Approximations in the use of pressure coordinates

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(Manuscript received September 25, 1963)

ABSTRACT

Approximations in the use of any scalar dependent variable as a vertical coordinate in Dynamic Meteorology are derived. A connection with the barotropic approximation is made for the case of a pressure coordinate system where the dependent variable is pressure.

1. Introduction

The two obvious reasons for the introduction of pressure as a vertical coordinate (pressure coordinate system or p-system) into the dynamical equations of meteorology are firstly to try to simplify these equations and secondly to make the equations directly applicable to variables measured, as they generally are, on constant pressure surfaces. The latter is the more basic reason and it is found that, in moulding the equations to fit the technique of constant pressure analysis, simplifications do occur.

In constant pressure (c–p) analysis the variables are, ideally, completely specified at a particular time (t_0) on a number of levels, $p = P =$ constant, of the parameter P, the pressure (e.g. 1000 mb, 700 mb, 500 mb, etc. levels). In this way a composite picture of the atmosphere as a whole is obtained at the time t_0. A set of dynamical prediction equations is then required to relate this picture to that at some later time. The similarity to constant height (c–l) analysis is noted. Here, at time t_0, the variables are specified on a set of levels where the parameter Z, the height above mean sea level, is constant. Z corresponds to the vertical coordinate (z) of a Cartesian frame fixed to the earth's surface at mean sea level. In this case the particular prediction equations are the well known Cartesian equations of motion, vorticity, etc. applied to each of the particular values of $z = Z =$ constant chosen. It is a simple matter to fix z in the relevant equations for the c–l case. However, since the pressure is generally a function of the space variables x, y and z and of time t, we cannot fix the value of p in a similar manner for the c–p case.

In both cases the method of representation of dependent variables is the same. Their values on a particular level (c–p or c–l) surface are projected onto a sea level chart. These projections, together with a contour map of the level surface, are sufficient to give a complete picture of the variation of the variables over the level surface. In the c–l case the contour map is trivial: in the c–p case the contour lines are drawn on the same chart as the projected variables and are equivalent to the isobars of the constant height representation.

Summarizing, we have a set of c–p boundary and initial conditions, but no predictive equations directly applicable to them, and a set of equations (e.g. the equations of motion and continuity) applicable to c–l boundary and initial conditions which in general are not known.

Let D_j be the set of differential and other operators used in meteorology and let V_i be the set of variables encountered (functions of x, y, z and t). $V_i^{(P)}$ and $V_i^{(Z)}$ represent the sets of variables with pressure as a variable parameter taking a finite number of values (c–p analysis) and with height taken as a variable parameter with a finite number of values (c–l analysis) respectively. We have then a set of dynamic equations of the form

$$\sum_{ij} D_j V_i = \text{constant} \tag{1}$$

and two alternative sets of boundary conditions

$$V_i^{(P)} = \text{const.} \tag{2}$$

or
$$V_i^{(Z)} = \text{const.} \tag{3}$$

The equations

$$\sum_{ij} D_j V_i^{(Z)} = \text{const}^{\cdot} \tag{4}$$

to which the boundary conditions (3) apply can be obtained directly from equation (1). Since the boundary conditions, are generally in the form of equation (2) we need a set of equations

$$\sum_{ij} D_j V_i^{(P)} = \text{const.} \tag{5}$$

to which they can be applied.

To obtain these equations we must find relations

$$D_j V_i^{(P)} = \text{function of the } D_k V_i^{(Z)}. \tag{6}$$

The first comprehensive treatment of this problem was given by ELIASSEN (1949) and his results have since been used extensively in most branches of dynamic meteorology and in particular in the field of numerical forecasting. However, Eliassen's results, which are essentially identical to equation (8) in the next section, are not directly applicable to the equations $\sum_{ij} D_j V_i$ = constant or $\sum_{ij} D_j V_i^{(Z)}$ = constant since equation (8) is not in the required form (6). In the following sections approximate equations are derived which can be so applied. The degree of approximation involved is determined and the consequences in numerical forecasting are briefly discussed.

