# The stability of nongeostrophic perturbations in a baroclinic zonal flow

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(Manuscript received March 18, 1963)

ABSTRACT

The stability of a baroclinic zonal flow is studied by means of the method of small perturbations without imposing geostrophic balance on the perturbations. In other respects the present treatment of the stability problem is within the same framework as most other studies on the subject.

The removal of the geostrophic assumption makes the results applicable to low latitudes and provides thereby some insight into the conditions under which tropical disturbances may form. The theory leads to increased understanding of conditions favoring deviations from geostrophic flow and explains why observed perturbations are for the most part quasigeostrophic. Furthermore, it reconciles results of some previous studies which within the narrower framework of geostrophic theory appeared unconnected.

Two cases of generalization of previous results by other investigators are studied. The first of these extend results of EADY (1949) and FJÖRTOFT (1950), the second case extends results obtained by CHARNEY (1947), HOLMBOE (1959), and ÁRNASON (1961). Based on these two cases general stability criteria are proposed.

## 1. Introduction

The stability of baroclinic flow has been the subject of extensive research, notably by CHARNEY (1947), EADY (1949), FJÖRTOFT (1950), and KUO (1952, 1953), all of whom used the method of small perturbations to obtain special solutions to this problem. Common to these studies is the assumption that the wind and pressure perturbations are in geostrophic balance. Though in fair agreement with synoptic evidence, this assumption limits the possibility of studying departures from geostrophic wind and restricts the application of the solutions mentioned above to middle and high latitudes. Without imposing geostrophic balance, the baroclinic stability problem has been solved for the very special case of neutral static stability by FJÖRTOFT (1950), and by HOLMBOE (1959).

Because of varying assumptions, the results of the studies referred to differ in some respects and may even appear to be contradictory as in the case of the theory of relatively short waves. EADY (1949) concluded that in the special case of a Coriolis parameter independent of latitude, waves shorter than a certain length are stable owing to the restraining influence of the static stability. The same was found independently by FJÖRTOFT (1950) in the more general case of a varying Coriolis parameter. Surprisingly, this criterion, hereafter referred to as the "short-wave cut-off", is absent in the theories of CHARNEY (1947) and KUO (1952). Their results show that for a given vertical wind shear, all waves shorter than a certain length are unstable. This second criterion, which we shall refer to for the sake of brevity as the "long-wave cut-off", is a result of a variable Coriolis parameter.

While the absence of the latter criterion in Eady's theory is well understood, the lack of a short-wave cut-off in the theories of Charney and Kuo has not been fully explained, but it is presumably a result of differences in boundary conditions used in the theories being compared. This point has been dealt with at some length in a paper by GREEN (1960), who attempts to reconcile the results of Eady on the one hand with those of Charney and Kuo on the other. Green concludes that if the upper lid in Eady's model is removed, and the variation with latitude of the Coriolis parameter omitted ($\beta = 0$) in the theories of Charney and Kuo, then the posed problems become identical and yield the result that all waves are neutral.[1] Green's conclusion is easily verified in Eady's case (see Chapter 4 of this paper), but appears to contradict Kuo's stability criterion (KUO 1952, see his eq. (86) and the definition of $r$ used in connection with eq. (54)). Kuo's criterion implies that when $\beta$ approaches zero all waves become unstable, which is consistent with results from theories by FJÖRTOFT (1950) and HOLMBOE (1959), discussed later in this paper. By relaxing the restriction $\beta = 0$ in the theory of Eady, but retaining Eady's upper boundary condition (rigid horizontal plane), Green finds that the previously stable short waves become unstable. This extension of Eady's theory gives results which greatly resemble those of KUO (1952), except that even the longest waves are found slightly unstable; the latter agrees with later results obtained by BURGER (1962).

It is of interest to note that elimination of the short stable waves by admitting a variable Coriolis parameter does not occur in the theory of nongeostrophic perturbations for the case of neutral static stability, given by FJÖRTOFT (1950) and HOLMBOE (1959). In this case the short-wave cut-off is essentially the same whether one assumes a variable Coriolis parameter or not. A stability criterion derived by FJÖRTOFT (1950) by means of energy considerations, assuming quasigeostrophic motion but allowing a variable Coriolis parameter, contains the two kinds of cut-offs dealt with above. This is also the case with stability criteria derived for quasigeostrophic two-level models used in numerical weather prediction, THOMPSON (1961). Hence, theories on baroclinic stability differ in regard to the stability of short waves, but are mainly in agreement in requiring a long-wave cut-off in response to a variable Coriolis parameter. A noteworthy exception to the latter statement is the result obtained by BURGER (1962), whose mathematical analysis of the baroclinic stability problem, as formulated by Charney, admits no long-wave cut-off. As we shall discuss at the end of Chapter 7, this particular result is probably not physically meaningful. In the same chapter we shall also view the results of this study in relation to those obtained by CHARNEY & STERN (1962) for an internal baroclinic jet.

The present paper expands the framework laid in previous studies by offering special solutions to the baroclinic stability problem without imposing the geostrophic assumption. The removal of this assumption makes the results applicable to low latitudes and provides thereby some insight into the dynamics of incipient perturbations within the Tropics. The theory leads to increased understanding of conditions favoring deviations from geostrophic flow and explains why observed perturbations are for the most part quasigeostrophic. Furthermore, it reconciles results of some of the previous studies which within the narrower framework of geostrophic theory appeared unconnected.

In mathematical terms the solution of the baroclinic stability problem is equivalent to solving an eigenvalue problem. In the case of a frictionless fluid this problem consists of solving a second-order differential equation with appropriate boundary conditions leading to a characteristic equation from which stability criteria may be derived. The principal obstacle to the solution of the baroclinic stability problem for nongeostrophic perturbations is the complexity of the associated differential equation. It has four singular points, all of which are on the real axis, and is not reducible to any of the known types, If one is concerned only with obtaining results of meteorological interest, it is not necessary to solve the differential equation for a domain comprising all of the singular points. The domain of primary interest contains one singular point only, the point $c - U = 0$[2] Its boundary contains two singular points which, speaking in physical terms, correspond to a transition from moderately nongeostrophic motion to the highly nongeostrophic motion associated with inertia-gravity waves.

The solutions of the special cases dealt with in the present study are confined to this domain. To distinguish between quasigeostrophic and nongeostrophic motion, we have introduced as the independent variable of the differential equation a quantity $\varepsilon$ which is numerically equal to the ratio of horizontal wind divergence to

(1) Neutral and stable are used synonymously in this paper.

(2) The most common usage of symbols is being adhered to in this paper. For this reason, only some of the symbols used are explained in the text but all are listed and defined at the end of the next chapter.

the vertical component of vorticity. The quasigeostrophic motion is defined here by the inequality $|\varepsilon| \ll 1$, and the type of nongeostrophic motion we are concerned with is characterized by $|\varepsilon| < 1$.

Notwithstanding the restriction $0 \leqslant |\varepsilon| < 1$ of the range of interest, a solution fully covering this range and including the two points corresponding to the upper and lower boundaries of the fluid is not easily obtained. The complexity of the differential equation has led us to resort to the use of a power series expansion around the ordinary point $\varepsilon = 0$ as a means of obtaining approximate solutions. This has enabled us to find solutions which extend results obtained by Eady, Fjörtoft, Holmboe, and Charney, referred to earlier. Since a series solution is valid no further than the nearest singular point, which in this case is $c - U = 0$, we do not generally obtain results valid for the entire range $0 \leqslant |\varepsilon| < 1$. An alternative to the expansion about $\varepsilon = 0$ is an expansion around the singular point $c - U = 0$. Solution by means of this expansion would extend results obtained by KUO (1952), GREEN (1960), and BURGER (1962) for quasigeostrophic motion, but would not be valid if the point of expansion is so close to either of the singular points $\varepsilon = \pm 1$ that the upper and lower boundaries of the fluid would not be included in the range for which the expansion holds. The latter expansion has not been used here for two reasons: the associated series converges more slowly than the series obtained by expanding around the regular point $\varepsilon = 0$, and the additional circumstance that one of the fundamental solutions in nonanalytic leads to a characteristic equation the solution of which appears extremely laborious.

## 2. The governing equations

The earth's atmosphere is in this study replaced by a single layer of a compressible, inviscid fluid in hydrostatic balance and of infinite lateral extent. In describing the physics of this model, we shall apply the continuity equation, the adiabatic equation, and the equations for vorticity, divergence, and hydrostatic equilibrium. The coordinate system used is the $x$, $y$, $p$ system and the five dependent variables are $u$, $v$, $\omega$, $\Phi$, and $\vartheta$ (a list of symbols is given at the end of this chapter). We confine ourselves to the application of the theory of small harmonic perturbations superposed on a stationary and simple basic flow assumed to be geostrophic and in hydrostatic equilibrium. If we denote by $U$ the velocity of this flow (directed along the $x$-axis), then the geostrophic and hydrostatic equations may be written as follows:

$$U = -\frac{1}{f}\frac{\partial \bar{\Phi}}{\partial y}, \tag{2.1}$$

$$\frac{\partial \bar{\Phi}}{\partial p} = -\frac{R\bar{T}}{p} = -F\bar{\vartheta}, \tag{2.2}$$

where
$$F = \frac{R}{p}\left(\frac{p}{p_0}\right)^{R/c_p}; \tag{2.3}$$

the bar refers to the dependent variables characterizing the basic flow. It is further assumed that $U$ is independent of $y$ and is, for the time being, an arbitrary function of the pressure $p$. From (2.1) and (2.2) we derive the thermal wind equation

$$\frac{\partial U}{\partial p} = \frac{R}{pf}\frac{\partial \bar{T}}{\partial y} = \frac{F}{f}\frac{\partial \bar{\vartheta}}{\partial y}. \tag{2.4}$$

Since $U$ is independent of $y$ it follows from (2.4) that $\partial \bar{T}/\partial y$ and $\partial \bar{\vartheta}/\partial y$ are also independent of $y$. Furthermore,

$$\bar{\zeta} = \operatorname{div} \bar{\mathbf{v}} = \bar{\omega} = 0. \tag{2.5}$$

The small perturbations, also assumed independent of the $y$-coordinate, are described by the following system of linear equations:
Vorticity equation:

$$\frac{\partial \zeta}{\partial t} + U\frac{\partial \zeta}{\partial x} + \beta v + f\frac{\partial u}{\partial x} = 0. \tag{2.6}$$

Divergence equation:

$$f\zeta - \beta u - \frac{\partial^2 \Phi}{\partial x^2} - \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\frac{\partial u}{\partial x} - \frac{\partial \omega}{\partial x}\frac{\partial U}{\partial p} = 0. \tag{2.7}$$

Adiabatic equation:

$$\frac{\partial \vartheta}{\partial t} + U\frac{\partial \vartheta}{\partial x} + v\frac{\partial \bar{\vartheta}}{\partial y} + \omega\frac{\partial \bar{\vartheta}}{\partial p} = 0. \tag{2.8}$$

Hydrostatic equation:

$$\frac{\partial \Phi}{\partial p} + F\vartheta = 0. \tag{2.9}$$

Continuity equation:

$$\frac{\partial \omega}{\partial p} + \operatorname{div} \mathbf{v} = 0, \tag{2.10}$$

where $$\zeta = \frac{\partial v}{\partial x} \quad \text{and} \quad \operatorname{div} \mathbf{v} = \frac{\partial u}{\partial x}. \tag{2.11}$$

The non-linear equations from which the system (2.6), (2.7), and (2.8) is derived are not reproduced here, but they appear respectively as equations (2.1), (2.2), and (2.4) in the author's article of 1961 given in the list of references. We mention that owing to the simplicity of the basic flow and the lack of $y$-dependence of the perturbations, the term $\mathbf{k}\cdot\nabla \times \omega(\partial \mathbf{v}/\partial p)$ in the non-linear vorticity equation gives no contribution when linearized. Its counterpart, $-\operatorname{div} \omega(\partial \mathbf{v}/\partial p)$, in the divergence equation contributes the last term in eq. (2.7).

The algebraic manipulations involved in replacing $u$, $v$, $\Phi$, $\omega$, and $\vartheta$ in (2.6) to (2.10) by their harmonic counterparts, are given in Appendix I. Elimination of four of the five dependent variables leads to a single second-order differential equation in the amplitude function of $\omega$, denoted by $D$. Together with appropriate boundary conditions, this equation defines an eigenvalue problem, the solution of which is the main objective of this paper.

### List of Symbols

| Symbol | Description |
|---|---|
| $c$ | Phase speed of small perturbations. |
| $c_i$ | The imaginary part of the phase speed $c$. |
| $c_r$ | The real part of the phase speed $c$. |
| $c_p$ | Specific heat for constant pressure. |
| $f$ | Coriolis parameter. |
| $g$ | Acceleration of gravity. |
| $h$ | The square root of the Richardson number of the basic flow. |
| $i$ | $\sqrt{-1}$. |
| $k$ | $2\pi/L$, wave number. |
| $n$ | Number of waves around a latitude circle. |
| $p$ | Pressure, or index in a series expansion. |
| $p_0$ | Reference pressure (1000 mb). |
| $s$ | $g/c_p + \partial \bar{T}/\partial z$, static stability, assumed positive. |
| $t$ | Time coordinate. |
| $u$ | Velocity component along the $x$-axis (East). |
| $v$ | Velocity component along the $y$-axis (North). |
| $x, y$ | Horizontal coordinates toward East and North, respectively. Alternatively, $y = h\varepsilon$. |
| $z$ | Vertical coordinate and equal to $ZH_m$. |
| $A$ | Amplitude of $u$. |
| $B$ | Amplitude of $v$. |
| $C$ | Amplitude of $\Phi$. |
| $C_i$ | $f/k$, speed of inertia waves. |
| $C_g$ | $(Z_1/\pi)\ (RH_m s)^{\frac{1}{2}}$, speed of internal gravity waves. |
| $D$ | Amplitude of $\omega$. |
| $E$ | Amplitude of $\vartheta$. |
| $F$ | $(R/p)\left(\frac{p}{p_0}\right)^{R/c_p}$. |
| $H_m$ | $R\bar{T}/g$. |
| $L$ | Wave-length. |
| $P_0, P_1, P_2$ | Defined by eq. (20) in Appendix I. |
| $R$ | Gas constant for dry air. |
| $T$ | Temperature. |
| $U$ | Speed of the basic flow. |
| $U_1$ | Speed at the bottom of the layer. |
| $U_2$ | Speed at the top of the layer. |
| $U_c$ | $\beta/k^2$. |
| $\bar{U}$ | $\frac{1}{2}(U_1 + U_2)$, mean speed of the basic flow. |
| $Z$ | $= \ln(p_0/p)$, used as vertical coordinate. |
| $\alpha$ | $Z_1(RH_m s)^{\frac{1}{2}}/2C_i = \frac{1}{2}h[(U_2 - U_1)/C_i]$. |
| $\alpha_1$ | $\alpha h^{-1} = \frac{1}{2}[(U_2 - U_1)/C_i]$. |
| $\alpha_2$ | $\alpha_1(1 + \frac{1}{4}\varkappa_1^2)$. |
| $\beta$ | $df/dy$. |
| $\gamma$ | $\frac{1}{2}h(\varepsilon_1 + \varepsilon_2)$ where subscripts 1 and 2 refer to the lower and upper boundaries. |
| $\delta$ | Horizontal wind divergence. |
| $\varepsilon$ | $\Delta/C_i$. |
| $\zeta$ | Vertical component of wind vorticity. |
| $\vartheta$ | Potential temperature. |
| $\varkappa_1$ | $U_c/C_i$. |

| | |
|---|---|
| $\varkappa_2$ | $U'/C_i$. |
| $\pi$ | 3.1416. |
| $\Phi$ | Geopotential. |
| $\varphi$ | Latitude. |
| $\omega$ | Vertical $p$-velocity. |
| $\Delta$ | $c - U + U_c$ |
| $\Gamma(\alpha)$ | See eq. (4.2). |
| $\mathbf{k}$ | Unit vector directed upward along the vertical. |
| $\mathbf{v}$ | Horizontal wind vector. |
| div $\mathbf{v}$ | Horizontal wind divergence. |

## 3. The eigenvalue problem

It is shown in Appendix I that elimination of all but one of the variables of the system (2.6) to (2.11) leads to the following second-order differential equation in the amplitude function $D$ (eq. (19) in the appendix):

$$P_0 D'' + (P_0 + P_1) D' - P_2 \varepsilon^2 D = 0, \qquad (3.1)$$

where a prime means differentiation with respect to $Z = \ln(p_0/p)$; $P_0$, $P_1$, and $P_2$ are certain functions of the independent variable $Z$ and are defined in Appendix I. Equation (3.1), together with appropriate boundary conditions, constitutes the eigenvalue problem to be solved. Since (3.1) does not appear to be reducible to a known type, we do not expect to find solutions in terms of elementary functions or known transcendental functions. Under these circumstances, the simplest way to obtain a solution is by means of a power-series expansion which is the mathematical tool used subsequently in this paper.

