\partial v}{\partial t} + 2\Omega u \cos\theta - 2\Omega w \sin\theta = \frac{-1}{\varrho r' \sin\theta}\frac{\partial P}{\partial \lambda}, \qquad (1)$$

$$\frac{\partial u}{\partial t} - 2\Omega v \cos\theta = \frac{-1}{\varrho r'}\frac{\partial P}{\partial \theta}, \qquad (2)$$

$$\frac{\partial P}{\partial r'} = -\varrho g \qquad (3)$$

and the continuity equation is

$$\frac{1}{r' \sin\theta}\left[\frac{\partial v}{\partial \lambda} + \frac{\partial u \sin\theta}{\partial \theta}\right] + \frac{\partial w}{\partial r'} = 0. \qquad (4)$$

The coordinates r', θ, λ and the associated velocities w, u, v, correspond to directions along the radius, the co-latitude and the azimuth of the sphere. P is the hydrostatic pressure, Ω the angular frequency of rotation of the Earth (0.728×10^{-4} sec^{-1}), ϱ the density of the fluid (1.03 gm cm^{-3}) and g the acceleration due to gravity (980 cm sec^{-2}).

The depth of the ocean being very small compared with the radius of the Earth, r' can be replaced by the constant terrestrial radius, R (6370 km), and $\partial/\partial r'$ by $\partial/\partial z$, where z is measured upwards along the local vertical. Furthermore, since in long waves the horizontal velocities are in general much larger than the vertical velocities, and since we are now working near the pole, where θ is small, it is assumed that

$$U \cos\theta \gg W \sin\theta,$$

(W being a vertical velocity scale) so that only one Coriolis parameter is retained in (1).

Let η be a small vertical displacement from the equilibrium water surface ($\eta < H$); the pressure gradients can then be replaced by gradients of elevation. Integrating the pressure equation (3), and neglecting the constant atmospheric pressure one has

$$P = g\varrho(\eta - z). \qquad (5)$$

Surface displacements will then replace pressure in the momentum equations.

The momentum and continuity equations are now integrated from the bottom ($z = -H$) to the surface ($z = \eta$) of the ocean. The velocity component perpendicular to the bottom vanishes at the bottom itself:

$$w + \frac{u}{R}\frac{\partial H}{\partial \theta} = 0 \quad \text{at} \quad z = -H. \qquad (6)$$

At the surface, a linear boundary condition is used:

$$w = \frac{\partial \eta}{\partial t} \quad \text{at} \quad z = \eta. \qquad (7)$$

Since viscous effects have been neglected altogether, slippage is allowed to occur at the boundaries. Because the density is constant, the horizontal velocities are independent of the depth, and integration does not change the form of the momentum equations. The continuity equation, however, transforms, using conditions (6) and (7) into (10) below.

In spherical coordinates, the integrated equations governing the motions are then

$$\frac{\partial v}{\partial t} + 2\Omega u \cos\theta = \frac{-g}{R \sin\theta}\frac{\partial \eta}{\partial \lambda}, \qquad (8)$$

$$\frac{\partial u}{\partial t} - 2\Omega v \cos\theta = \frac{-g}{R}\frac{\partial \eta}{\partial \theta}, \qquad (9)$$

$$\frac{1}{R \sin\theta}\left[\frac{\partial vH}{\partial \lambda} + \frac{\partial uH \sin\theta}{\partial \theta}\right] + \frac{\partial \eta}{\partial t} = 0. \qquad (10)$$

3. The polar plane approximation

The surface of the sphere is projected, in a simple orthographic manner (Fig. 1), on to a plane tangent to the sphere at the pole. This is done mostly to simplify the mathematics, but also with the aim of establishing in polar regions an approximation technique comparable to the mid-latitude beta-plane.

Plane polar coordinates r and ϕ in the plane of projection are related to the spherical coordinates as follows:

$$dr = R\cos\theta\, d\theta, \quad d\phi = d\lambda. \qquad (11)$$

Equations (8)–(10) become in the plane of projection

$$\frac{\partial v}{\partial t} + 2\Omega u \cos\theta = \frac{-g}{r}\frac{\partial \eta}{\partial \phi}, \qquad (12)$$

$$\frac{\partial u}{\partial t} - 2\Omega v \cos\theta = -g\cos\theta\frac{\partial\eta}{\partial r}, \qquad (13)$$

$$\frac{1}{r}\frac{\partial vH}{\partial\phi} + \frac{\cos\theta}{r}\frac{\partial urH}{\partial r} + \frac{\partial\eta}{\partial t} = 0. \qquad (14)$$

The variables u, v, and η retain their previous significance.