2. Relations between the differential operators

Instead of restricting the discussion to constant pressure analysis only, we will consider constant scalar analysis for a general scalar function

$$\xi(x,y,z,t) = \Xi = \text{constant}.$$

We will use the following terms: q = any of the independent variables x, y or t. Cartesian variables have been chosen for simplicity.

$A = A(x,y,z,t)$ is any scalar dependent variable. Since the extension to vector quantities is straightforward only the scalar case will be presented in detail.

$Z_\Xi = Z_\Xi(x, y, \Xi, t)$ denotes the height of the chosen constant scalar surface.

The parameters corresponding to the variables p, z and ξ will be written P, Z and Ξ so that a particular constant pressure surface for example would be specified by $p = P$ = constant.

The values of A on the surface $\xi = \Xi$ will be

$$A_{Z\Xi} = A_{Z\Xi}(x, y, Z_\Xi, t) = A_{Z\Xi}(x, y, Z_\Xi(x, y, \Xi, t), t)$$

or in an implicit form

$$A_\Xi = A_\Xi(x, y, \Xi, t).$$

Since $\quad A_\Xi(x, y, \Xi, t) = A_{Z\Xi}(x, y, Z_\Xi, t)$

we have
$$\frac{\partial A_\Xi}{\partial q} = \frac{\partial}{\partial q}(A_{Z\Xi}) + \frac{\partial}{\partial Z_\Xi}(A_{Z\Xi})\frac{\partial Z_\Xi}{\partial q}. \tag{7}$$

If $[f]_\alpha$ represents the value of f for $z = \alpha$ then

$$\frac{\partial}{\partial q}(A_{Z\Xi}) = \left[\frac{\partial A}{\partial q}\right]_{Z\Xi}$$

and
$$\frac{\partial}{\partial Z_\Xi}(A_{Z\Xi}) = \left[\frac{\partial A}{\partial z}\right]_{Z\Xi}$$

giving
$$\frac{\partial}{\partial q}(A_\Xi) = \left[\frac{\partial A}{\partial q}\right]_{Z\Xi} + \left[\frac{\partial A}{\partial z}\right]_{Z\Xi}\frac{\partial Z_\Xi}{\partial q}. \tag{8}$$

The right-hand side of (8) depends on the variable heights of the constant-ξ surface. The available dynamic equations do not involve time and space derivatives of a variable when these derivatives are taken at different heights for each instant and position (x,y): in particular the derivatives at different heights of a constant scalar surface. The equations however do contain derivatives taken at varying heights of a constant level surface $z = Z$ = constant. It is necessary then to relate terms of the form $[\partial A/\partial q]_{Z\Xi}$ to terms like $[\partial A/\partial q]_Z$.

To do this we use a Taylor series expansion in powers of $\Delta z = Z_\Xi - Z$, where Z is chosen appropriately between the highest and lowest values of Z_Ξ.

$$\left[\frac{\partial A}{\partial q}\right]_{Z\Xi} = \left[\frac{\partial A}{\partial q}\right]_Z + \left[\frac{\partial}{\partial z}\left(\frac{\partial A}{\partial q}\right)\right]_{Z\Xi}\Delta z - \frac{1}{2!}\left[\frac{\partial^2}{\partial z^2}\left(\frac{\partial A}{\partial q}\right)\right]_{Z\Xi}(\Delta z)^2 + \dots \tag{9}$$

Assuming the surface $\xi = \Xi$ to be quasihorizontal we can neglect powers of Δz greater than one. The accuracy of this step will be discussed later. Substitution of (9) into (8) gives

$$\frac{\partial A_\Xi}{\partial q} = \left[\frac{\partial A}{\partial q}\right]_Z + \left[\frac{\partial}{\partial z}\left(\frac{\partial A}{\partial q}\right)\right]_{Z\Xi}\Delta z + \frac{\partial Z_\Xi}{\partial q}\left[\frac{\partial A}{\partial z}\right]_Z + \frac{\partial Z_\Xi}{\partial q}\left[\frac{\partial}{\partial z}\left(\frac{\partial A}{\partial z}\right)\right]_{Z\Xi}\Delta z$$