To facilitate comparison with special solutions of (3.1) known from the literature and discussed below, we shall at this point make a few simplifications. Let us first restrict $U$ to be a linear function of $Z$, i.e. $U' =$ const.; next let us omit the term $(c - U) U'$ as compared with $RH_m s$ in the expression for $P_2$, eq. (20) in Appendix I. This can be done safely as long as we are not concerned with an extreme vertical wind shear and a phase speed approaching the speed of gravity waves. The third simplification we shall make is to omit $P_0$ as compared to $P_1$ in the second term of (3.1). As the derivation of eq. (3.1) shows (see Appendix I), the appearance of the term $P_0 D'$ is closely related to the particular choice of vertical coordinate, the aim of which is to define a static stability approximately independent of the vertical coordinate. With these three approximations included eq. (3.1) may be written

$$\varepsilon(\varepsilon - \varkappa_1)(1 - \varepsilon^2) D'' + (2\varepsilon - \varkappa_1)\varkappa_2 D' - \frac{RH_m s}{C_i^2} \varepsilon^2 D = 0. \qquad (3.2)$$

The special case $U' = 0$ has been dealt with elsewhere by the author (ÁRNASON, 1961). In this case (3.1) is satisfied by elementary functions and with the boundary conditions $D = 0$ at $Z = 0$, $Z_1$ there are three neutral solutions for the phase velocity $c$, one of which corresponds to quasigeostrophic motion. The two remaining phase velocities are those of inertia-gravity waves. A fourth solution is Rossby's well-known formula for the speed of non-divergent waves, which is, however, a singular solution in that it does not follow from the general frequency equation. Rossby's special case is easily derived from eqs. (2) and (6) in Appendix I by postulating $D \equiv 0$. We mention that in the special case $U' = 0$ the omission of the term $P_0 D'$ in (3.1) has a negligible effect upon the phase speed; its main effect is the modification of the vertical structure of the perturbations.

A second special case of (3.1) is that of vanishing static stability, i.e. $P_2 = 0$, studied by FJÖRTOFT (1950) and by HOLMBOE (1959). Here both external and internal gravity waves have been precluded, the former through the specified boundary conditions and the latter by the assumption of zero static stability.[1] In this case there are two solutions for the phase speed, $c$, which may be complex owing to the baroclinicity of the basic flow, thus allowing perturbations to grow with time; these are combined vorticity-inertia waves. The associated wind may be nongeostrophic to a considerable degree as will be discussed in some detail in Chapter 5.

The third special case we shall mention is the case where we initially make use of the geostrophic assumption by assuming $|\varepsilon| < 1$ in the first term of (3.2) which then becomes

$$\varepsilon(\varepsilon - \varkappa_1) D'' + (2\varepsilon - \varkappa_1) D' + \frac{RH_m s}{C_i^2} \varepsilon^2 D = 0. \qquad (3.3)$$

[1] Both Fjörtoft and Holmboe used a $z$-system and imposed the boundary condition that the vertical velocity vanishes at $z = 0$ and at some finite height $z = z_1$.

This equation, or rather the corresponding equation for the amplitude function $B$ of the meridional wind component, $v$, has been solved by CHARNEY (1947) and later under more general conditions by KUO (1952) for boundary conditions which differ from those used by Fjörtoft, Holmboe, and Árnason. The corresponding eigenvalue problem leads to two solutions for the phase speed, $c$, which under certain conditions are complex, thus admitting amplifying perturbations. EADY (1949) also solved (3.3) in the special case $\varkappa_1 = 0$ for the boundary conditions $D = 0$ at the upper and lower boundaries of the fluid layer. While Charney's and Kuo's solutions to (3.3) are in the form of confluent hyper-geometric series, Eady, because of the assumption $\varkappa_1 = 0$, obtained elementary functions as solutions to (3.3). Although in many respects similar to those of Charney and Kuo, Eady's results differ on one important point: he finds that waves shorter than a certain wave-length are always stable, while Charney and Kuo found no such restriction on the short waves. This difference has been discussed by GREEN (1960), who attributes it to the assumption $\varkappa_1 = 0$ and to difference in the upper boundary condition.

In solving eq. (3.2) it is convenient for our purpose to replace $Z$ by $\varepsilon$ as the independent variable for reasons that will subsequently become clear. With this transformation, eq. (3.2) takes the form

$$\varepsilon(\varepsilon - \varkappa_1)(1 - \varepsilon^2) D'' - (2\varepsilon - \varkappa_1) D' - h^2 \varepsilon^2 D = 0, \tag{3.4}$$

where a prime now means differentiation with respect to $\varepsilon$, and

$$h^2 = \frac{RH_m s}{\left(\frac{dU}{dZ}\right)^2} = Z_1^2 \frac{RH_m s}{(U_2 - U_1)^2} \tag{3.5}$$

is the Richardson number of the basic flow and will be considered independent of $\varepsilon$. Because of subsequent references to some of the special cases of (3.4), referred to above, we have compiled in Table 1 a listing of the forms of (3.4) solved by various investigators along with the solutions and frequency equations obtained, and the assumptions made.

The main mathematical difficulty in solving (3.4) is the occurrence of singularities. Inspection reveals that the points $\varepsilon = 0$, $\varepsilon = \varkappa_1$, $\varepsilon = 1$, and $\varepsilon = -1$ are all regular singular points, but that the point $\varepsilon = 0$ is an apparent singularity, i.e., the solutions to (3.4) are analytic at that point. Consequently, the application of a power series expansion around $\varepsilon = 0$ is straightforward and converges rapidly for small $\varepsilon$ and is valid as far as the nearest singular point. This is fortunate, since we are most interested in solutions to (3.4) for a range of $\varepsilon$ well within the unit circle. That this is the case follows from the fact (see eq. (15) in Appendix I) that $\varepsilon$ is equal in magnitude to the ratio $\delta/\zeta$, where $\delta$ is the horizontal wind divergence of the perturbation and $\zeta$ the relative wind vorticity. While this ratio is equal to or greater than unity for inertia and gravity waves, it is considerably less than unity for synoptic waves with which we are primarily concerned. By restricting $\varepsilon$ to the range $0 \leqslant |\varepsilon| < 1$, we eliminate in practice the two singular points $\varepsilon = \pm 1$. The remaining obstacle to a sufficiently general power-series solution to (3.4) is the singularity $\varepsilon = \varkappa_1$ beyond which the power series around $\varepsilon = 0$ becomes invalid. Unfortunately, $\varepsilon$ may well exceed $\varkappa_1$ which is clear from the definition

$$\varepsilon = \frac{\Delta}{C_i} = \frac{c - U}{C_i} + \varkappa_1 \tag{3.6}$$

and the fact that the real part of $c - U$ can be positive. A power-series solution around $\varepsilon = 0$ is therefore not a general solution without being extended beyond the point $\varepsilon = \varkappa_1$. Such an extension requires a suitable method for analytical continuation. This, however, is beyond the scope of this article, where we shall confine ourselves to special cases not involving the singularity $\varepsilon = \varkappa_1$. The non-dimensional number $\varkappa_1$ is important in the baroclinic stability problem; it is shown in Fig. 1 in relation to latitude and the number of waves around a latitude circle.

Before leaving the subject of singular points, it may be of interest to mention that under certain conditions, the point $\varepsilon = \varkappa_1$ is an apparent singularity only. More specifically this is the case when $h = 0$ [1] which is borne out by the solutions to the special cases 3 and 5 given in Table 1; both solutions are analytic when

[1] This is strictly true only if $1 - \varepsilon^2$ in the first term of (3.4) is either replaced by $1 - \varepsilon(\varepsilon - \varkappa_1)$ or by 1.

TABLE 1. *Six special cases of the differential equation (3.4), the fundamental solutions, and the frequency equations appropriate to the given boundary conditions (see the text).*

The occurring notations are explained in the text or in the list of symbols at the end of Chapter 2. The exceptions are $\nu_1$ and $\nu_2$ which are the roots of $1-\varepsilon(\varepsilon-\varkappa_1)=0$ and $r_1$ which is defined by

$$r_1^2=\tfrac{1}{4}+\frac{RH_m s}{(c-U)(C_i^2-\Delta^2)}.$$

| No. | Name | Differential equation | Solutions | Frequency equations | Assumptions |
|---|---|---|---|---|---|
| 1 | Charney (1947) | $\varepsilon(\varepsilon-\varkappa_1)\frac{d^2D}{d\varepsilon^2}-(2\varepsilon-\varkappa_1)\frac{dD}{d\varepsilon}-h^2\varepsilon^2D=0$ | Transcedental functions | $c_r\leqslant U_0$. Not derived | $c_r\leqslant U$; $\lvert\varepsilon\rvert<1$ |
| | Kuo (1952) | | | $c_r-U_0\sim\frac{dU}{dz},\quad c_i\sim\frac{C_i}{h}$ | $\lvert\varepsilon\rvert<1$; $h=\text{Const.}$ |
| 2 | Eady (1949) | $\varepsilon\frac{d^2D}{d\varepsilon^2}-2\frac{dD}{d\varepsilon}-h^2\varepsilon D=0$ | $D_1=(1+h\varepsilon)e^{-h\varepsilon}$<br>$D_2=(1-h\varepsilon)e^{-h\varepsilon}$ | $c=\bar{U}\pm i\frac{C_i}{h}\Gamma(\alpha)=\bar{U}\pm i\frac{\Gamma(\alpha)}{2\alpha}(U_2-U_1)$<br>$\Gamma(\alpha)=[(\alpha-\tanh\alpha)(\coth\alpha-\alpha)]^{\frac{1}{2}}$ | $\varkappa_1=0$<br>$\lvert\varepsilon<1$<br>$h=\text{Const.}$ |
| 3 | Fjörtoft (1950) | $\varepsilon(\varepsilon-\varkappa_1)\frac{d^2D}{d\varepsilon^2}-(2\varepsilon-\varkappa_1)\frac{dD}{d\varepsilon}=0$ | $D_1=\text{Const.}$<br>$D_2=\frac{\varepsilon^3}{3}-\frac{\varkappa_1}{2}\varepsilon^2$ | $c=\bar{U}-\frac{1}{2}\beta/k^2\pm iC_i\left[\frac{\alpha_1^2}{3}-\frac{\varkappa_1^2}{4}\right]^{\frac{1}{2}}$<br>$=\bar{U}-\frac{1}{2}\beta/k^2\pm i[\frac{1}{12}(U_2-U_1)^2-\frac{1}{4}(\beta/k^2)^2]^{\frac{1}{2}}$ | $h=0;\ \lvert\varepsilon\rvert<1$ |
| 4 | Fjörtoft (1950) | $\varepsilon(1-\varepsilon^2)\frac{d^2D}{d\varepsilon^2}-2\frac{dD}{d\varepsilon}=0$ | $D_1=\text{Const.}$<br>$D_2=2\varepsilon+\ln\frac{1-\varepsilon}{1+\varepsilon}$ | $c=\bar{U}\pm iC_i\Gamma(\alpha_1)=\bar{U}\pm i\frac{\Gamma(\alpha_1)}{2\alpha_1}(U_2-U_1)$<br>$\alpha_1=\alpha h^{-1}$ | $h=0;\ \varkappa_1=0$ |
| 5 | Holmboe (1959) | $\varepsilon(\varepsilon-\varkappa_1)(1-\varepsilon(\varepsilon-\varkappa_1))\frac{d^2D}{d\varepsilon^2}-(2\varepsilon-\varkappa_1)\frac{dD}{d\varepsilon}=0$ | $D_1=\text{Const.}$<br>$D_2=2\varepsilon+\left(1+\frac{\varkappa_1^2}{4}\right)^{-\frac{1}{2}}\ln\frac{\varepsilon-\nu_1}{\varepsilon-\nu_2}$ | $c=\bar{U}-\frac{1}{2}\beta/k^2\pm iC_i\left[\Gamma^2(\alpha_2)-(1-\alpha_1^2)\frac{\varkappa_1^2}{4}\right]^{\frac{1}{2}}$<br>$\alpha_2=\alpha_1\left(1+\frac{\varkappa_1^2}{4}\right)$ | $h=0$ |
| 6 | Árnason (1961) | $(\varepsilon-\varkappa_1)(1-\varepsilon^2)\left[\frac{d^2D}{dZ^2}+\frac{dD}{dZ}\right]-\frac{RH_m s}{C_i^2}\varepsilon D=0$ | $D_1=e^{-\frac{1}{2}Z}\sin r_1 Z$<br>$D_2=e^{-\frac{1}{2}Z}\cos r_1 Z$ | $\Delta[g'H-(c-U)\Delta]+(c-U)C_i^2=0$<br>$\Delta=c-U+\beta/k^2$ | $U=\text{Const.}$<br>$s=\text{Const.}$ |

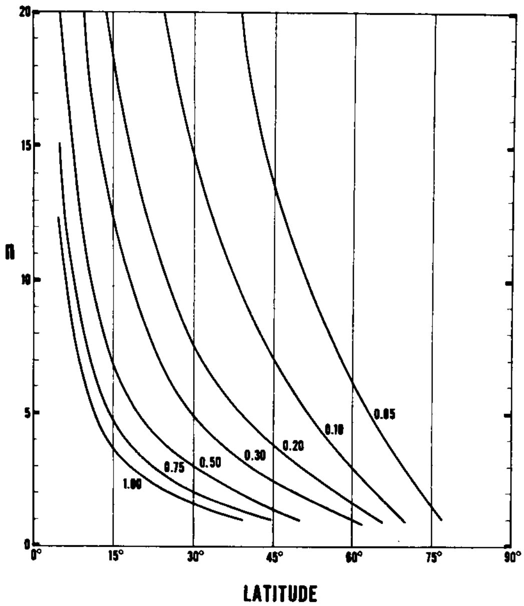


FIG. 1. Isopleths for $\varkappa_1$ in relation to latitude and the number of waves, $n$, around a latitude circle.