The working approximation, to be introduced in the analysis after all differentiations have been performed, will consist of neglecting the second and higher powers of r/R with respect to unity: $(r/R)^2 < 1$. The influence of the terrestrial curvature is then retained only to first order in r/R. The polar plane approximation will differ from the mid-latitude beta-plane approximation in that the derivative of the Coriolis parameter will be a linear function of r:

$$\left.\begin{aligned} f &\simeq 2\Omega, \\ \frac{df}{dr} &\simeq -\frac{2\Omega r}{R^2}. \end{aligned}\right\} \qquad (15)$$

This approximation should be valid over the entire Arctic Ocean, since $(r/R)^2$ is less than 0.1 even in the southernmost corner of the basin (Beaufort Sea, 72° N).

4. Zonal waves

Let us now look for solutions of (12)–(14) which have the form of zonally propagating waves for which the displacement from the equilibrium surface can be written as

$$\eta = F(r)e^{i(\omega t - s\phi)}. \qquad (16)$$

ω is the frequency of the wave, s its constant azimuthal wave number and $F(r)$ an amplitude function to be determined. The following will hold for the derivatives of the displacement (16):

$$\frac{\partial\eta}{\partial t} = i\omega\eta,$$

$$\frac{\partial\eta}{\partial\phi} = -is\eta,$$

$$\frac{\partial\eta}{\partial r} = -i\sigma\eta, \quad \sigma = \frac{i}{F}\frac{dF}{dr}. \qquad (17)$$

The velocities u and v must have a similar temporal and azimuthal dependence, so that they can be eliminated from (12) and (13):

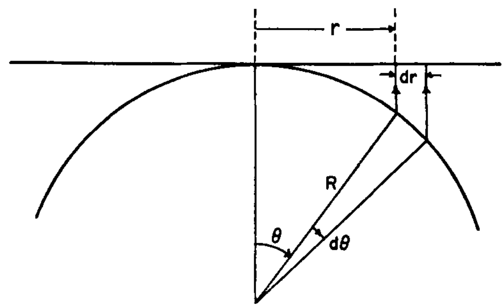

FIG. 1. The orthogonal polar projection.

$$u = \frac{g\eta(-\omega\sigma\cos\theta + i2\Omega\cos\theta/r)}{4\Omega^2\cos^2\theta - \omega^2}, \qquad (18)$$

$$v = \frac{g\eta(-\omega s/r - i2\Omega\sigma\cos^2\theta)}{4\Omega^2\cos^2\theta - \omega^2}. \qquad (19)$$

Substitution of the expressions (18) and (19) for the velocities in the continuity equation (14), together with the subsequent application of the approximation $(r/R)^2 < 1$, yields a differential equation for σ:

$$(4\Omega^2 - \omega^2)\left[-\sigma^2 - i\frac{d\sigma}{dr} + \frac{i\sigma r}{R^2} - \frac{s^2}{r^2} - \frac{i\sigma}{r}\left(1 + \frac{r}{H}\frac{dH}{dr}\right) - \frac{2\Omega}{\omega}\frac{s}{rH}\frac{dH}{dr}\right] - i\frac{8\Omega^2\sigma r}{R^2} - \frac{2\Omega s}{\omega R^2}(4\Omega^2 + \omega^2)$$
$$= (4\Omega^2 - \omega^2)^2\frac{\omega}{gH}. \qquad (20)$$

Substituting in (20) for σ in terms of $F(r)$ (17), the amplitude equation follows:

$$\frac{d^2F}{dr^2} + \frac{dF}{dr}\left[\frac{1}{r} + \frac{(4\Omega^2 + \omega^2)r}{(4\Omega^2 - \omega^2)R^2} + \frac{1}{H}\frac{dH}{dr}\right] - F\left[\frac{s^2}{r^2} + \frac{2\Omega s}{\omega R^2}\frac{(4\Omega^2 + \omega^2)}{(4\Omega^2 - \omega^2)} + \frac{(4\Omega^2 - \omega^2)}{gH} + \frac{2\Omega s}{\omega r H}\frac{dH}{dr}\right] = 0. \qquad (21)$$