$$= \left[\frac{\partial A}{\partial q}\right]_Z + \left[\frac{\partial A}{\partial z}\right]_Z\frac{\partial Z_\Xi}{\partial q} + \Delta z\left\{\left[\frac{\partial}{\partial q}\left(\frac{\partial A}{\partial z}\right)\right]_{Z\Xi} + \frac{\partial Z_\Xi}{\partial q}\left[\frac{\partial}{\partial z}\left(\frac{\partial A}{\partial z}\right)\right]_{Z\Xi}\right\}$$

$$= \left[\frac{\partial A}{\partial q}\right]_Z + \left[\frac{\partial A}{\partial z}\right]_Z\frac{\partial Z_\Xi}{\partial q} + \Delta z\left\{\frac{\partial B_\Xi}{\partial q}\right\}, \qquad (10)$$

where $B = \partial A/\partial z$.

Higher order terms in Δz have been neglected.

Except for the term in Δz the expression (10) is the same as that given by ELIASSEN (1949), PERKINS and GUSTAFSON (1951) and others, and which is the basis of the dynamical equations in the p-system. However, the approximation involved in neglecting terms involving Δz has been overlooked.

Neglecting the Δz term gives

$$\frac{\partial A_\Xi}{\partial q} = \left[\frac{\partial A}{\partial q}\right]_Z + \left[\frac{\partial A}{\partial z}\right]_Z\frac{\partial Z_\Xi}{\partial q}. \qquad (11)$$

From the equations for $q \equiv x$ and $q \equiv y$ we obtain the vector analogue of (11)

$$\nabla_h \otimes \mathbf{A}_\Xi = [\nabla_h \otimes \mathbf{A}]_Z + \nabla_h Z_\Xi \otimes \left[\frac{\partial \mathbf{A}}{\partial z}\right]_Z, \qquad (12)$$

where ∇_h is the horizontal del operator,

$$\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y},$$

and $\otimes$ denotes either a vector or a scalar product.

Differentiating A_Ξ with respect to the parameter Ξ gives, similar to equation (7)

$$\frac{\partial A_\Xi}{\partial \Xi} = \frac{\partial A_{Z\Xi}}{\partial \Xi} + \frac{\partial A_{Z\Xi}}{\partial Z_\Xi}\frac{\partial Z_\Xi}{\partial \Xi}$$

$$= \left[\frac{\partial A}{\partial z}\right]_{Z\Xi}\frac{\partial Z_\Xi}{\partial \Xi} \quad \text{since } \frac{\partial A_{Z\Xi}}{\partial \Xi} \equiv 0$$

$$= \left[\frac{\partial A}{\partial z}\right]_Z \Big/ \frac{\partial \xi}{\partial z}$$

to order zero in powers of Δz. The suffix Ξ can be dropped from A_Ξ without ambiguity in this case since A must be written in the form dependent on Ξ explicitly in order that partial differentiation with respect to Ξ may be performed. Hence we have

$$\frac{\partial A}{\partial \Xi} = \left[\frac{\partial A}{\partial z}\right]_Z \Big/ \frac{\partial \xi}{\partial z}. \qquad (13)$$

3. Basic differential formulae

The fundamental equations are (11) and (12). Rewriting them we have

$$\nabla_h \times \mathbf{A}_\Xi = \nabla_h \times \mathbf{A} + \nabla_h Z_\Xi \times \frac{\partial \mathbf{A}}{\partial z}, \qquad (14\text{A})$$

$$\nabla_h \cdot \mathbf{A}_\Xi = \nabla_h \cdot \mathbf{A} + \nabla_h Z_\Xi \cdot \frac{\partial \mathbf{A}}{\partial z}, \qquad (14\text{B})$$

$$\nabla_h A_\Xi = \nabla_h A + (\nabla_h Z_\Xi)\frac{\partial A}{\partial z}, \qquad (14\text{C})$$

$$\frac{\partial A_\Xi}{\partial t} = \frac{\partial A}{\partial t} + \frac{\partial Z_\Xi}{\partial t}\frac{\partial A}{\partial z} \qquad (14\text{D})$$

where the suffixes Z have been dropped from the right-hand sides for simplicity and to align the notation more closely with that of other authors.