$\varepsilon = \varkappa_1$. This might possibly be interpreted to mean that solutions for small $h$ are meaningful somewhat beyond the point $\varepsilon = \varkappa_1$ and lead to tolerably accurate eigenvalues.

Returning to the use of power-series expansion around $\varepsilon = 0$, we shall in the following chapter use this method to generalize the case $\varkappa_1 = 0$, dealt with separately by Eady and Fjörtoft, cf. Table 1. The latter obtained results valid for nongeostrophic perturbations, but the severe restriction $h = 0$ casts some doubts as to the applicability of his solution to the atmosphere. Eady took into account the observed mean static stability of the atmosphere but his results are only valid for quasigeostrophic motion. By relaxing the constraints made by these two investigators, we shall obtain results allowing both geostrophic and nongeostrophic motion and applicable to an atmosphere of constant static stability. The remaining assumption $\varkappa_1 = 0$ resolves the problem of the singularity discussed above.

This restriction will be dropped in Chapter 6 which deals with an extension of the cases studied by CHARNEY (1947) and by the author (1961). A new restriction, $|\varepsilon| < \varkappa_1$, is imposed by necessity, however, since the power-series expansion around $\varepsilon = 0$ becomes invalid beyond the singular point $\varepsilon = \varkappa_1$.

## 4. Solution for the case $\varkappa_1 = 0$

Two special solutions for this case were given in the previous chapter as cases 2 and 4 in Table 1. For vanishing vertical motion at the lower ($\varepsilon = \varepsilon_1$) and upper ($\varepsilon = \varepsilon_2$) boundaries both lead to a frequency equation that may be expressed in the form

$$c = \bar{U} \pm i \frac{\Gamma(\alpha)}{2\alpha}(U_2 - U_1), \qquad (4.1)$$

where $U_2$ and $U_1$ are the speeds of the basic flow at respectively the top and the bottom of the fluid layer, $\bar{U} = \frac{1}{2}(U_1 + U_2)$, and

$$\Gamma(\alpha) = [(\alpha - \tanh\alpha)(\coth\alpha - \alpha)]^{\frac{1}{2}}. \qquad (4.2)$$

The difference between the two cases lies in the meaning of the number $\alpha$ in (4.2); in case 2, studied by Eady, $\alpha$ is defined as

$$\alpha = \frac{Z_1}{2} \frac{\sqrt{RH_m s}}{C_i} = \frac{h}{2} \frac{U_2 - U_1}{C_i} \qquad (4.3)$$

and in case 4, dealt with by Fjörtoft

$$\alpha = \alpha_1 = \frac{1}{2} \frac{U_2 - U_1}{C_i}; \qquad (4.4)$$

hence, $$\alpha = h\alpha_1. \qquad (4.5)$$

It follows readily from (4.2) that $\Gamma(\alpha) = 0$ for $\alpha = 0$ and for $\alpha = 1.2$ which limits the range of amplifying waves to $0 < \alpha < 1.2$; hence, the waves are neutral when $\alpha \geq 1.2$. In the latter case the waves are too short to be amplified; the function $\Gamma(\alpha)$ is plotted against $\alpha$ in Fig. 2.

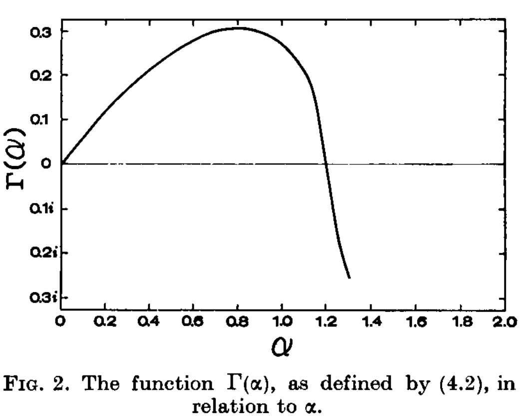


FIG. 2. The function $\Gamma(\alpha)$, as defined by (4.2), in relation to $\alpha$.

As an indication of the feasibility of using power series in solving (3.4), this method was successfully applied to the cases discussed above. The following series were obtained as solutions to case 2:

$$\left.\begin{aligned} u_1 &= a_0 + a_2\varepsilon^2 + \dots + a_{2p}\varepsilon^{2p} + \dots, \\ u_2 &= \varepsilon^3[b_0 + b_2\varepsilon^2 + \dots + a_{2q}\varepsilon^{2q}\dots], \end{aligned}\right\} \quad (4.6)$$

where

$$\left.\begin{aligned} a_2 &= -\frac{h^2}{2}a_0; \quad a_4 = \frac{h^2}{4}a_2 = -\frac{h^4}{8}a_0; \\ a_{2p} &= \frac{h^2}{2p(2p-3)}a_{2p-2}, \\ b_2 &= \frac{h^2}{10}b_0; \quad b_4 = \frac{h^2}{28}b_2 = \frac{h^4}{280}b_0; \\ b_{2q} &= \frac{h^2}{2q(2q+3)}b_{2q-2}. \end{aligned}\right\} \quad (4.7)$$

The series solution to case 4 is

$$v_2 = \varepsilon^3\left[1 + \frac{3}{5}\varepsilon^2 + \frac{3}{7}\varepsilon^4 + \dots + \frac{3}{2m+1}\varepsilon^{2m-2} + \dots\right], \quad (4.8)$$

The series $u_1$ and $u_2$ were truncated, retaining the first five terms of the former and the first four of the latter, and computed for the range $0 \leq \alpha \leq 2$. Comparison with their exact counterparts, cf. Table 1,(1) is given in Table 2. In the same table comparison between values of the infinite $v_2$-series and the sum of its first four terms is given for the range $0 \leq \alpha \leq 0.9$. Finally, Table 2 compares exact values of $c_i/(U_2 - U_1) = \Gamma(\alpha)/2\alpha$, computed by means of (4.2), with corresponding values obtained from the truncated series. The table shows that in the case $|\varepsilon| < 1$, the truncated series are quite accurate for the entire range $0 \leq \alpha \leq 2$; in the nongeostrophic case, $h = 0$, the accuracy is adequate up to at least $\alpha_1 = 0.8$ (the true value of $\Gamma(\alpha_1)/2\alpha_1$ for $\alpha_1 = 0.9$ is 0.167, while that obtained by means of the truncated $v_2$-series is 0.176). Based on this, we may anticipate a power-series solution to the more general case, dealt with below, to be sufficiently accurate for the same range of $\alpha$ (and corresponding range of $\varepsilon$).

By setting $\varkappa_1 = 0$, eq. (3.4) becomes

$$\varepsilon(1-\varepsilon^2)D'' - 2D' - h^2\varepsilon D = 0 \quad (4.9)$$

which we shall solve by substituting for $D$ the power series

$$D = a_0\varepsilon^r + a_1\varepsilon^{r+1} + \dots + a_n\varepsilon^{r+n} + \dots, \quad (4.10)$$

where $r$ must be either 0 or 3 if (4.10) is to satisfy (4.9). By comparing terms of equal power in $\varepsilon$, we find the two independent solutions

$$D_1 = a_0 + a_2\varepsilon^2 + \dots + a_{2p}\varepsilon^{2p} + \dots, \quad (4.11)$$

$$D_2 = \varepsilon^3[b_0 + b_2\varepsilon^2 + \dots + b_{2q}\varepsilon^{2q} + \dots]. \quad (4.12)$$

Here,

$$\left.\begin{aligned} a_2 &= -\frac{h^2}{2}a_0; \quad a_4 = -\left(\frac{h^2}{8} + \frac{1}{4}\right)h^2 a_0; \\ a_{2p} &= \left(\frac{h^2}{2p(2p-3)} + \frac{p-1}{p}\right)a_{2p-2} \end{aligned}\right\} \quad (4.13)$$

and

$$\left.\begin{aligned} b_2 &= \left(\frac{h^2}{10} + \frac{3}{5}\right)b_0; \quad b_4 = \left(\frac{h^2}{10} + \frac{3}{5}\right) \\ &\times \left(\frac{h^2}{28} + \frac{5}{7}\right)b_0; \\ b_{2q} &= \left(\frac{h^2}{2q(2q+3)} + \frac{2q+1}{2q+3}\right)b_{2q-2} \end{aligned}\right\} \quad (4.14)$$

It follows from the expressions for $a_{2p}$ and $b_{2q}$ that the series (4.11) and (4.12) converge absolutely for $|\varepsilon| < 1$ which we would expect, since $\varepsilon = \pm 1$ are the nearest singular points.(2) By

(1) One verifies readily that $u_1 = 0.5(D_1 + D_2)$ and $u_2 = 1.5(D_1 - D_2)$, where $D_1$ and $D_2$ are the solutions in case 2. Similarly $v_2 = -1.5D_2$, where $D_2$ is the solution in case 4.

(2) If we introduce $x = \varepsilon^2$ as a new independent variable in (4.9) this equation is transformed to

$$x(1-x)D'' - \tfrac{1}{2}(1+x)D' - \frac{h^2}{4}D = 0. \quad (4.9\text{a})$$

Moreover, by substituting $v = D\varepsilon^{-3}$ and then introducing $x = \varepsilon^2$ as a new independent variable, (4.9) takes the form

$$x(1-x)v'' + \tfrac{1}{2}(5-7x)v' - \tfrac{1}{2}\left(3 + \frac{h^2}{2}\right)v = 0. \quad (4.9\text{b})$$

Obviously (4.11) is the solution to (4.9a) and $D_2\varepsilon^{-3}$, given by (4.12), is identical with $v$. Both (4.9a) and (4.9b) are special cases of the differential equation of Gauss for the hypergeometric function

$$z(1-z)y'' + [\gamma - (\alpha+\beta+1)z]y' - \alpha\beta y = 0,$$

where $z$ is the independent variable and $y$ the dependent variable.

TABLE 2.

The columns with headings $u_1$, $u_2$, and $v_2$ give the true values, with three decimal accuracy, of the infinite series $u_1$, $u_2$, and $v_2$ defined by (4.6) and (4.8), respectively. The immediately following columns, without headings, show the values of the truncated series. Similarly the column $\Gamma(\alpha)/2\alpha$ gives the values derived from the exact solution (4.2); the corresponding values derived by means of the truncated series are given in the seventh and the last columns. $v_2$-values in parantheses correspond to $\alpha = 0.9$.

| | Case number 2 ($\lvert\varepsilon\rvert \ll 1$) | | | | | | Case number 4 ($h=0$) | | |
|---|---|---|---|---|---|---|---|---|---|
| $\alpha$ | $u_1$ | | $u_2$ | | $\Gamma(\alpha)/2\alpha$ | | $v_2$ | | |
| 0.0 | 1.000 | 1.000 | 0.000 | 0.000 | 0.289 | 0.289 | 0.000 | 0.000 | 0.289 |
| 0.2 | 0.980 | 0.980 | 0.008 | 0.008 | 0.282 | 0.283 | 0.008 | 0.008 | 0.284 |
| 0.4 | 0.917 | 0.917 | 0.065 | 0.065 | 0.264 | 0.264 | 0.071 | 0.071 | 0.268 |
| 0.6 | 0.803 | 0.803 | 0.224 | 0.224 | 0.235 | 0.235 | 0.279 | 0.278 | 0.240 |
| 0.8 | 0.627 | 0.627 | 0.546 | 0.546 | 0.194 | 0.194 | 0.897 | 0.843 | 0.198 |
| 1.0 | 0.368 | 0.368 | 1.104 | 1.104 | 0.137 | 0.137 | (1.717) | (1.417) | 0.157 |
| 1.2 | −0.001 | −0.001 | 1.990 | 1.990 | 0.013 | 0.012 | | | 0.130 |
| 1.4 | −0.515 | −0.515 | 3.321 | 3.320 | 0.133 *i* | 0.132 *i* | | | |
| 1.6 | −1.223 | −1.223 | 5.245 | 5.241 | 0.185 *i* | 0.182 *i* | | | |
| 1.8 | −2.188 | −2.188 | 7.969 | 7.941 | 0.221 *i* | 0.217 *i* | | | |
| 2.0 | −3.492 | −3.490 | 11.693 | 11.659 | 0.250 *i* | 0.243 *i* | | | |

setting $h=0$ in (4.13) and (4.14), $D_1$ is reduced to a constant and $D_2$ becomes identical to (4.8) which is absolutely convergent for $|\varepsilon|<1$. By omitting $(p-1)/p$ and $(2q+1)/(2q+3)$ in (4.13) and (4.14) respectively, $D_1$ and $D_2$ become the same as $u_1$ and $u_2$ above. With the boundary conditions $aD_1+bD_2=0$ for $\varepsilon=\varepsilon_1$, $\varepsilon_2$ ($a$ and $b$ are arbitrary non-zero constants) the frequency equation may be written

$$D_2(y_1)\,D_1(y_2)=D_2(y_2)\,D_1(y_1), \qquad (4.15)$$

where $y=h\varepsilon$ has been chosen for the sake of convenience as a new independent variable. In solving (4.15), we retain the first five terms in $D_1$ and the first four in $D_2$. Values for the corresponding coefficients, appropriate to the new independent variable, are given as Table 1 in Appendix II. Rather than solve directly for the unknown phase speed $c$, we replace $y_1$ and $y_2$ by the new variables $\alpha$ and $\gamma$ defined by

$$\alpha=\tfrac{1}{2}(y_1-y_2) \quad \gamma=\tfrac{1}{2}(y_1+y_2). \qquad (4.16)$$

It is readily verified that $\alpha$ is independent of $c$ and identical with the $\alpha$ defined by (4.3); $\gamma$ may be written as

$$\gamma=\frac{h}{C_i}(c-\bar{U})=\frac{h}{C_i}(c_r-\bar{U})+\frac{h}{C_i}ic_i. \qquad (4.17)$$