The description of the problem is completed by the following boundary conditions. First, the amplitude $F(r)$ must remain finite everywhere within the basin and its boundaries; secondly, the velocity component perpendicular to the boundary at $r = r_1$ must vanish: $u(r_1) = 0$. This last condition is expressed in terms of $F(r)$ through (18):

$$\frac{dF}{dr} = \frac{2\Omega s}{\omega r_1} F \quad \text{at} \quad r = r_1. \tag{22}$$

To simplify the notation, the following non-dimensional variables and abbreviations will be introduced:

$$\left.\begin{aligned} \omega' &= \frac{\omega}{2\Omega}, \\ x &= r/r_1, \\ H'(x) &= H(x)/H(0), \\ \varepsilon &= \frac{(1+\omega'^2)\, r_1^2}{(1-\omega'^2)\, R^2}, \\ M &= \frac{4\Omega^2 R^2}{gH}, \\ \delta &= \frac{s\varepsilon}{\omega'} + (1-\omega'^2)\, M \frac{R^2}{r_1^2}. \end{aligned}\right\} \tag{23}$$

Gravitational effects are represented by M, curvature effects by ε; the relative magnitudes of expressions containing these quantities will decide which of the two, gravity or curvature of the Earth, is most influential in a particular type of motion. The amplitude equation together with the boundary conditions are then reformulated in terms of (23).

$$x^2 \frac{d^2F}{dx^2} + x \frac{dF}{dx}\left(1 + \varepsilon x^2 + \frac{x}{H'}\frac{dH'}{dx}\right) - F\left(s^2 + \delta x^2 + \frac{xs}{\omega' H'}\frac{dH'}{dx}\right) = 0, \tag{24}$$

$$F(x) \text{ is finite for } 0 \leqslant x \leqslant 1, \tag{25}$$

$$\frac{dF}{dx} = \frac{sF}{\omega'} \quad \text{at} \quad x = 1. \tag{26}$$

5. Types of solutions

An analysis of the solutions of (24) using the Method of Signatures (LeBlond, 1964) shows that the solutions satisfying the boundary conditions (25) and (26) can be divided into two classes: gravity waves, of period shorter than a half pendulum day ($\omega' > 1$, M term dominant in δ), and long period vorticity waves ($\omega' < 1$, ε term dominant in δ). It is already well known that such gravity waves on a rotating sphere are modified by the bottom topography and the beta-effect and can propagate in either zonal direction. The influence of bottom topography on the long period vorticity waves is most easily understood through the conservation of potential vorticity theorem (Veronis, 1963)

$$\frac{f+\xi}{H+\eta} = \text{constant}. \tag{27}$$

In an open ocean with a flat bottom, Rossby waves propagate only towards the west (Rossby, 1939); similarly, in a closed polar basin, only eigensolutions with $s < 0$ will exist when H = constant. This will also be true when there is a depth variation working in the same direction as the beta-effect (the depth increasing away from the pole) or in the opposite direction but with less influence than the beta-effect. In the first instance the frequency of the wave is increased by the presence of a depth gradient, in the second it is decreased.

If, however, the depth variation counteracts and overbalances the variation of Coriolis parameter, so that the net influence of both is like a negative beta-effect, then Rossby waves will propagate towards the east only ($s > 0$). The steeper the depth gradient, the higher the frequency of such waves will be.

For any given analytical bottom configuration $H'(x)$, the Method of Signatures allows one to determine whether the net effect of bottom slopes will impose eigensolutions of the western or the eastern propagating type. However, only in very simple cases, such as the ones studied below, can explicit analytical expressions be obtained for the eigenfrequencies in which the influence of the parameters of the problem can be readily apprehended.

We are not interested here in gravity waves and will consider only long period vorticity waves in basins with simple bottom topography.