Putting $A = \xi$ in (14C) we have, since $\nabla_h \xi_\Xi \equiv 0$,

$$\nabla_h Z_\Xi = -\nabla_h \xi \Big/ \frac{\partial \xi}{\partial z}. \qquad (15)$$

Using the approximate form of the third equation of motion in Cartesian coordinates, viz.

$$\frac{\partial p}{\partial z} = -\varrho g, \qquad (16)$$

where ϱ is the density and g the acceleration due to gravity, we find for the p-system the familiar equation

$$\nabla_h Z_P = (\nabla_h p)/\varrho g. \qquad (17)$$

Putting $A = \xi$ in (14D) gives

$$\frac{\partial Z_\Xi}{\partial t} = -\frac{\partial \xi}{\partial t} \Big/ \frac{\partial \xi}{\partial z}.$$

Hence

$$\frac{D_h Z_\Xi}{Dt} = -\frac{\partial \xi}{\partial t} \Big/ \frac{\partial \xi}{\partial z} - \mathbf{v} \cdot \nabla_h \xi \Big/ \frac{\partial \xi}{\partial z}$$

$$= -\left(\frac{D\xi}{Dt} - v_z\frac{\partial \xi}{\partial z}\right) \Big/ \frac{\partial \xi}{\partial z},$$

where D/Dt is the derivative following the motion

$$\left(\equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right) \quad \text{and} \quad \frac{D_h}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla_h.$$

v_z is the vertical component of the velocity $\mathbf{v}$.

The vertical ξ-velocity is then

$$\frac{D\xi}{Dt} = -\frac{\partial \xi}{\partial z}\left(\frac{D_h Z_\Xi}{Dt} - v_z\right). \tag{18}$$

Using (14C), (14D), (15) and (18) gives

$$\frac{D_h A_\Xi}{Dt} = \frac{DA}{Dt} - \frac{D\xi}{Dt}\left(\frac{\partial A}{\partial z} \bigg/ \frac{\partial \xi}{\partial z}\right).$$

In the p-system, where $\omega = Dp/Dt$, then

$$\frac{DA}{Dt} = \frac{D_h A_P}{Dt} - \frac{\omega}{\varrho g}\frac{\partial A}{\partial z}. \tag{19}$$

The continuity equation takes on a new form.

$$\begin{aligned}
\frac{\partial \omega}{\partial z} &= \frac{\partial}{\partial z}\left(\frac{Dp}{Dt}\right) = \frac{\partial}{\partial z}\left(\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla_h p + v_z \frac{\partial p}{\partial z}\right) \\
&= \frac{D}{Dt}\left(\frac{\partial p}{\partial z}\right) + \frac{\partial \mathbf{v}}{\partial z} \cdot \nabla_h p + \frac{\partial v_z}{\partial z}\frac{\partial p}{\partial z} \\
&= -g\frac{D\varrho}{Dt} + \varrho g[\nabla_h \cdot \mathbf{v}_P - \nabla_h \cdot \mathbf{v}] - \varrho g \frac{\partial v_z}{\partial z} \\
&= -g\left[\frac{D\varrho}{Dt} + \varrho \nabla \cdot \mathbf{v}\right] + \varrho g[\nabla_h \cdot \mathbf{v}_P],
\end{aligned}$$

using (14B), (15) and (17).

$$\frac{D\varrho}{Dt} + \varrho \nabla \cdot \mathbf{v} = 0$$

becomes

$$\frac{\partial \omega}{\partial z} = \varrho g \nabla_h \cdot \mathbf{v}_P. \tag{20}$$

In the p-system equation (13) becomes

$$\frac{\partial A}{\partial P} = -\frac{1}{\varrho g}\frac{\partial A}{\partial z}. \tag{21}$$

Using the notation $\nabla_P \otimes \mathbf{A} \equiv \nabla_h \otimes \mathbf{A}_P$ employed by Eliassen and others, equation (20) becomes

$$\nabla_P \cdot \mathbf{v} + \frac{\partial \omega}{\partial P} = 0 \tag{22}$$

from which it appears that the atmospheric flow is non-divergent. The notation used above is slightly misleading and the approximations involved, as found in the previous section and evaluated in the following section, in this form of the continuity equation are generally not mentioned. The fact that large scale flow in the atmosphere is approximately non-divergent has been long established.