The arithmetic details of arranging terms in (4.15) will be omitted; the resulting equation is

$$A\gamma^2+B\gamma^4+C\gamma^6+D\gamma^8+E\gamma^{10}+E_1\gamma^{12}+E_2\gamma^{14}+E_3\gamma^{16}+F=0, \qquad (4.18)$$

where $A$, $B$, $C$, $D$, $E$, $F$ are all functions of $\alpha$ and of the coefficients of the truncated powers series (4.11) and (4.12). The expressions for $F$ and the four largest coefficients in (4.18) together with their numerical values are given in Appendix II; the remaining coefficients may safely be omitted. We are interested in the smallest root of (4.18) which in the first approximation is $-F/A$; successive approximations by means of Newton's method with

TABLE 3. $\Gamma_1(\alpha)/2\alpha$ *in relation to* $\alpha$ *and* $h$; "*h large*" *corresponds to the case studied by Eady* ($|\varepsilon|<1$).

| $\alpha$ | $h=1$ | $h=2$ | $h=5$ | $h=7$ | $h$ large |
|---|---|---|---|---|---|
| 0.0 | 0.2887 | 0.2887 | 0.2887 | 0.2887 | 0.2887 |
| 0.2 | 0.2765 | 0.2810 | 0.2824 | 0.2824 | 0.2823 |
| 0.4 | 0.2423 | 0.2589 | 0.2636 | 0.2640 | 0.2644 |
| 0.6 | 0.1881 | 0.2235 | 0.2231 | 0.2340 | 0.2349 |
| 0.8 | 0.1123 | 0.1744 | 0.1906 | 0.1921 | 0.1936 |
| 1.0 | 0.0801 *i* | 0.1028 | 0.1316 | 0.1341 | 0.1366 |
| 1.2 | 0.1522 *i* | 0.0907 *i* | 0.0379 *i* | 0.0274 *i* | 0.0054 *i* |
| 1.4 | 0.1981 *i* | 0.1586 *i* | 0.1384 *i* | 0.1359 *i* | 0.1333 *i* |
| 1.6 | 0.2318 *i* | 0.2002 *i* | 0.1881 *i* | 0.1865 *i* | 0.1847 *i* |
| 1.8 | 0.2527 *i* | 0.2308 *i* | 0.2237 *i* | 0.2224 *i* | 0.2213 *i* |
| 2.0 | 0.2629 *i* | 0.2552 *i* | 0.2535 | 0.2502 *i* | 0.2497 *i* |

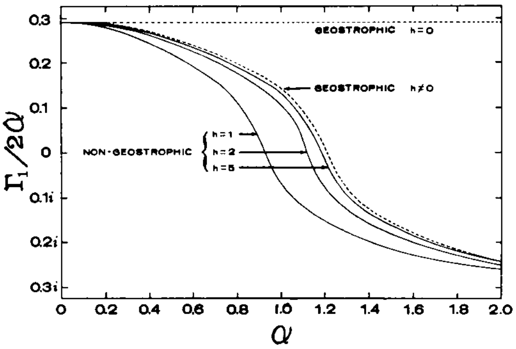


FIG. 3. $\Gamma_1/2\alpha$ in relation to $\alpha$ for $h=1$, 2, 5 in the non-geostrophic case, $\varkappa_1=0$; it shows also the corresponding functions for the geostrophic cases $h=0$ and $h\neq 0$.

$-F/A$ as the first estimate leads after a few iterations to the desired solution $\gamma^2$ which is real, since the coefficients in (4.18) are all real; $\gamma$ is therefore either real or purely imaginary. It follows from (4.17) and (4.3) that

$$c=\bar{U}+\frac{C_i}{h}\gamma=\bar{U}\pm i\frac{\Gamma_1(\alpha)}{2\alpha}(U_2-U_1), \quad (4.19)$$

where $\Gamma_1=(-\gamma^2)^{\frac{1}{2}}$. Numerical values for $\Gamma_1(\alpha)/2\alpha$ are given in Table 3. Note that $\Gamma_1(\alpha)$ corresponds to $\Gamma(\alpha)$, defined by (4.2). For increasing $h$, $\Gamma_1(\alpha)$ approaches $\Gamma(\alpha)$. Fig. 3 shows $\Gamma_1(\alpha)/2\alpha$ in relation to $\alpha$ for the range $0\leqslant\alpha\leqslant 2$ for $h=1$, 2 and 5. The two additional curves labelled "geostrophic" show $\Gamma(\alpha)/2\alpha$ and the more special case $\Gamma(\alpha)/2\alpha=1/\sqrt{12}$; the latter is the value we obtain by setting $\varkappa_1=0$ in the expression for $c$ in case number 3 of Table 1. Note that there is no short-wave cut-off.

## 5. Discussion of results from the case $\varkappa_1=0$

The solution of this case, given in the previous chapter, extends the results obtained by Eady, employing the geostrophic assumption $|\varepsilon|<1$, and those obtained by Fjörtoft, assuming $h=0$. Conditions for instability are given by Eady in terms of the parameter $\alpha$, defined by (4.3); $\alpha$ may alternatively be expressed as the ratio of the speed of internal gravity waves, $C_g'=(Z_1/\pi)\sqrt{RH_m s}$, and $C_i$ as follows:

$$\alpha=\frac{\pi C_g'}{2C_i}. \quad (5.1)$$

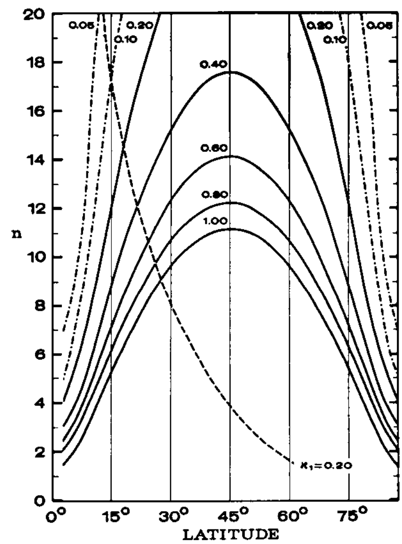


FIG. 4. Isopleths for the critical static stability, $s$ °C/100 m, corresponding to $\alpha=1.2$ and $Z_1=0.69315$ in Eady's theory. For large Richardson's numbers the figure applies to the nongeostrophic case, dealt with in Chapter 4. The vertical coordinate, $n$, is the number of waves around a latitude circle. The portion to the left of the line $\varkappa_1=0.20$ does presumably not apply.

Note that for a given thickness of the fluid layer and for $h\neq 0$, $\alpha$ is independent of the windshear $U_2-U_1$ and depends exclusively on the static stability of the basic flow and on the wave-length of the perturbation. The smaller the static stability, the wider is the range of waves that may intensify, and it would appear that the longest waves are the most likely to amplify. In reality, however, the theory does not apply to very long waves because the assumption $\varkappa_1=0$ does not admit such waves. This is particularly true for low and middle latitudes; for latitudes higher than 60° the theory is presumably valid also for waves of low wave numbers, cf. Fig. 1. The critical static stability is shown in Fig. 4 in relation to latitude and the number of waves around a latitude circle. We have chosen $Z_1=0.69315$, corresponding to a layer extending from 1000 to 500 mb; for a thicker layer the critical stability is correspondingly smaller. Note that 45° is clearly the most favored latitude for a high wave number, $n$, to occur. The thick dashed line in Fig. 4 corresponds to $\varkappa_1=0.20$; if we may assume that the theory is approximately valid for $\varkappa_1$ not exceeding this value, the portion of the

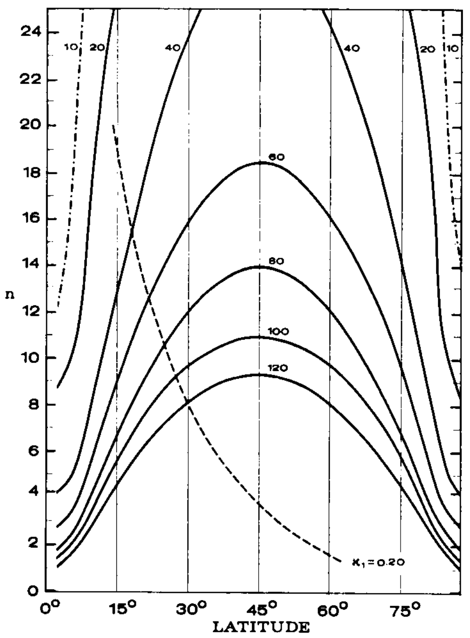


FIG. 5. Isopleths for the critical wind shear, $U_2 - U_1$ m/sec, corresponding to $\alpha_1 = 1.2$ in Fjörtoft's theory.

diagram to the right of this line applies, and the one to the left is to be disregarded as not valid.

It is of interest to examine the extent to which the results obtained in this special case are in agreement with the initial assumption $|\varepsilon| < 1$; to this end we study the maximum value $\varepsilon$ may take within the layer. It follows from (4.1) and (4.3) that

$$\varepsilon = \frac{c - U}{C_i} = \frac{\bar{U} - U}{C_i} \pm i\frac{\Gamma(\alpha)}{h}. \qquad (5.2)$$

At the lower and upper boundaries, the real part of $\varepsilon$ is numerically $\frac{1}{2}(U_2 - U_1)/C_i = \alpha^{(1)}/h$. For a given $h$, $|\varepsilon|$ will exceed unity if $\alpha > h$ and the theory has then become inconsistent with the assumption $|\varepsilon| < 1$. In the atmosphere, typical values of $h$ range from three to ten which means that the theory is presumably applicable to most of the range of developing waves, $0 < \alpha < 1.2$. It is less certain whether it can be extended to $\alpha$-values much greater than 1.2, i.e., to short stable waves; these become increasingly nongeostrophic near the upper and lower boundaries as $\alpha$ increases. The theory, of course, holds best for large $h$-values, as shown by Eady and is a surprisingly good approximation for values of $h$ as low as unity.

As $h$ approaches zero, Eady's theory breaks down and is to be replaced by the solution given by Fjörtoft. Here the parameter $\alpha_1 = \frac{1}{2}[(U_2 - U_1)/C_i] = \alpha/h$ replaces $\alpha$ in the previous case. Near the upper and lower boundaries, the real part of $\varepsilon$ is numerically equal to $\alpha_1$ and the motion of the most unstable wave, corresponding to $\alpha_1 = 0.8$, is therefore highly nongeostrophic. As $\alpha_1$ increases this becomes more pronounced, and when $\alpha_1$ reaches the cut-off value 1.2, the horizontal circulation near the boundaries is in a direction opposite to that of the geostrophic circulation. By then there is no net conversion of potential to kinetic energy and the perturbations develop no further. For values of $\alpha_1$ in the neighborhood of unity, the orbital frequency is close to the inertia frequency $f$. In physical terms this means that there is a near resonance between the free inertia oscillations and those drawing energy from the horizontal temperature gradient of the basic flow. In practice, wind shears required to give values of $\alpha_1$ approaching unity rarely occur in the troposphere. This is clear from Fig. 5 where the critical $U_2 - U_1$ values corresponding to $\alpha_1 = 1.2$ are given in relation to latitude and the wave number $n$.

In the general case, $\varkappa_1 = 0$, we found that for $h$-values in excess of 5, the theory leads to essentially the same results as derived by Eady, and that the motion is in the first approximation geostrophic for all latitudes. The modification caused by not assuming $|\varepsilon| < 1$ initially is in the direction of reducing the growth rate and lowering the maximum admissible value for the static stability. For $h \geqslant 5$ these effects are practically negligible, and the critical stability values given in Fig. 4 are therefore applicable to the nongeostrophic case. It follows from (4.19) that the growth rate $kc_i$ is

$$kc_i = k\frac{\Gamma_1(\alpha)}{2\alpha}(U_2 - U_1), \qquad (5.3)$$

where $U_2 - U_1$ is to be taken as positive. Equation (5.3) applies equally well to all latitudes and may therefore be used to discuss the possibility of the occurrence of unstable tropical disturbances. The condition $\alpha < 1.2$, and the additional requirement that $\varkappa_1$ be small,

(1) $(U_2 - U_1)/C_i$ is a thermal Rossby number.

restricts this possibility to waves corresponding to a small range of $n$. It follows from Fig. 4 that if we may assume the theory to be valid for values of $\varkappa_1$ not exceeding 0.20, then a static stability of 0.1°C per 100 meters will admit slightly unstable waves at 15° latitude, provided these waves extend vertically no further than to the 500 mb level. A static stability that small is not typical of the low latitudes, and we may therefore conclude that baroclinic perturbations are much less likely to occur at low latitudes than at middle or high latitudes. To the extent they occur in the Tropics, they are presumably shallow and grow slowly because of the small wind shear.

If we allow for release of energy through condensation in the basic flow, synoptic-scale perturbations are more likely to occur than under dry-adiabatic conditions. The basic flow within the intertropical convergence (ITC) zone is probably at times close to saturation owing to large-scale ascent caused by the zones' overall convergence or induced by upper-air large-scale disturbances. If low-level perturbations are formed within this zone, the associated ascent of an air parcel will be moist-adiabatic and the conditions may resemble those postulated by setting $h = 0$. Assuming the basic flow to be slightly baroclinic, these perturbations may persist. As a matter of fact, the tropical troposphere is statically unstable for moist-adiabatic ascent up to about 500 mb during the hurricane season (JORDAN, 1958) which accounts for widespread shower activities. As shown by HAQUE (1952), LILLY (1960), and KUO (1961), the preferred scale is in this case that of a cumulus cloud, and the release of latent heat does not lead to the amplification of synoptic scale systems in the absence of vertical wind shear. The occurrence of a slight wind shear, approximately in geostrophic balance, may considerably change this picture, since a synoptic scale system would now draw kinetic energy from a new source and not depend on the release of latent heat alone. Once such a system is formed, the organized large-scale release of latent heat may be instrumental in the further development. Because these systems are associated with small values of the Richardson number, their field of motion may be markedly nongeostrophic. This depends on the size of the vertical wind shear as compared with the speed of inertia waves; for latitudes 8° to 12°, this ratio is typically somewhere between one-half and one, and the ratio $\zeta/\delta$ therefore ranges from one-fourth to one-half.