6. Flat Bottom solutions

In a basin with a flat bottom, planetary waves can propagate only towards the west (LeBlond, 1964). For such low frequencies as we expect to find, we can neglect εx^2 with respect to unity and, in the absence of depth gradients, the amplitude equation reduces to

TABLE 1. *Eigenfrequencies (ω') of zonal planetary waves in a symmetrical polar ocean with a flat bottom, as calculated with the help of the polar plane approximation.*

$s =$	-1	-2	-3	-5	-8
$n = 1$	.00324	.00360	.00359	.00320	.00261
2	.00099	.00139	.00155	.00163	.00152
3	.00048	.00073	.00088	.00101	
4	.00028	.00045	.00057	.00069	
5	.00018	.00031	.00040	.00051	

$$x^2 \frac{d^2F}{dx^2} + x\frac{dF}{dx} - F(s^2 + \delta x^2) = 0. \qquad (28)$$

For $\omega' < 1$, $s < 0$, the constant δ is negative and the solution which is regular at the pole is the Bessel function

$$F(x) = J_k(\sqrt{|\delta|}\, x). \qquad (29)$$

k will henceforth denote $|s|$, and assumes here the values 1, 2, 3, ... Using (29), the boundary condition (26) becomes

$$\frac{J_{k-1}(\sqrt{|\delta|})}{J_k(\sqrt{|\delta|})} = \frac{k}{\sqrt{|\delta|}}\left(1 + \frac{s}{k\omega'}\right). \qquad (30)$$

For small ω', the right hand side of (30) is considerably larger than unity, and one finds that the value of δ which satisfies the boundary condition differs from the root of the denominator of the left hand side only in the third significant figure (JAHNKE & EMDE, 1945). The eigenfrequencies will then be approximated by

$$\omega'_{s,n} = \frac{k}{M + (R^2/r_1^2)\,\beta_{k,n}^2}. \qquad (31)$$

The constant $\beta_{k,n}$ is the nth root of the Bessel function J_k. The ratio (R^2/r_1^2) takes the value 20 for a basin of 1500 km radius. Some eigenfrequencies calculated from (31) are listed in Table 1.

Substitution of (29) into (16), (18) and (19) gives for the amplitude and the horizontal velocities of the waves

$$\eta = cJ_k(\sqrt{|\delta|}\, x)\, e^{i(\omega t - s\phi)}, \qquad (32)$$

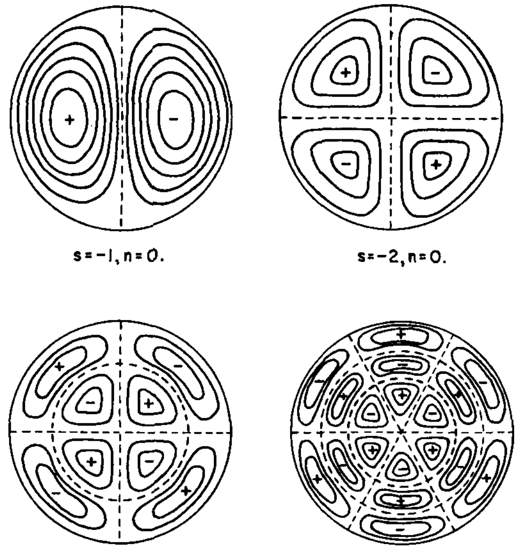

FIG. 2. Contours of surface displacement for zonal planetary waves in a basin with a flat bottom. The patterns rotate clockwise with angular velocity $\frac{1}{k}\omega'_{s,n}$.

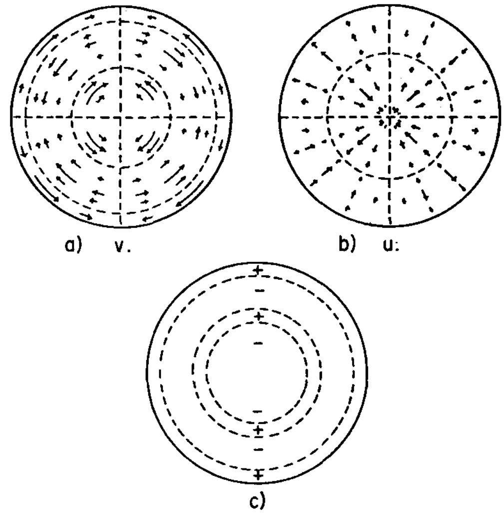

FIG. 3. (a) and (b) The zonal (v) and radial (u) components of the velocity field for $s = -2$, $n = 1$. The patterns rotate clockwise with angular velocity $\frac{1}{2}\omega'_{2,1}$. (c) The direction in which the local velocity vector rotates: − for clockwise, + for counterclockwise rotation.