4. Magnitude of errors

Since $\partial A/\partial q$ generally has no singularities the series expansion for $\partial A_\Xi/\partial q$ will always converge. The error introduced by neglecting all powers of Δz greater than zero is to be found.

If we write equation (9) in the form

$$\left[\frac{\partial A}{\partial q}\right]_Z = \sum_{n=0}^{\infty} a_n(A),$$

where $a_n(A)$ depends on A, then

$$\frac{|a_{n+1}|}{|a_n|} \approx \frac{\Delta z}{n+1}\left\{\frac{\partial^{n+1}}{\partial z^{n+1}}\left(\frac{\partial A}{\partial q}\right) \bigg/ \frac{\partial^n}{\partial z^n}\left(\frac{\partial A}{\partial q}\right)\right\}.$$

When referring to orders of magnitude we shall use the convention of regarding quantities differing by a factor of 10 as differing by one order of magnitude. The symbol $\sim$ will be used to equate quantities of the same order of magnitude. Thus, following BURGER (1960) and others we have

$$\frac{\partial^n}{\partial z^n}\left(\frac{\partial A}{\partial q}\right) \sim A \bigg/ \left(\frac{\lambda_z}{2}\right)^n \left(\frac{\lambda_q}{2}\right),$$

where $\frac{1}{2}\lambda_q$ is a characteristic half-wavelength in the q direction (written H and S for vertical and horizontal directions by Burger who gives $H \sim 10^4$ m).

Hence $$\frac{|a_{n+1}|}{|a_n|} \sim \frac{2\Delta z}{\lambda_z (n+1)} \sim \frac{\Delta z}{10^4 (n+1)}.$$

Thus we have

$$\frac{\partial A_\Xi}{\partial q} = \left[\frac{\partial A}{\partial q}\right]_Z + \left[\frac{\partial A}{\partial z}\right]_Z \frac{\partial Z_\Xi}{\partial q}$$

to within $$100\frac{\Delta z}{10^4}\% = \frac{\Delta z}{10^2}\%. \tag{23}$$

In particular, for the p-system and at the 500 mb. level, Δz is of the order of half the maximum difference in heights of the 500 mb surface over the area of analysis (i.e. $\Delta z \sim 10^3$ m), so that errors of up to 10 %, though generally smaller, are inherent in the use of equations (11), (12) and (13).

It can be shown that the error is of the same order of magnitude as the error introduced when barotropic flow (i.e. density a function of pressure only, so that the vertical shear of the geostrophic wind velocity is zero) is assumed.

Consider the derivation of the equation

$$\nabla_h Z_P = \frac{1}{\varrho g} \nabla_h p.$$

Suppose that for each x, y and t the constant-ξ surface passing through Z is $\Xi_0(x,y,Z,t)$ where Z is, as previously defined, between the height extremes of the surface $\xi = \Xi$.

$$\text{Since} \quad \frac{\partial \xi_\Xi}{\partial q} \equiv 0 = \left[\frac{\partial \xi}{\partial q}\right]_{Z\Xi} + \frac{\partial Z_\Xi}{\partial q}\left[\frac{\partial \xi}{\partial z}\right]_{Z\Xi} \tag{24}$$

$$\text{we have} \quad \left[\frac{\partial \xi}{\partial q}\right]_Z = -\frac{\partial Z_{\Xi_0}}{\partial q}\left[\frac{\partial \xi}{\partial z}\right]_Z.$$

Expanding $\partial Z_{\Xi_0}/\partial q$ in terms of $\Delta\Xi = \Xi - \Xi_0$ gives

$$-\left[\frac{\partial \xi}{\partial q}\Big/\frac{\partial \xi}{\partial z}\right]_Z = \sum_{n=0}^{\infty} \frac{(\Delta\Xi)^n}{n!}\frac{\partial^n}{\partial \Xi^n}\left(\frac{\partial Z_\Xi}{\partial q}\right) \tag{25}$$

$$\therefore \left[\frac{\partial \xi}{\partial q}\right]_Z = -\sum_{n=0}^{\infty}\left[\frac{\partial \xi}{\partial z}\right]_Z \frac{(\Delta\Xi)^n}{n!}\frac{\partial^n}{\partial \Xi^n}\left(\frac{\partial Z_\Xi}{\partial q}\right) = \sum_{n=0}^{\infty} b_n$$

which we can compare with

$$\left[\frac{\partial \xi}{\partial q}\right]_Z = \sum_{n=0}^{\infty} a_n(\xi).$$