## 6. Solution for the case $0 < \varkappa_1 < 1, |\varepsilon| < \varkappa_1$

The differential equation for this case is (3.4) which we shall solve by substituting for $D$ the power series

$$D = a_0 \varepsilon^r + \ldots + a_n \varepsilon^{r+n} + \ldots \qquad (6.1)$$

It is readily shown that $r$ must be either 0 or 2, and the solutions may therefore be written

$$D_1 = a_0 + a_1 \varepsilon + a_2 \varepsilon^2 + \ldots + \ldots a \varepsilon^p + \ldots, \qquad (6.2)$$

$$D_2 = \varepsilon^2 [b_0 + b_1 \varepsilon + b_2 \varepsilon^2 + \ldots + a_a \varepsilon^q + \ldots], \qquad (6.3)$$

where

$$a_1 = 0; \quad a_2 = -\frac{h^2}{2} a_0; \quad a_3 = 0; \quad a_4 = -\frac{h^2}{8} a_0,$$

$$b_0 = -\frac{3\varkappa_1}{2} b_1; \quad b_2 = -\frac{3\varkappa_1}{8} b_1; \quad b_3 = \left(\frac{h^2}{10} + \frac{1}{2}\right) b_1$$

and in general

$$a_{p+1} = \frac{p(p-3)}{\varkappa_1 (p+1)(p-1)} a_p + \frac{p-2}{p+1} a_{p-1} - \frac{h^2 + (p-2)(p-3)}{\varkappa_1 (p+1)(p-1)} a_{p-2}; \qquad (6.4)$$

$$b_{q+1} = \frac{(q+2)(q-1)}{\varkappa_1 (q+3)(q+1)} b_p + \frac{q}{q+3} b_{q-1} - \frac{h^2 + q(q-1)}{\varkappa_1 (q+3)(q+1)} b_{q-2} \qquad (6.5)$$

Note that for $b_1 = \frac{1}{3}$ the first two terms of (6.3) are identical with the solution to case number 3 in Table 1; for $h = 0$, (6.2) is reduced to the constant $a_0$. In deriving the characteristic equation, we apply the same boundary conditions as in the case $\varkappa_1 = 0$ and make also the substitution $y = h\varepsilon$, so eq. (4.15) applies also to the present case. Expressions for the coefficients appropriate to the new independent variable are readily obtained and are given as eqs. (1) and (2) in Appendix III, where the coefficients have been denoted by $a_p'$ and $b_q'$ to distinguish them from those given by (6.4) and (6.5). The first sixteen coefficients for each of the series (6.2) and (6.3) were computed with eight decimal accuracy for $\varkappa_1 = 0.10$, 0.15, 0.20, 0.25, 0.35, 0.50, 0.75, 1.00 and for $h = 1$, 2, 3, 5, 7, 10, 15. It is not feasible to give all the numerical values here but those corresponding to the pairs $\varkappa_1 = 0.10$, $h = 1$; $\varkappa_1 = 0.10$, $h = 10$; $\varkappa_1 = 0.50$, $h = 1$; $\varkappa_1 = 0.50$, $h = 10$ are given in Table 1 of Appendix III. It follows from the table that the convergence is quite rapid in the first five terms and becomes gradually slower for increasing $p$ and $q$. For very large $p$ and $q$,

$$a'_{p+1} \to \frac{1}{\varkappa_1 h} a'_p$$

and an analogous relation applies to $b'_{q+1}$ and $b'_q$.

In solving eq. (4.15) the first nine and the first seven terms of (6.2) and (6.3), respectively, were retained. The resulting characteristic equation is

$$A_1 \gamma + A_2 \gamma^2 + A_3 \gamma^3 + \dots A_{18} \gamma^{18} + F = 0, \qquad (6.6)$$

where $\gamma$ and $\alpha$ have the same meaning as in the case $\varkappa_1 = 0$, cf. (4.16). The coefficients $A_1$, $A_2$, ..., $F$ are functions of $a'_p$, $b'_q$, and $\alpha$: approximate expressions for $A_1$, $A_2$, $A_3$, $A_4$, $A_5$, $A_6$, and $F$ are given in Appendix III.

For $\alpha = 0$, $F$ in (6.6) is zero, and $\gamma = 0$ (which corresponds to $c = \bar{U} - \beta/k^2$) is a root; another root is approximately equal to $-A_1/A_2$ which corresponds to $c = \bar{U}$. This latter root is identical with one of the three roots for the special case $U' = 0$, $s = 0$ (see case 6 in Table 1); the first root, of course, gives the Rossby solution, which is the fourth root of the case $U' = 0$, as discussed on page 209. For $\alpha \neq 0$ we are interested in the two roots which in the first approximation are obtained by solving

$$A_1 \gamma_0 + A_2 \gamma_0^2 + F = 0, \qquad (6.7)$$

i.e.,

$$\gamma_0 = -\frac{A_1}{2A_2} \pm i \sqrt{\frac{F}{A_2} - \left(\frac{A_1}{2A_2}\right)^2} \qquad (6.8)$$

With $\gamma_0$ as a first estimate, we apply Newton's successive approximation method to (6.6), and a few iterations lead to the desired solution $\gamma$. It follows from the definition of $\gamma$, cf. (4.16) and (4.17), that

$$c_0 = \bar{U} - U_c + \frac{C_i}{h}\gamma_0 = \bar{U} - \left(1 + \frac{A_1}{2\varkappa_1 h A_2}\right)\beta/k^2$$
$$\pm i \frac{C_i}{h} \sqrt{\frac{F}{A_2} - \left(\frac{A_1}{2A_2}\right)^2}, \qquad (6.9)$$

where $c_0$ is an approximate value for $c$ corresponding to $\gamma_0$. If we compare (6.9) to (4.19), $(F/A_2)^{\frac{1}{2}}$ corresponds to $\Gamma_1(\alpha)$, whereas $A_1/2A_2$ is a new term introduced by $\varkappa_1$ being different from zero. It is readily seen that $(A_1/2hA_2)^2$ corresponds to $\frac{1}{4}\varkappa_1^2$ and $(1 - \alpha_1^2)\frac{1}{4}\varkappa_1^2$ in the respective frequency equations of cases 3 and 5 in Table 1 of Chapter 3, and that

$$\left(1 + \frac{A_1}{2\varkappa_1 h A_2}\right)\beta/k^2$$

in (6.9) is to be compared with $\frac{1}{2}\beta/k^2$ in the same equations. As it turns out, $A_1/2\varkappa_1 h A_2$ is always negative; it is practically independent of $h$ and $\varkappa_1$, equals $-0.5$ for $\alpha = 0$, and decreases numerically for increasing $\alpha$.

Let the true solution, $\gamma$, of (6.6), obtained from the estimate $\gamma_0$ by means of successive approximation, be written as

$$\gamma = a \pm ib = a \pm i \sqrt{b^2 + a^2 - a^2}. \qquad (6.10)$$

Then $a$ corresponds to $-\frac{1}{2}(A_1/A_2)$ in (6.8) and $b^2 + a^2$ corresponds to $F/A_2$. To facilitate comparison with the case $\varkappa_1 = 0$ and with the special cases given in Table 1, we set

$$-\frac{a}{\varkappa_1 h} = \tfrac{1}{2}G(\alpha) \quad \text{and} \quad b^2 + a^2 = \Gamma_2^2(\alpha)$$

and then replace (6.9) by the more accurate frequency equation

$$c = \bar{U} - (1 - \tfrac{1}{2}G(\alpha))\beta/k^2$$
$$\pm i \frac{C_i}{h} \sqrt{\Gamma_2^2(\alpha) - \frac{h^2}{4} G^2(\alpha)\varkappa_1^2}$$
$$= \bar{U} - (1 - \tfrac{1}{2}G(\alpha))\beta/k^2$$
$$\pm i \sqrt{\frac{\Gamma_2^2(\alpha)}{(2\alpha)^2}(U_2 - U_1)^2 - \frac{G^2(\alpha)}{4}(\beta/k^2)^2}. \qquad (6.11)$$

The functions of $G(\alpha)$ and $G^2(\alpha)$ are shown in Fig. 6 for $h = 10$ and for $\varkappa_1 = 0.20$; $G(\alpha)$ is essentially independent of both $h$ and $\varkappa_1$. It fol-

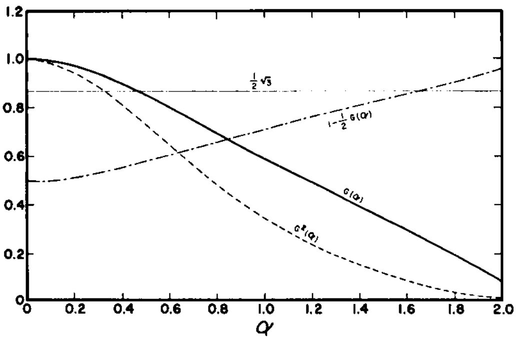


FIG. 6. The curves for $G(\alpha)$, $G^2(\alpha)$ and $1-\frac{1}{2}G(\alpha)$ corresponding to $\varkappa_1=0.20$, $h=10$. The intersection of $1-\frac{1}{2}G(\alpha)$ and the straight horizontal line $\frac{1}{2}\sqrt{3}$ gives the lowest of $\alpha$ for which (6.15) is compatible with (6.14).

lows from the figure that the speed reduction term, $-\frac{1}{2}\beta/k^2$, which appears in the geostrophic theory for neutrally stratified fluid, is an underestimate and the more so the larger $\alpha$. The function $G^2(\alpha)$, which occurs as a coefficient of $\beta/k^2$ in the radicand in (6.11) is to be compared with $(1-\alpha_1^2)$ in the case $h=0$, dealt with by Holmboe (case 5 in Table 1); the behavior of these two functions is similar in that they both decrease rapidly with increase in their respective arguments.

Unlike $G(\alpha)$ the function $\Gamma_2(\alpha)$ varies with both $\varkappa_1$ and $h$, but this variation is most notable for small values of both these parameters. For

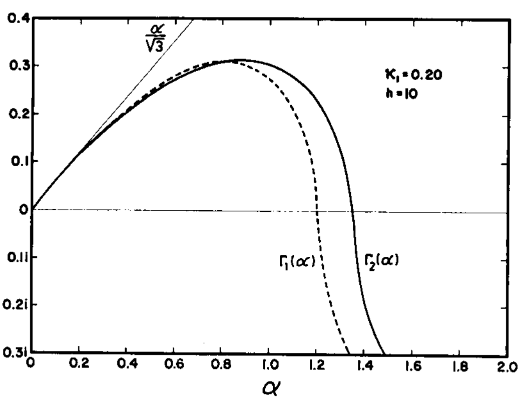


FIG. 7. The thick solid curve shows $\Gamma_2(\alpha)$ corresponding to $h=10$ and $\varkappa_1=0.20$ as a function of $\alpha$, and the dashed curve gives $\Gamma_1(\alpha)$ for $h=10$ ($\varkappa_1=0$). The thin solid straight line, $\alpha/\sqrt{3}$, is the corresponding curve for the geostrophic case $h=0$, $\varkappa_1=0$ (case 3 in Table 1).

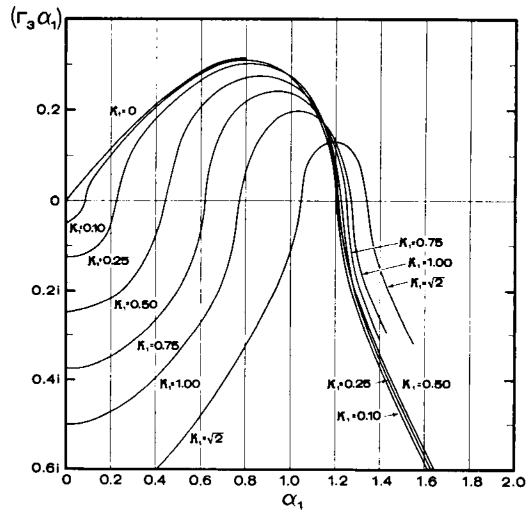


FIG. 8. $\Gamma_3=[\Gamma^2(\alpha_2)-(1-\alpha_1^2)\frac{1}{4}\varkappa_1^2]^{\frac{1}{2}}$ as a function of $\alpha_1$ (case 5 in Table 1) for various values of $\varkappa_1$

increasing values of $\varkappa_1$ and $h$, $\Gamma_2(\alpha)$ becomes insensitive to their variations and resembles very much $\Gamma_1(\alpha)$ of the case $\varkappa_1=0$; both are plotted in Fig. 7. Quite clearly the frequency equation (6.11) combines the main features of the theory for $\varkappa_1=0$ and that for the nongeostrophic case, $h=0$. The radicand, $\Gamma_3(\alpha)$, of the frequency equation for the latter case is plotted in Fig. 8 for selected values of $\varkappa_1$, and in Fig. 9 we have plotted the corresponding radicand, $\Gamma_4(\alpha)$, from (6.11) for $h=5$.

Before we can draw safe conclusions from (6.11), we must decide for which values of $\alpha$ it represents a true solution to the eigenvalue

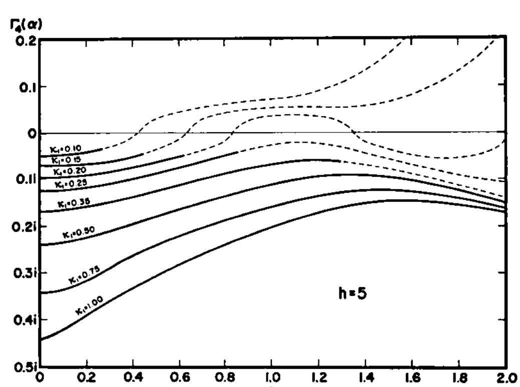


FIG. 9. $\Gamma_4(\alpha)$ corresponding to $h=5$. The dashed parts of the curves are not valid.

problem at hand. As discussed earlier, the power-series solutions (6.2) and (6.3) are valid only if the real part of $c - U$ is less, or at the most equal, to zero for all values of $U$. For $\alpha = U_2 - U_1 = 0$, $h \neq 0$ (6.11) has the two solutions $c_1 = U - \beta/k^2$ and $c_2 = U$, since $G(\alpha) = 1$ in this case. As $\alpha$ increases both roots approach the same value

$$c_{1,2} = \bar{U} - (1 - \tfrac{1}{2}G(\alpha))\beta/k^2 \qquad (6.12)$$

when the radicand in (6.11) becomes zero. At the lower boundary of the fluid, $c - U$ has its maximum value (assuming a westerly wind shear), $c - U_1$, which, when we substitute for $c$ from (6.12), gives

$$c - U_1 = \tfrac{1}{2}(U_2 - U_1) - (1 - \tfrac{1}{2}G(\alpha))\beta/k^2. \qquad (6.13)$$

Hence, the condition $c - U_1 \leqslant 0$ requires

$$U_2 - U_1 \leqslant 2(1 - \tfrac{1}{2}G(\alpha))\beta/k^2. \qquad (6.14)$$

For small $\alpha$, $1 - \frac{1}{2}G(\alpha)$ is close to 0.5 and increases for increasing $\alpha$ as shown in Fig. 6. In calculating the radicand in (6.11), we find that it becomes zero when

$$U_2 - U_1 = 1.72\,\beta/k^2. \qquad (6.15)$$

Different values for $\varkappa_1$ and $h$ give proportionality factors that deviate but slightly and irregularly from the mean value 1.72, although there is a trend toward a lower value when $\varkappa_1$ is above 0.35. This result is practically identical with that obtained in the nongeostrophic case, $h = 0$, and both lead to essentially the same result as obtained in the geostrophic case $h = 0$, i.e.,

$$U_2 - U_1 = \sqrt{3}\,\beta/k^2. \qquad (6.16)$$

Unfortunately, however, the stability criterion (6.15) is obtained by extending the series solutions (6.2) and (6.3) beyond their circle of convergence. This follows from the computed values of $\alpha$ for which the radicand in (6.11) is zero, and from the corresponding $G(\alpha)$ which taken together show that (6.14) has been violated before the radicand in (6.11) vanishes. A more detailed analysis shows that the critical $\alpha$, beyond which (6.11) becomes invalid, is

$$\alpha = \alpha_1 = [1 - \tfrac{1}{2}G(\alpha_1)]\varkappa_1 h \qquad (6.17)$$

and that the radicand of (6.11) becomes zero for $\alpha_2 = 0.86\ \varkappa_1 h$ if $\varkappa_1 h \leqslant 1.1$.[1] It follows from Fig. 6 that $1 - \frac{1}{2}G(\alpha)$ is smaller than 0.86 for all values of $\alpha$ less than 1.1 (the corresponding value of $1 - \frac{1}{2}G(\alpha)$ for $\alpha = 1.1$ is 0.74) and *consequently all admissible waves are neutral.* A question of interest is the length of these neutral waves. Quite clearly, some of these waves are neutral because of insufficient wind shear, cf. (6.16), but others appear to be neutral even for a very large wind shear. A closer study of the radicand in (6.11) reveals that sufficiently long waves are stable for values of $h$ typical of the atmosphere. This is borne out by Fig. 9 which shows that some of the growth curves have a maximum while still within the domain for which they are valid. A study of this figure and of similar graphs corresponding to other values for $h$ points to a threshold value, $e$, for the product $\varkappa_1 h$ which cannot be exceeded by unstable waves; this value appears to be in the range 1.1 to 1.3. Hence, we have the relation

$$\varkappa_1 h \leqslant e \qquad (6.18)$$

for unstable waves. By introducing the wave number $n$, (6.18) may be written

$$n \geqslant \frac{\cos^2\varphi}{e\sin\varphi}\,h. \qquad (6.19)$$

This relationship shows that for a given $h$, low wave numbers are least favored at low latitudes.