$$u = \frac{-icg\omega'}{2\Omega r_1} \times \left[\frac{d}{dx} J_k(\sqrt{|\delta|}\, x) - \frac{s}{\omega' x} J_k(\sqrt{|\delta|}\, x)\right] e^{i(\omega t - s\phi)}, \quad (33)$$

$$v = \frac{gc}{2\Omega r_1} \times \left[\frac{d}{dx} J_k(\sqrt{|\delta|}\, x) - \frac{s\omega'}{x} J_k(\sqrt{|\delta|}\, x)\right] e^{i(\omega t - s\phi)}. \quad (34)$$

In the above expressions, the constant c has the dimensions of a length, and is determined by the total energy of the system. Sketches of amplitude and velocity contours are found in Figs. 2 and 3. There is a node of velocity and amplitude at the pole, n nodal circles of amplitude and zonal velocity (v), but only $n-1$ nodal circles of radial velocity (u); there are also k nodal diameters.

At a fixed point, the tip of the velocity vector traces an ellipse over a cycle, the ellipticity being given by

$$1 - \frac{|u|}{|v|} = 1 - \omega' \times \left| \frac{\frac{d}{dx} J_k(\sqrt{|\delta|}\, x) - \frac{s}{\omega' x} J_k(\sqrt{|\delta|}\, x)}{\frac{d}{dx} J_k(\sqrt{|\delta|}\, x) - \frac{s\omega'}{x} J_k(\sqrt{|\delta|}\, x)} \right|. \quad (35)$$

The direction in which the ellipse is traced varies radially, and, as one progresses from the pole, bands where the velocity vector rotates clockwise alternate with bands where it rotates counterclockwise (Fig. 3c).

The vertical vorticity component, ξ, defined by

$$\xi = \frac{1}{r}\left[\cos\theta \frac{\partial vr}{\partial r} - \frac{\partial u}{\partial \phi}\right] \quad (36)$$

becomes, when u and v are expressed in terms of (33) and (34),

$$\xi = \frac{-gc}{2\Omega r_1^2} J_k(\sqrt{|\delta|}\, x)\, e^{i(\omega t - s\phi)}. \quad (37)$$

The vorticity then has the same functional dependence as the surface displacement, but is 180° out of phase with it; the energy fluctuates between the forms of kinetic energy of rotation and gravitational potential energy. Note that although we have neglected viscosity altogether, there does not arise any singularity in vorticity at $x = 0$, as sometimes happens when such a simplification is made.

To first order, when the pressure is hydrostatic and the velocities independent of depth, the average rate of energy transfer due to the wave across a vertical plane of unit width is

$$\langle \varrho g H \eta \mathbf{v} \rangle. \quad (38)$$

The brackets indicate time averaging over a cycle. The average radial energy transport vanishes. The zonal component is proportional to

$$\frac{1}{2}\frac{dF^2}{dx} - \frac{s\omega' F^2}{x}. \quad (39)$$

The total energy transport is then proportional to the integral of (39) from the pole to the boundary:

$$F^2(1) - s\omega' \int_0^1 \frac{F^2}{x}\, dx. \quad (40)$$

At the boundary, the amplitude is very small; as a matter of fact, we have approximated the eigenfrequencies by those values of frequency which make the amplitude vanish at $x = 1$. The second term of (40) will then dominate, and the net average energy transport will be towards the east, in a direction oppposite to that of phase propagation. This agrees with Longuet-Higgins' (1964a) general results on energy transport in planetary waves.

We have just described the planetary wave eigensolutions of a symmetric polar basin with a flat bottom. These results are, however, subject to the approximation $(r/_R)^2 < 1$, which will be referred to as the "polar plane approximation". It is interesting to compare the above results with similar results derived entirely in the spherical geometry, so as to provide an *a posteriori* vindication of the analyses performed with the help of the polar plane approximation. Such a comparison can be made with the results obtained by Longuet-Higggins (1964b) in which a completely different approach was used.