Then

$$\frac{|b_n|}{|a_n|} \approx \frac{(\Delta\Xi)^n}{n!}\frac{\partial \xi}{\partial z}\frac{\partial^n}{\partial \Xi^n}\left(\frac{\partial Z_\Xi}{\partial q}\right)\Big/\frac{(\Delta z)^n}{n!}\frac{\partial^n}{\partial z^n}\left(\frac{\partial \xi}{\partial q}\right)$$

$$\approx \left(\frac{\Delta\Xi}{\Delta z}\right)^n \frac{\partial \xi}{\partial z}\frac{\partial^n}{\partial z^n}\left(\frac{\partial \xi}{\partial q}\Big/\frac{\partial \xi}{\partial z}\right)\Big/\left(\frac{\partial \xi}{\partial z}\right)^n \frac{\partial^n}{\partial z^n}\left(\frac{\partial \xi}{\partial q}\right)$$

using equations (13) and (24).

Thus

$$\frac{|b_n|}{|a_n|} \sim \left(\frac{\partial \xi}{\partial z}\right)^{n+1}\left(\frac{\partial \xi}{\partial q}\right)\left(\frac{\lambda_z}{2}\right)^n \Big/ \frac{\partial \xi}{\partial z}\left(\frac{\lambda_z}{2}\right)^n\left(\frac{\partial \xi}{\partial z}\right)^n \frac{\partial \xi}{\partial q} = 1$$

as one would expect.

Now from equation (25) it is evident that the neglect of the term of order one in powers of $\Delta\Xi$ (which is equivalent to neglecting terms of order one in powers of Δz) is equivalent to assuming that

$$\frac{\partial}{\partial \Xi}\left(\frac{\partial Z_\Xi}{\partial q}\right) = 0.$$

In the p-system this implies

$$\frac{\partial}{\partial P}\left(\frac{\partial Z_P}{\partial q}\right) = 0$$

or, if $\mathbf{G}$ is the geostrophic wind, then

$$\frac{\partial \mathbf{G}}{\partial P} = 0$$

so that the vertical shear of the geostrophic wind is zero, which is true only if the atmosphere is barotropic.

From the argument leading to equation (23) it can be seen that the degree of approximation in all equations of the p-system is the same. It follows then that the use of these equations is no better than assuming barotropic conditions to hold. Thus, all numerical forecasting models which make use of pressure coordinates, have inherent in them an approximation which is of the order of the barotropic flow approximation. We would expect, then, that models which are more sophisticated than the simple "equivalent barotropic" model (where the less restrictive assumption of no wind direction change with height is made) would yield results which were not significantly better.

5. Conclusions

The usual expressions used in deriving dynamical equations for a coordinate system where the vertical cartesian coordinate is replaced by a scalar dependent variable are found to be accurate only to within $10^{-2}\Delta z$ % where Δz (metres) is half the height difference between the highest and lowest points of the particular constant scalar surface under consideration. For the p-system this approximation is shown to be equivalent to the barotropic assumption and consequently one would expect an "equivalent, barotropic" numerical prediction model to be as good as any other model where pressure coordinates are used.

REFERENCES

BURGER, A. P., 1960, *First Steps in Dynamical Weather Prediction*, Weather Bureau, Union of South Africa, WB25.

ELIASSEN, A., 1949, The quasi-static equations of motion with pressure as an independent variable. *Geofysiske Publikasjoner*, *17* No. 3, pp. 1–44.

PERKINS, D. T. and GUSTAFSON, A. F., 1951, Differential relations between quantities projected from meteorological surfaces. *J. Meteor.*, *8*, No. 6 pp. 418–420.