Since all the admissible waves consistent with the assumption $c_r \leqslant U_1$ are neutral, we conclude that developing waves move with a speed which is greater than the surface speed, $U_1$, of the basic flow. In the case of a very small value of $\varkappa_1$, this speed is slightly less than $\bar{U}$, cf. Chapter 4. With increasing $\varkappa_1$ the "steering level" becomes lower but is still above the ground for the critical wave $\varkappa_1 h = e$.

## 7. Stability criteria

It was shown in the previous chapter that there is an upper limit for $\alpha$, given by (6.17), beyond which the series solution (6.1) is not valid. Because of this restriction, the solution

(1) The radicand in (6.11) may have two zeros; the one under discussion is the first one. For $\alpha = \varkappa_1 h = 1.1$ the two roots coincide and there are no real roots for $\varkappa_1 h > 1.1$.

given in the last chapter is insufficient to provide general stability criteria and to deal with growth rates of developing waves.

Even though we lack a *complete* solution to the eigenvalue problem posed in Chapter 3, we shall propose general criteria that will appear plausible in view of the results of the various special cases dealt with in this paper. These results show that we may distinguish between three stability criteria. The first of these is the long-wave cut-off derived by Charney, Fjörtoft, and Kuo (cases 1 and 3 in Table 1) by means of a geostrophic theory, and by Holmboe without the geostrophic assumption (case 5). This criterion, which is a result of the variation of the Coriolis parameter with latitude, appears to be essentially independent of the geostrophic assumption. This follows from the special case $h = 0$, studied by HOLMBOE (1959), and is also borne out by the more general study made in the previous chapter. This study suggests that the form

$$U_2 - U_1 \geqslant \sqrt{3}\,\beta/k^2, \quad (7.1)$$
[1]

of case 3 of Table 1, is approximately valid for all values of $h$. Fig. 10 portrays the relationship (7.1) in which the total wind shear, $U_2 - U_1$, is plotted against latitude and the wave number, $n$.

The second criterion is the short-wave cut-off derived by EADY (case 2 in Table 1) and independently by FJORTOFT (1950) by means of energy considerations; the latter's criterion is

$$(U_2 - U_1)\left(1 - \frac{4}{\pi^2}\alpha^2\right) \geqslant \sqrt{3}\,\beta/k^2 \quad (7.3)$$

and contains both the criteria we have mentioned. It follows from the theory of Chapter 4 that, in the case $\beta = 0$, the short-wave cut-off does not depend crucially on the geostrophic assumption and according to (7.3) it is independent of $\beta$. This, as we have discussed in the introduction to this paper, is contrary to the findings of GREEN (1960) who concluded in the case of $\beta \neq 0$ that the short waves were unstable. The theory of the previous chapter is insufficient to resolve this controversy but it suggests the existence of a short-wave cut-off, cf. (6.18) and Fig. 9.

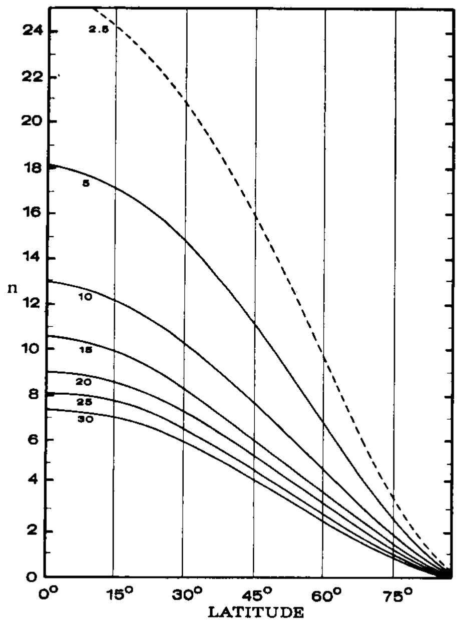


FIG. 10. Eq. (7.1) in terms of isopleths for $U_2 - U_1$ in relation to $n$ and to latitude.

The third stability criterion was first derived by FJÖRTOFT (1950) in the very special case $h = \beta = 0$ and was later extended by HOLMBOE (1959) to include an arbitrary $\beta$. This criterion was obtained by not imposing the geostrophic assumption and is therefore absent in the geostrophic versions of these two cases. When $h = \beta = 0$ the criterion is

$$U_2 - U_1 \leqslant 2.4\,C_i \quad (7.4)$$

and is only slightly altered in the case of a non-zero $\beta$ as appears from Fig. 8. Given a wavelength and a latitude, (7.4) sets an upper limit for the wind shear, while (7.1) sets a lower limit. *Thus unstable waves are confined to a certain range of wind shear.*

From these two cases it would appear that a criterion of the type (7.4) could not be obtained from quasigeostrophic theory. This is so in the special case of neutral static stability;

---

(1) The corresponding criterion derived by KUO (1952) is approximately

$$\frac{dU}{dz} \geqslant \frac{Rs\beta L}{2\pi f\left[2(RsH_m)^{\frac{1}{2}} - \dfrac{f}{2\pi}L\right]}, \quad (7.2)$$

where $dU/dz$ is the vertical wind shear. The upper boundary condition used by Kuo and Charney differs from that used in the other cases dealt with in the present paper. This presumably accounts for the differences between the criteria (7.2) on the one hand and (7.1) and (7.3) on the other.

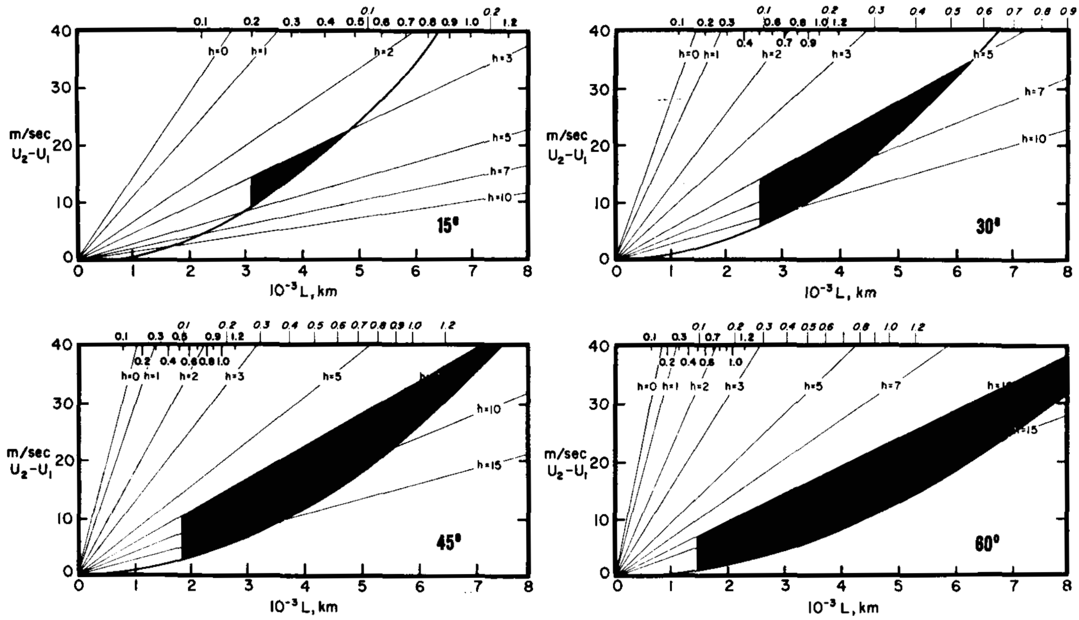

FIG. 11. Stability diagrams portraying (7.5) and (7.1) for latitudes 15°, 30°, 45°, and 60°. The shaded areas indicate an assumed range of conditions at that latitude admitting growing waves. The numbers at the top refer to the static stability, $s$, in units of °C/100 m. The bold type applies when $Z_1 = \ln(1000/500)$ and the light (slanted) type when $Z_1 = \ln(1000/200)$.

when $h$ is no longer zero, a criterion similar to (7.4) is obtained from the theory of Chapter 4 and may be shown to be inherent in Eady's theory. Referring to Fig. 8, which is based on case 5 in Table 1, the second point of intersection between each of the curves $\Gamma_3$ and the $\alpha_1$-axis represents the transition from unstable to stable waves, approximately in accordance with (7.4). The corresponding points in the theory of Chapter 4 and that of Eady are shown in Figs. 3 and 2, respectively, as the intersections between the $\alpha$-axis and the $h$-labelled growth curves. The values of $\alpha$ at the points of intersection are in the range 0.9 to 1.2, and it follows from (4.3) that the corresponding stability criterion is

$$U_2 - U_1 \leqslant \frac{2\alpha(h)}{h} C_i, \tag{7.5}$$

where $\alpha(1) = 0.9$ and $\alpha(h)$ approaches 1.2 with increasing $h$. We cannot determine here the extent to which this criterion is modified when the assumption $\beta = 0$ is relaxed beyond what has already been said above.

For the purpose of constructing plausible stability diagrams we shall, as a working hypothesis, assume that (7.5) is valid for the case of an arbitrary $\beta$. We have therefore in Fig. 11 plotted (7.1) and (7.5) in terms of $U_2 - U_1$ against the wave-length $L$ for latitudes 15°, 30°, 45°, and 60° and for a number of values of the parameter $h$. Additionally, we have indicated by means of a static stability scale the cut-off value in $L$ that corresponds to a given static stability (see Chapter 4); this has been done for two different values of $Z_1$. For the sake of illustration we have shown, by shading, areas of instability corresponding to certain lower limits for $h$ and for the static stability, $s$, appropriate for the latitude being considered. As these limits have been chosen, it is quite apparent that the lower the latitude, the smaller is the area of instability. At 15°, for example, a $s$-value of 0.2°C or more per 100 m would altogether preclude waves shorter than about 3000 km, and a static stability of 0.5°C or more per 100 m admits no developing waves if $h \geqslant 3$. Fig. 12 is similar to Fig. 11 except that the wave-length $L$ has been replaced by the number of waves, $n$, around a latitude circle. The figure shows that for Richardson's number typical of the atmosphere, wave number three or less is stable at least south of latitude 60°.

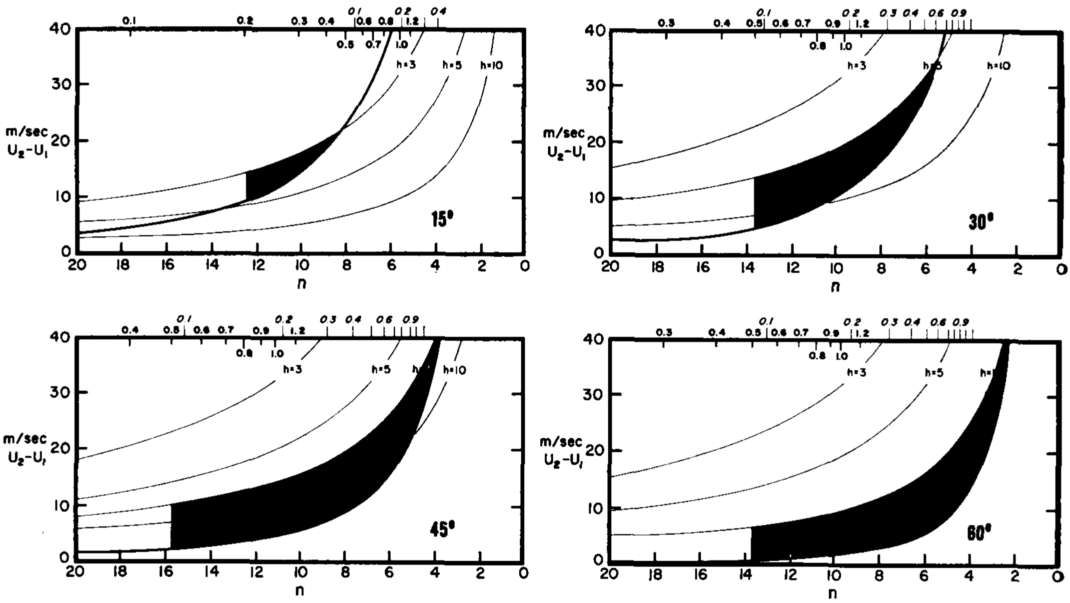


FIG. 12. The same as Fig. 11 except that the wave-length $L$ has been replaced by the wave number $n$.

The stability criterion (6.18) is derived independently of the criteria (7.1) and (7.5) and therefore useful for comparison. For this purpose we eliminate $U_2 - U_1$ between (7.1) and (7.5) and obtain

$$\varkappa_1 h \leqslant \frac{2\alpha(h)}{\sqrt{3}} \tag{7.6}$$

which is identical with (6.18) except for the right side. We noted above that $\alpha(h)$ ranges from 0.9 for $h = 1$ to 1.2 for large value of $h$, and this corresponds to a range of 1.1 to 1.4 for the right side of (7.6). This is practically the same as the values given earlier for the constant $e$ in (6.18). This criterion, in the form (6.19), is shown graphically in Fig. 13.