7. Comparison with results on the sphere

By assuming that the surface displacements have a negligible influence on the vorticity balance, LONGUET-HIGGINS (1964*b*) has formulated the problem of planetary waves in two dimensions, and has solved it in terms of a stream function. This approximation of nearly two-dimensional flow will be very good provided the wave length is not comparable to the radius of the Earth. In our notation, the stream function, ψ, characterizing eastward travelling zonal planetary waves in a polar basin on a sphere is given by Longuet-Higgins as

$$\psi = P_\nu^k (\cos\theta). \tag{41}$$

$P_\nu^k (\cos\theta)$ is the Legendre function of argument $\cos\theta$. The constant ν is a positive real number satisfying the boundary condition

$$P_\nu^k(\cos\theta_1) = 0, \quad \theta_1 = \sin^{-1} r_1/R. \tag{42}$$

The frequency of the planetary wave is then given by

$$\omega'_{k,\nu} = \frac{k}{\nu(\nu+1)}. \tag{43}$$

This relation is not unlike (31), but ν is not as easily evaluated as $\beta_{k,n}$. The Legendre function $P_\nu^k(\cos\theta)$, when x is real and k (but not ν) is an integer, can be expressed in terms of gamma functions and the hypergeometric series, ${}_2F_1$: $P_\nu^k(\cos\theta) =$

$$\frac{(-2)^k}{k!}\frac{\Gamma(\nu+k+1)}{\Gamma(\nu-k+1)}(1-\cos\theta)^{k/2}$$
$${}_2F_1[1+k+\nu,\, k-\nu;\, k+1;\, \tfrac{1}{2}-\tfrac{1}{2}\cos\theta]. \tag{44}$$

The expression (44) will have roots in ν only when the hypergeometric series vanishes. It is clear from the behaviour of such series (ERDELYI *et al.*, 1953, ch. III) that there is an infinity of progressively larger values of ν which make the series vanish for constant k and $\cos\theta$. The same multiplicity of solutions then exists as found in the polar plane. The first root was calculated for a few values of k by computing the sum of the first 20 terms of the series for increasing values of ν until a change of sign occurred (Table 2, Column 2).

The eigenfrequencies thus calculated differ from their equivalents in the polar plane (Table 2, Column 1) by small but appreciable amounts, especially at low wave numbers. Since both sets of results are only estimates of the exact eigenfrequencies, departing from the true values because of the approximations used in solving the problem, we can discover the relative distorting effects of the two fundamental approximations (Longuet-Higgins' or the polar plane) by applying them both to the problem simultaneously. Longuet-Higgins' approximation that the surface displacements can be neglected will then be superimposed on the polar plane approximation used in this work.

TABLE 2. *Comparison of eigenfrequencies (ω') of zonal planetary waves in a symmetrical polar ocean with a flat bottom, as calculated from the polar plane approximation (P–P) and Longuet-Higgins' approximation (L.–H.).*

s	$\omega'_{s,1}$ *P–P*	*L.–H.*	*P–P* and *L.–H.*
−1	.00324	.00344	.00342
−2	.00366	.00378	.00380
−3	.00359	.00369	.00369
−5	.00320	.00324	.00326
−8	.00261	.00268	.00268
−12	.00222	.00224	.00224
−16	.00179	.00179	.00179

For a two-dimensional problem in spherical coordinates, the velocities can be expressed in terms of a stream function, Ψ:

$$u = \frac{1}{R\sin\theta}\frac{\partial\Psi}{\partial\lambda}, \tag{45}$$

$$v = \frac{-1}{R}\frac{\partial\Psi}{\partial\theta}. \tag{46}$$

Taking the curl of the momentum equations (8) and (9), and using the continuity equation (10) with the depth now treated as constant, we have the vorticity equation

$$\frac{\partial}{\partial t}\left\{\frac{1}{R^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\lambda^2} + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial\Psi}{\partial\theta}\right]\right\} - \frac{1}{R^2\sin\theta}\frac{df}{d\theta}\frac{\partial\Psi}{\partial\lambda} = 0. \tag{47}$$