As briefly noted in the introduction, BURGER's (1962) mathematical analysis of the baroclinic stability properties of the model formulated by CHARNEY (1947) shows that waves of all lengths are essentially unstable. In an attempt to reconcile this result with the results obtained in this study, we observe that Burger's analysis depends crucially on a certain parameter, denoted by $a$ (see the expression following eq. (18) in his article). In the notations used in the present paper, this parameter has the definition

$$a = 1 - \tfrac{1}{2}\left[\varkappa_1 h + \frac{C_i}{(RH_m s)^{\frac{1}{2}}}\right]. \tag{7.7}$$

Solution of (7.7) with respect to $\varkappa_1 h$ gives

$$\varkappa_1 h = 2(1-a) - \frac{C_i}{(RH_m s)^{\frac{1}{2}}} \tag{7.8}$$

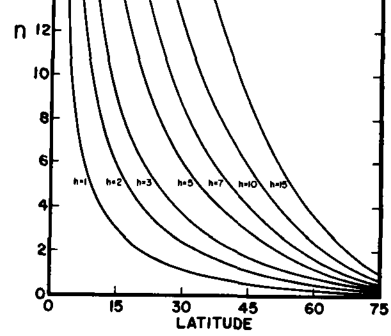


FIG. 13. Portrays the stability criterion (6.19) for $e = 1.2$.

which resembles (7.6). For values of $a$ in the range $0 < a < 1$ waves may amplify exponentially; since $\varkappa_1 h$ is a positive quantity the largest value $a$ can take is $a = a_0 = 1 - \frac{1}{2}[C_i/(RH_m s)^{\frac{1}{2}}]$ and (7.8) may be written

$$\varkappa_1 h = 2(a_0 - a). \tag{7.9}$$

It follows from both Burger's and Kuo's analyses that the waves become neutral for $a = 0$ corresponding to $\varkappa_1 h = 2a_0 < 2$. This is analogous to the criterion (6.18) of the present paper. What is surprising in Burger's analysis is that it shows waves corresponding to values of $a < 0$ to become amplified rather than remain neutral as was assumed by Charney and Kuo and found to be the case for the models studied here. As there is no doubt as to the correctness of Burger's mathematical analysis, his lack of a genuine long-wave cut-off shows the importance of the choice of the upper boundary condition. More important, perhaps, is to observe that a value of $a$ appreciably less than zero leads to a value for $\varkappa_1$ (assuming some reasonable value for $h$) corresponding to a very long wave and to which quasigeostrophic theory does not apply accurately (Burger, 1958; Árnason, 1961). We may, therefore, doubt whether values for $a$ appreciably less than zero are admissible on physical grounds. This implies that in physical terms Burger's work may not extend upon the results obtained by Charney (1947) and Kuo (1952) who both found $a = 0$ represent the transition from amplifying to neutral waves.

The results of the study on baroclinic stability reported here are limited in their applications to the atmospheric troposphere where observed conditions are often similar to those of the theoretical model. This applies in particular to incipient cyclone waves as we observe them on surface weather maps, embedded in a predominantly zonal flow whose north–south temperature gradient is moderate to strong at the ground and relatively uniform both laterally and vertically. Synoptic studies of such waves show that they are of limited vertical extent, probably not reaching much above the 500 mb level. It appears, therefore, that the boundary conditions used in this study are essentially in agreement with synoptic knowledge.

Our results do not, however, apply to internal baroclinic jets, discussed by Charney & Stern (1962). In contrast to the uniform basic flow dealt with in the previous chapters, the internal jet is by its very nature associated with large variation in both lateral and vertical shear and in static stability. Because of this, realistic upper and lower boundary condition will be quite different from those appropriate for a flow of uniform wind shear, and we would expect its stability characteristics to differ from those given in this paper. As shown by Charney & Stern, the occurrence of lateral shear, although it represents an additional source of energy, may act as a constraint on the release of potential energy through baroclinic overturning. We conclude by observing that in spite of a number of studies on stability conditions for barotropic and baroclinic flows, treated separately, our knowledge of the stability of a flow of arbitrary wind shear is as yet inadequate. The study by Charney & Stern is an outstanding contribution towards a better understanding of the stability of such a flow.

## 8. Summary and conclusions

The present study extends the framework of previous investigations of the baroclinic stability problem by neither imposing geostrophic balance on the perturbations nor unduly restricting the static stability of the basic flow. Otherwise, the framework is the same as in most other studies, i.e., a straight zonal flow, varying linearily with height, upon which small perturbations are superposed.

In mathematical terms the solution of the problem of baroclinic stability is equivalent to solving an eigenvalue problem the formulation of which is given in Chapter 3. It is shown in this chapter that the main obstacle to a general solution is the occurrence of singular points in the associated differential equation. This equation appears not to be reducible to any of the standard types except in special cases, some of which are listed in Table 1 of Chapter 3 along with the solutions. It is shown in the general case that by introducing an appropriate nondimensional parameter, $\varepsilon$, as the independent variable of the differential equation one can conveniently distinguish between quasigeostrophic motion and motion which to a varying degree is nongeostrophic.

The mathematical tool used to solve the eigenvalue problem is a power-series expansion around the regular point $\varepsilon = 0$. This leads to two cases of generalization of previous results by other investigators. Chapter 4 deals with the first of these cases which extends results of EADY (1949) and FJÖRTOFT (1950) obtained under the assumption of a Coiolis parameter independent of latitude. The expanded framework of nongeostrophic theory makes it possible to reconcile these results which within the narrower framework of geostrophic theory appeared unconnected.

The second case, which is treated in Chapter 6, extends the results obtained by CHARNEY (1947), HOLMBOE (1959), and ÁRNASON (1961). Because of the limited convergence of the power series used, sufficiently general stability criteria cannot be obtained directly. By combining the two cases of nongeostrophic solutions obtained in this paper, *plausible* general stability criteria are proposed which are applicable to all latitudes; this is the subject of Chapter 7.

The study made in the previous chapters leads to the following conclusions:

(*a*) Theories on baroclinic stability are essentially in agreement in requiring a long-wave cut-off in response to a Coriolis parameter varying with latitude, but differ in regard to the stability of relatively short waves.

(*b*) Developing waves of small amplitudes are quasigeostrophic at all latitudes if the Richardson number of the basic flow is 25 or higher. An exception to this is possibly the Equator and its immediate neighborhood, where the theory does not apply.

(*c*) Deviations from geostrophic flow are favored by

(1) A small Richardson number (i.e., a small static stability, large wind shear, or both). If the static stability is zero, developing perturbations may be highly nongeostrophic.

(2) A large value of $\varkappa_1$ (i.e., low latitude, large wave-length, or both, and

(3) Nearness to the upper and lower boundaries.

(d) For a given static stability there is a certain critical wave-length, $L_{c,1}$ such that all waves *shorter* than $L_{c,1}$ are stable; $L_{c,1}$ increases with decreasing latitude. If we replace $L_{c,1}$ by $n_{c,1}$ ($n$ is the number of waves around a latitude circle), then $n_{c,1}$ is largest for latitude 45°.

(*e*) For a given Richardson number, there is a certain critical wave-length $L_{c,2}$ such that all waves *longer* than $L_{c,2}$ are stable. $L_{c,2}$ decreases with decreasing latitude.

(*f*) Because $L_{c,1}$ increases and $L_{c,2}$ decreases with decreasing latitude, we may find a certain latitude, where $L_{c,1} = L_{c,2}$. Below this latitude waves will not amplify.

(*g*) Waves corresponding to $n \leqslant 3$ appear to be stable, under conditions typical of the troposphere, if they occur at latitudes lower than 60°.

(*h*) For our boundary conditions of two rigid horizontal planes, developing waves move with a speed which is less than the speed of the basic flow at the midlevel but greater than the speed at the lower boundary.

## Acknowledgements

Thanks are extended to Dr. Barry Saltzman and Dr. Edward A. Newburg for carefully reading the manuscript and for suggesting changes leading to improved clarity; these changes were gladly incorporated.

Thanks are also due to Mr. Ronald McDonald for help in machine programming and computations, and to Mrs. Lori M. Maston for typing the manuscript.

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## Appendix I

The perturbation quantities $u$, $v$, $\omega$, $\Phi$, and $\vartheta$, assumed independent of $y$, are expressed in terms of $x$, $p$, and $t$ as follows:

$$\left.\begin{aligned} u &= A(p)e^{ik(x-ct)}, \quad v = B(p)e^{ik(x-ct)}, \\ \Phi &= C(p)e^{ik(x-ct)}, \quad \omega = D(p)e^{ik(x-ct)}. \\ \vartheta &= E(p)e^{ik(x-ct)}, \end{aligned}\right\} \qquad (1)$$

We substitute for $u$, $v$, $\omega$, $\Phi$, and $\vartheta$ from (1) into the system (2.6), (2.7), (2.8), (2.9), (2.10), utilizing (2.11), and obtain (in the same order)

$$\Delta B + \frac{if}{k}A = 0, \qquad (2)$$

$$\Delta A - \frac{if}{k}B - C + \frac{i}{k}U'D = 0, \qquad (3)$$

$$\frac{c-U}{F}C' - \frac{i}{k}\frac{\partial\bar\vartheta}{\partial y}B - \frac{i}{k}\bar\vartheta' D = 0, \qquad (4)$$

$$C' + FE = 0, \qquad (5)$$

$$D' + ikA = 0, \qquad (6)$$

$E$ has been eliminated from (4) by means of (5); a prime means differentiation with respect to $p$, and $\Delta$ is defined by

$$\Delta = c - U + \beta/k^2 = c - U + U_c. \qquad (7)$$

Elimination of $B$ between (2) and (3) and between (2) and (4) gives respectively

$$\frac{\Delta^2 - C_i^2}{\Delta}A - C + \frac{i}{k}U'D = 0, \qquad (8)$$

$$\frac{c-U}{F}C' - \frac{C_i^2 U'}{\Delta F}A - \frac{i}{k}\bar\vartheta' D = 0, \qquad (9)$$

where $C_i = f/k$ is the speed of inertia waves. By means of (6), $A$ is eliminated from (8) and (9), and we obtain the following expressions for $C$ and $C'$ respectively:

$$C = \frac{i}{k}\left(\frac{\Delta^2 - C_i^2}{\Delta}D' + U'D\right) = -\frac{i}{k}\frac{Q_1 D' - Q_2 D}{\Delta} \qquad (10)$$

and
$$\frac{c-U}{F}C' = \frac{i}{k}\left[\frac{C_i^2 U'}{\Delta F}D' + \bar\vartheta' D\right], \qquad (11)$$

where
$$Q_1 = C_i^2 - \Delta^2 \quad Q_2 = \Delta U'. \qquad (12)$$

Elimination of $C$ between (10) and (11) gives the following second-order differential equation in $D$:

$$\frac{c-U}{F}\left[\frac{Q_1 D' - Q_2 D}{\Delta}\right]' + \frac{C_i^2 U'}{\Delta F}D' + \bar\vartheta' D = 0. \qquad (13)$$

By carrying out the differentiation of the square bracket in (13), utilizing (12), and arranging terms, we obtain

$$\Delta(c-U)(C_i^2 - \Delta^2)D'' + (c-U)C_i^2 \left(1 + \frac{\Delta}{c-U}\right)U'D' + \Delta^2[F\bar\vartheta' - (c-U)U'']D = 0. \qquad (14)$$

At this point we define the non-dimensional number

$$\varepsilon = \frac{\Delta}{C_i} \qquad (15)$$

(or, by virtue of (2), $\varepsilon = -iu/v = -i\delta/\zeta$) together with

$$\varkappa_1 = \frac{U_c}{C_i}; \quad \varkappa_2 = \frac{U'}{C_i};$$

$$\varkappa_3^2 = -\frac{1}{C_i^2}[F\bar\vartheta' - (c-U)U''] \qquad (16)$$

and introduce into (14); this givet

$$\varepsilon(\varepsilon-\varkappa_1)(1-\varepsilon^2)D''+(2\varepsilon-\varkappa_1)\varkappa_2 D'-\varkappa_3^2\varepsilon^2 D=0. \tag{17}$$

It follows from the definitions for $F$, eq. (2.3), and for the potential temperature $\bar{\vartheta}$ that

$$F\bar{\vartheta}'=\frac{R}{p^2}\left(\frac{\partial T}{\partial \ln p}-\frac{RT}{c_p}\right)=-\frac{R}{p^2}\left(\frac{RT}{c_p}+\frac{\partial T}{\partial Z}\right), \tag{18}$$

where $Z=\ln(p_0/p)$. In the atmosphere $RT/c_p+\partial T/\partial Z$ varies slowly along the vertical as compared with $p^2$ (except when the tropopause is passed through) and may for practical purposes be replaced by an appropriate mean value. It is now convenient to take $Z$ as the independent variable instead of $p$, and (17) becomes

$$P_0 D''+(P_0+P_1)D'-P_2\varepsilon^2 D=0, \tag{19}$$

where

$$\left.\begin{aligned} P_0&=\varepsilon(\varepsilon-\varkappa_1)(1-\varepsilon^2);\\ P_1&=(2\varepsilon-\varkappa_1)\varkappa_2\\ C_i^2 P_2&=R\left(\frac{RT}{c_p}+\frac{\partial T}{\partial Z}\right)+(c-U)(U'+U'')\\ &=RH_m s+(c-U)(U'+U'')\end{aligned}\right\} \tag{20}$$

and $$H_m=\frac{RT}{g};\quad s=\frac{g}{c_p}+\frac{\partial T}{\partial z};\quad z=ZH_m. \tag{21}$$

Note that a prime now means differentation with respect to $Z$, which implies that $\varkappa_2$, as defined by (16), $P_0$, $P_1$ and $P_2$ are all non-dimensional numbers.

## Appendix II

Expression for the last term, $F$, and the first four coefficients, $A$, $B$, $C$, and $D$ of (4.18), are given below.

$$\begin{aligned}A=3&+(-2a_2'+10b_2')\alpha^2+(-3a_4'+a_2'b_2'+21b_4')\alpha^4\\&+(-4a_4'b_2'+8a_2'b_4'+36b_6')\alpha^6+(-5a_6'b_2'\\&-a_4'b_4'+7a_8'+19a_2'b_6')\alpha^8+(-6a_6'b_4'-2a_8'b_2'\\&+6a_4'b_6')\alpha^{10}+(-7a_8'b_4'-3a_6'b_6')\alpha^{12}\\&-8a_8'b_6'\alpha^{14}.\end{aligned}$$

**Table 1.** *The first seven and the first six coefficients of the series* (4.11) *and* (4.12), *respectively corresponding to the independent variable* $y=h\varepsilon$.

The prime is used to distinguish these coefficients from those corresponding to $\varepsilon$ as independent variable.

| | $h$ | | | | |
|---|---|---|---|---|---|
| | 1 | 2 | 5 | 7 | 10 |
| $a_0'$ | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 |
| $-a_2'$ | 0.50000 | 0.50000 | 0.50000 | 0.50000 | 0.50000 |
| $-a_4'$ | 0.38750 | 0.18750 | 0.13500 | 0.13010 | 0.12750 |
| $-a_6'$ | 0.27084 | 0.04167 | 0.01110 | 0.00900 | 0.00793 |
| $-a_8'$ | 0.20990 | 0.00885 | 0.00061 | 0.00036 | 0.00026 |
| $-a_{10}'$ | 0.17092 | 0.00190 | 0.00003 | 0.00001 | 0.00001 |
| $-a_{12}'$ | 0.14402 | 0.00041 | 0.00000 | 0.00000 | 0.00000 |
| $b_0'$ | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 |
| $b_2'$ | 0.70000 | 0.25000 | 0.12400 | 0.11224 | 0.10600 |
| $b_4'$ | 0.52500 | 0.05357 | 0.00797 | 0.00564 | 0.00454 |
| $b_6'$ | 0.41806 | 0.01141 | 0.00040 | 0.00019 | 0.00012 |
| $b_8'$ | 0.34680 | 0.00246 | 0.00002 | 0.00001 | 0.00000 |
| $b_{10}'$ | 0.29611 | 0.00054 | 0.00000 | 0.00000 | 0.00000 |

$$\begin{aligned}B=5b_2'&+a_2'+(35b_4'-5a_2'b_2'+3a_4')\alpha^2+(-14a_2'b_4'\\&+6a_4'b_2'-6a_6'+12\,6b_6')\alpha^4+(6a_4'b_4'+10a_6'b_2'\\&-22a_8'-6a_2'b_6')\alpha^6+(-5a_8'b_2'+15a_6'b_4'\\&-29a_4'b_6')\alpha^8+(-3a_6'b_6'+21a_8'b_4')\alpha^{10}\\&+28a_8'b_6'\alpha^{12}.\end{aligned}$$

$$\begin{aligned}C=&-a_4'+3a_2'b_2'+7b_4'+(-4a_4'b_2'+8a_6'+84b_6')\alpha^2\\&+(-10a_6'b_2'+14a_8'+14a_4'a_4'-42a_2'b_6')\alpha^4\\&+(-20a_6'b_4'+20a_8'b_2'+36a_4'b_6')\alpha^6+(-35a_8'b_4'\\&+25a_6'b_6')\alpha^8-56a_8'b_6'\alpha^{10}.\end{aligned}$$

$$\begin{aligned}D=&-3a_6'+a_4'b_2'+5a_2'b_4'+9b_6'+(-11a_4'b_4'+5a_6'b_2'\\&+21a_2'b_6'+5a_8')\alpha^2+(-25a_8'b_2'+15a_6'b_4'\\&-9a_4'b_6')\alpha^4+(-45a_6'b_6'+35a_8'b_4')\alpha^6\\&+70a_8'b_6'\alpha^8.\end{aligned}$$

$$\begin{aligned}F=\alpha^2[1&+(a_2'+b_2')\alpha^2+(a_4'+a_2'b_2'+b_4')\alpha^4+(a_6'+a_4'b_2'\\&+a_2'b_4'+b_6')\alpha^6+(a_8'+a_6'b_2'+a_4'b_4'+a_2'b_6')\alpha^8\\&+(a_8'b_2'+a_6'b_4'+a_4'b_6')\alpha^{10}+(a_8'b_4'+a_6'b_6')\alpha^{12}\\&+a_8'b_6'\alpha^{14}].\end{aligned}$$

Numerical values for these quantities are given in Table 2.

TABLE 2. *Numerical values for A, B, C, D, and F of eq.* (4.18) *for selected values of h.*

| | *A* | *B* | *C* | *D* | *F* | *A* | *B* | *C* | *D* | *F* |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | $h=1$ | | | | | $h=2$ | | |
| 0.0 | 3.00000 | 3.00000 | 3.00000 | 3.00000 | 0.00000 | 3.00000 | 0.75000 | 0.18749 | 0.04690 | 0.00000 |
| 0.2 | 3.33972 | 3.85055 | 4.36764 | 2.83599 | 0.04031 | 3.14252 | 0.75286 | 0.22012 | 0.04274 | 0.03958 |
| 0.4 | 4.63909 | 7.50684 | 8.54520 | 2.40121 | 0.16396 | 3.60150 | 1.10638 | 0.31922 | 0.03106 | 0.15247 |
| 0.6 | 7.99458 | 17.42574 | 15 66486 | 1.81841 | 0.36420 | 4.47864 | 1.68760 | 0.48682 | 0.01428 | 0.31355 |
| 0.8 | 16.03191 | 39.76096 | 25.99608 | 0.80127 | 0.47670 | 5.96339 | 2.77367 | 0.72132 | −0.00382 | 0.44803 |
| 1.0 | 34.52355 | 82.27464 | 42.18409 | −3.17429 | −0.94077 | 8.34877 | 4.68215 | 1.01247 | −0.01994 | 0.34450 |
| 1.2 | 82.51776 | 142.11624 | 114.25659 | −19.67323 | −13.82010 | 12.04475 | 7.87867 | 1.34838 | −0.03611 | −0.58794 |
| 1.4 | 252.99220 | 145.79081 | 202.86343 | −75.26337 | −96.86639 | 17.65983 | 12.96156 | 1.74055 | −0.07050 | −3.99703 |
| 1.6 | 992.46027 | −266.67898 | 610.81290 | −231.51034 | −552.63204 | 26.46215 | 20.45592 | 2.32480 | −0.17732 | −14.45747 |
| 1.8 | 4145.76478 | −2368.36314 | 1837.95318 | −614.48524 | −2749.51516 | 42.18543 | 30.06758 | 3.58162 | −0.47971 | −44.53396 |
| 2.0 | 16251.97792 | −9968.39712 | 5146.22320 | −1459.90730 | −12164.69376 | 76.69649 | 38.66012 | 6.79125 | −1.22054 | −128.10560 |
| | | | $h=5$ | | | | | $h=7$ | | |
| 0.0 | 3.00000 | 0.12000 | 0.00479 | 0.00024 | 0.00000 | 3.00000 | 0.06120 | 0.00122 | 0.00001 | 0.00000 |
| 0.2 | 3.09042 | 0.12748 | 0.00526 | 0.00018 | 0.03939 | 3.08562 | 0.06476 | 0.00131 | −0.00002 | 0.03937 |
| 0.4 | 3.37167 | 0.15131 | 0.00662 | 0.00002 | 0.14960 | 3.35135 | 0.07600 | 0.00160 | −0.00010 | 0.14932 |
| 0.6 | 3.87485 | 0.19595 | 0.00875 | −0.00018 | 0.30190 | 3.82473 | 0.09676 | 0.00188 | −0.00021 | 0.30087 |
| 0.8 | 4.65563 | 0.26937 | 0.01130 | −0.00032 | 0.43080 | 4.55518 | 0.13030 | 0.00207 | −0.00029 | 0.42918 |
| 1.0 | 5.80014 | 0.38414 | 0.01350 | −0.00025 | 0.40005 | 5.61942 | 0.18163 | 0.00172 | −0.00030 | 0.40311 |
| 1.2 | 7.43376 | 0.55879 | 0.01403 | 0.00021 | 0.06153 | 7.13016 | 0.25813 | 0.00020 | −0.00014 | −0.03073 |
| 1.4 | 9.73466 | 0.81958 | 0.01084 | 0.00122 | −1.47947 | 9.24945 | 0.37011 | −0.00351 | 0.00027 | −1.34685 |
| 1.6 | 12.95542 | 1.20179 | 0.00095 | 0.00294 | −4.85304 | 12.20948 | 0.53145 | −0.01083 | 0.00100 | −4.41311 |
| 1.8 | 17.46087 | 1.74908 | −0.00919 | 0.00540 | −12.05591 | 16.34613 | 0.76009 | −0.02373 | 0.00209 | −10.78735 |
| 2.0 | 23.79696 | 2.50748 | −0.05313 | 0.00843 | −27.05472 | 22.15488 | 1.07776 | −0.04466 | 0.00355 | −23.22048 |

TABLE 1. $a_p'$ *and* $b_q'$ *for p and q ranging from 0 to 15 for selected values of* $\varkappa_1$ *and h.*

The number following the four decimal fractions denotes the power of 10 by which the fraction should be multiplied. For instance, 0.1481 +2 means $0.1481\ 10^2$ and 0.6077 −3 means $0.6077\ 10^{-3}$.

| | $\varkappa_1=0.10$ $h=1$ | $\varkappa_1=0.10$ $h=10$ | $\varkappa_1=0.50$ $h=1$ | $\varkappa_1=0.50$ $h=10$ | | $\varkappa_1=0.10$ $h=1$ | $\varkappa_1=0.10$ $h=10$ | $\varkappa_1=0.50$ $h=1$ | $\varkappa_1=0.50$ $h=10$ |
|---|---|---|---|---|---|---|---|---|---|
| $a_0'$ | 1.0000 | 1.0000 | 1.0000 | 1.0000 | $b_0'$ | −0.1500 | −0.1440 +1 | −0.7500 | −0.7500 + 1 |
| $a_1'$ | 0.0000 | 0.0000 | 0.0000 | 0.0000 | $b_1'$ | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| $a_2'$ | −0.5000 | −0.5000 | −0.5000 | −0.5000 | $b_2'$ | −0.3750 −1 | −0.3750 −2 | −0.1875 | −0.1875 − 1 |
| $a_3'$ | 0.0000 | 0.0000 | 0.0000 | 0.0000 | $b_3'$ | 0.6000 | 0.1050 | 0.6000 | 0.1050 |
| $a_4'$ | −0.1250 | −0.1250 −2 | −0.1250 | −0.1250 −2 | $b_4'$ | −0.4354 | −0.4354 −3 | −0.1771 | −0.1771 − 3 |
| $a_5'$ | 0.6667 | 0.3367 −1 | 0.1333 | 0.6733 −2 | $b_5'$ | −0.1757 +1 | 0.4961 −3 | 0.3000 | 0.7018 − 3 |
| $a_6'$ | 0.2715 +1 | 0.1402 −1 | 0.4861 −1 | 0.5549 −3 | $b_6'$ | −0.1315 +2 | −0.2338 +2 | −0.2857 | −0.4442 − 3 |
| $a_7'$ | 0.1481 +2 | 0.7443 −2 | 0.2190 | 0.1035 −3 | $b_7'$ | −0.8250 +2 | −0.1472 −2 | 0.1151 −1 | −0.5100 − 4 |
| $a_8'$ | 0.8517 +2 | 0.3588 −2 | 0.1693 | −0.1812 −4 | $b_8'$ | −0.5567 +3 | −0.1019 −2 | −0.5069 | −0.1249 − 4 |
| $a_9'$ | 0.5373 +3 | 0.2038 −2 | 0.3131 | −0.3900 −5 | $b_9'$ | −0.3920 +4 | −0.6944 −3 | −0.3796 | −0.7366 − 6 |
| $a_{10}'$ | 0.3607 +4 | 0.1267 −2 | 0.3058 | −0.1021 −5 | $b_{10}'$ | −0.2866 +5 | −0.4958 +3 | −0.9509 | −0.5548 − 7 |
| $a_{11}'$ | 0.2540 +5 | 0.8555 −3 | 0.4652 | −0.1156 −6 | $b_{11}'$ | −0.2160 +6 | −0.3662 −3 | −0.1083 +1 | 0.1913 − 7 |
| $a_{12}'$ | 0.1857 +6 | 0.6077 −3 | 0.5306 | −0.1344 −7 | $b_{12}'$ | −0.1668 +7 | −0.2786 −3 | −0.1922 +1 | 0.4367 − 8 |
| $a_{13}'$ | 0.1399 +7 | 0.4487 −3 | 0.7702 | −0.2063 −9 | $b_{13}'$ | −0.1315 +8 | −0.2171 −3 | −0.2605 +1 | 0.9748 − 9 |
| $a_{14}'$ | 0.1081 +8 | 0.3413 −3 | 0.9942 | 0.1516 −9 | $b_{14}'$ | −0.1055 +9 | −0.1725 −3 | −0.4229 +1 | 0.1484 − 9 |
| $a_{15}'$ | 0.8518 +8 | 0.2659 −3 | 0.1463 +1 | 0.5427 −10 | $b_{15}'$ | −0.8594 +9 | −0.1394 −3 | −0.6286 +1 | 0.2259 −10 |

## Appendix III

When $y = h\varepsilon$ is introduced as a new independent variable in equations (6.2) and (6.3), the new sets of coefficients are

$$a_0' = 1; \quad a_1' = 0; \quad a_2' = -0.5;$$

$$a_3' = 0; \quad a_4' = -\frac{1}{8h^2};$$

$$b_0' = -\frac{3\varkappa_1 h}{2}; \quad b_1' = 1;$$

$$b_2' = -\frac{3\varkappa_1}{8h}; \quad b_3' = \left(\frac{1}{10} + \frac{1}{2h^2}\right)$$

and in general

$$a_{p+1}' = \frac{p(p-3)}{\varkappa_1 h(p+1)(p-1)} a_p' + \frac{p-2}{h^2(p+1)} a_{p-1}' - \frac{h^2 + (p-2)(p-3)}{\varkappa_1 (p+1)(p-1)h^3} a_{p-2}'; \qquad (1)$$

$$b_{q+1}' = \frac{(q+2)(q-1)}{\varkappa_1 h(q+3)(q+1)} b_q' + \frac{q}{h^2(q+3)} b_{q-1}' - \frac{h^2 + q(q-1)}{\varkappa_1 (q+3)(q+1)h^3} b_q' - 2. \qquad (2)$$

We have primed the new coefficients to distinguish them from those corresponding to the independent variable $\varepsilon$; moreover, $a_0$ and $b_1$ in the series (6.2) and (6.3) have been set equal to unity. The values of the first 16 coefficients in each series corresponding to selected values for $\varkappa_1$ and $h$ are given in Table 1. Approximate expressions for $A_1$, $A_2$, $A_3$, $A_4$, $A_5$, $A_6$, and $F$ of eq. (6.6) are given below.

$$A_1 = 2b_0' + 4b_2'\alpha^2 - (b_2' - 6b_4' + 2b_0'a_4')\alpha^4 - (2b_4' - 2a_5' + 4b_0'a_6' - 8b_6')\alpha^6.$$

$$A_2 = 3 + (1 + 10b_3')\alpha^2 + (21b_5' - 3a_4' - 0.5b_3' - b_0'a_5')\alpha^4.$$

$$A_3 = 4b_2' + (2b_2' + 20b_4' + 4a_4'b_0')\alpha^4.$$

$$A_4 = -0.5 + 5b_3' + (35b_5' + 2.5b_3' + 3a_4' - 5b_0'a_5')\alpha^2.$$

$$A_5 = -b_2' + 6b_4' - 2a_4'b_0' + (6a_5' + 56b_6' + 2b_4' + 4a_6'b_0')\alpha^2.$$

$$A_6 = -1.5b_3' + 7b_5' - a_4' - 3a_5'b_0' + (8a_6' - 4a_4'b_3' + 4a_5'b_2')\alpha^2.$$

$$F = \alpha^2[1 - (0.5 - b_3')\alpha^2 - (0.5b_3' - b_5' - a_4' - a_6' - a_5'b_0')\alpha^4 - (0.5b_5' - a_4'b_3' + a_5'b_2' + b_0'a_7')\alpha^6 + (a_8' - b_2'a_7' + a_6'b_3' - a_5'b_4' + a_4'b_5')\alpha^8].$$