This is the equation solved by Longuet-Higgins. Transforming to the polar plane through the relations (11), (47) becomes

$$\frac{\partial}{\partial t}\left\{\frac{1}{r^2}\frac{\partial^2\Psi}{\partial\phi^2}+\frac{\partial^2\Psi}{\partial r^2}+\frac{1}{r}\frac{\partial\Psi}{\partial r}\frac{d}{dr}(r\cos\theta)\right\}+\frac{2\Omega}{R^2}\cos\theta\frac{\partial\Psi}{\partial\phi}=0. \quad (48)$$

Let us look for vorticity waves of the form

$$\Psi=\Psi_0(r)e^{i(\omega t-s\phi)}. \quad (49)$$

We can now substitute (49) into (48); with $x=r/r_1$ and $(r/_R)^2<1$, the following amplitude equation in ψ_0 is obtained

$$x^2\frac{d^2\Psi_0}{dx^2}+x\frac{d\Psi_0}{dx}-\Psi_0\left(s^2+\frac{sx^2r_1^2}{\omega' R^2}\right)=0. \quad (50)$$

Equation (50) resembles (28) very much; in fact, the two differ only in the definition of the constant δ. Putting now

$$\delta^*=\frac{sr_1^2}{\omega' R^2}, \quad (51)$$

we recover the fact that in the two-dimensional flow hypothesis there are no gravity influences (no M term in δ^*). This is of course a direct consequence of neglecting the surface elevation, there being no departures from the equilibrium surface on which gravity can act.

With $s<0$, (50) has for solution

$$\Psi_0(x)=J_k(\sqrt{|\delta^*|}\,x). \quad (52)$$

The boundary condition is now

$$\Psi_0(1)=J_k(\sqrt{|\delta^*|})=0, \quad (53)$$

so that the eigenfrequencies are given by

$$\omega'_{s,n}=\frac{k}{20\beta_{k,n}^2}. \quad (54)$$

Some of the eigenvalues (54) are listed in Table 2, Column 3. These frequencies, calculated using both basic approximations, differ little from those computed with the aid of Longuet-Higgins' approximation alone. We must then conclude that the application of the polar plane approximation results only in a small distortion and that it will yield the best estimates of the eigenfrequencies. As one can see from Table 2, the distorting effect of the two approximations is strongest at low wave numbers and diminishes rapidly as k increases.

It then appears that the eigenfrequencies calculated from (31) differ only in the third significant figure from the results which could be obtained through much labour from a stricter analysis. Moreover, since the planetary waves here studied, which, because of their intimate dependence on the Earth's curvature, are most likely to be distorted by any departures from the strict spherical shape, are found to suffer only minor distortions, then all other types of wave motion, which are less dependent on the terrestrial curvature, will be correspondingly less affected by the polar plane approximation. This approximation will then provide results departing very little from those computed on the sphere, and can be applied with confidence to the study of oceanographic phenomena in the polar ocean. It then provides a reliable analogue of Rossby's beta-plane in high latitudes.

8. Sloping bottom

Although the type of possible solutions in the presence of radial slopes can be ascertained by using the Method of Signatures, explicit analytical solutions of (24) together with its boundary conditions (25) and (26) can be obtained only for very simple bottom topographies. A particularly simple and instructive bottom profile is

$$H'(x)=(1+px^2/_2), \quad (55)$$

where $p/2<1$. The functional dependence of the depth is then the same as that of the Coriolis parameter, and the influence of the two can be directly compared. For very low frequencies and small p, the amplitude equation (24) is approximated by

$$x^2\frac{d^2F}{dx^2}+x\frac{dF}{dx}-F\left[s^2+\left(\delta+\frac{ps}{\omega'}\right)x^2\right]=0. \quad (56)$$

This is of the same form as (28) and will have planetary wave solutions of the type (29) for negative $(\delta+ps/\omega')$:

$$F(x)=J_k\left[\sqrt{\left|\delta+\frac{ps}{\omega'}\right|}\,x\right]. \quad (57)$$

More explicitly, the constant $(\delta+ps/\omega')$ can be expanded as

TABLE 3. *Some eigenfrequencies of zonal planetary waves for a symmetrical polar ocean with bottom profile $H' = 1 + px^2/_2$.*

	$p = 0.1$ $s < 0$	$p = -0.1$ $s > 0$
k	$\omega'_{s,1}$	
1	0.0095	0.00318
2	0.01100	0.00366
5	0.00963	0.00321

$$\delta + \frac{ps}{\omega'} \simeq \left(p + \frac{r_1^2}{R^2}\right)\frac{s}{\omega'} + M\frac{r_1^2}{R^2}. \tag{58}$$

We can then see the role played by a bottom configuration of the form (55): if p is positive, so that the depth increases towards the boundary, then s must be negative and propagation to the west in order to keep (58) negative. If p is negative, so that the depth decreases with x, and large enough to make $(p + r_1^2/_{R^2})$ negative also, then the wave number, s, will have to be positive to make (58) negative. This is exactly the behaviour expected from considerations of conservation of potential vorticity and also predicted by the Method of Signatures.

The boundary condition is now

$$\frac{J_{k-1}\left[\sqrt{\left|\delta + \frac{ps}{\omega'}\right|}\right]}{J_k\left[\sqrt{\left|\delta + \frac{ps}{\omega'}\right|}\right]} = \frac{k\left(1 + \frac{s}{k\omega'}\right)}{\sqrt{\left|\delta + \frac{ps}{\omega'}\right|}} \tag{59}$$

and, as in the flat bottom case, the eigenfrequency is closely approximated by the root of the denominator of the left hand side. This gives for the eigenvalues

$$\omega'_{s,n} = \frac{-s(p + r_1^2/R^2)}{\beta_{k,n}^2 + Mr_1^2/R^2}. \tag{60}$$

A few of the eigenfrequencies are listed in Table 3.

This simple example illustrates the influence of depth variations on long vorticity waves. The case of symmetric bottom slopes can be considered solved in principle, since it is always possible to ascertain the existence and the type of solutions. The evaluation of the eigenfrequencies may, however have to be done by numerical methods.

9. Conclusions

We have here studied the problem of long zonal waves in a symmetrical polar ocean. The analysis has been performed in a plane polar projection of the sphere, and only first order terms in the terrestrial curvature have been kept. Choosing for particular scrutiny the low frequency planetary eigensolutions, we have described these waves for a basin with a flat bottom. Comparison of these results with their equivalents on the sphere, emanating from a different approach due to LONGUET-HIGGINS (1964*b*), leads to the conclusion that the polar plane approximation used here is a very good one indeed and can be employed in the study of polar oceanographic phenomena with the same confidence that is bestowed upon the beta-plane approximation in the study of mid-latitude phenomena.

Acknowledgements

This work was conducted under the direction of Dr. R. W. Stewart as part of a doctoral dissertation presented at the University of British Columbia. I also wish to thank Dr. R. W. Burling for many useful suggestions and his careful reading of the manuscript.

REFERENCES

Coachman, L. K., 1962, On the water masses of the Arctic Ocean. Doctoral Dissert., Dept. of Oceanography, Univ. of Washington. (Unpublished manuscript.)

Eckart, C., 1960, *Hydrodynamics of Oceans and Atmospheres*, pp. 290. Pergamon Press, New York.

Erdelyi, A., Magnus, M., Oberhettinger, F., and Tricomi, F., 1953, *Higher transcendental Functions*, vol. I, pp. 302. McGraw-Hill, New York.

Gordienko, P. A., 1961, The Arctic Ocean. *Scientific American*, **204**, pp. 88–102.

Jahnke, E., and Emde, F., 1945, *Tables of Functions*, pp. 304. New York. Dover.

Le Blond, P. H., 1964, Planetary waves in a polar ocean. Doctoral Dissertation. Institute of Oceanography, University of British-Columbia.

Longuet-Higgins, M. S., 1964*a*, On group velocity and energy flux in planetary wave motions. *Deep Sea Research*, **11**, pp. 35–43.

Longuet-Higgins, M. S., 1964*b*, Planetary waves on a rotating sphere. *Proc. Roy. Soc.*, ser. A (in press).

Rossby, C. G., 1939, Relations between variations in the intensity of the zonal circulation of the atmosphere and the displacements of the semipermanent centres of action. *Jour. Marine Research*, **2**, pp. 38–55.

Veronis, G., 1963, On the approximations involved in transforming the equations of motion from a spherical surface to the beta-plane. I. Barotropic systems. *Jour. Marine Research*, **21**, pp. 110–125.