Objective analysis of the vorticity field within a region of no data

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ABSTRACT

Objective methods for estimating meteorological fields within so-called "data-holes" are compared. In particular, if Ψ is the streamfunction, a solution of the biharmonic equation $\nabla^4\Psi = 0$ subject to specified values of Ψ and $\partial\Psi/\partial n$ around the boundary of the hole is shown to provide a field whose mean square wind velocity error is comparable to that obtained by RICHARDSON (1961) for a hole 1200 km square using THOMPSON's (1961) technique of sequential Lagrangian advection of known vorticity into the hole from outside. A scheme is suggested, combining certain advantages of both methods, which involves the Lagrangian advection of vorticity and periodic adjustment of the vorticity field to conform with the known changes in Ψ and $\partial\Psi/\partial n$ around the boundary.

1. Introduction

The problem of how best to estimate the vorticity and height fields in a region of either no data or very sparse data when surrounded by areas of very good data coverage has been discussed in a group of three papers by THOMPSON, RICHARDSON & SMITH (1961). Studying barotropic forecasts in which absolute vorticity is conserved, Thompson showed that one way of estimating these fields was to advect good data from outside the region (or "hole"). The method used the known wind field outside and as good a wind field inside as could be deduced from the implied field of vorticity within the hole. In doing this the vorticity in the hole was initially assumed constant and equal to the value implied by the known circulation round the boundary, but as time went on more and more known vorticity was advected into the region and this was used to improve the estimate of the wind velocity field. By assuming the wind field to be non-divergent the velocity $\mathbf{V}$ and the vorticity ζ could be related through a streamfunction Ψ:

$$\mathbf{V} = \mathbf{k} \wedge \nabla\Psi, \quad \zeta = \nabla^2\Psi.$$

It was not immediately clear whether the error in Ψ within the hole could be significantly reduced as time went on by this scheme because the advection velocity itself is only an approximate field and the vorticity may be advected to the wrong position. Cyclogenesis will introduce further changes in ζ which cannot be envisaged by our simple barotropic scheme and which will prevent the error in Ψ falling to zero. Both these effects certainly play a part in delaying the improvement and the cyclogenesis is responsible for establishing a level of error below which one cannot hope to improve. However, Richardson showed in a series of numerical experiments that a significant improvement was achieved over the initial guess and therefore that Lagrangian advection of vorticity into the hole was very much worthwhile.

If Ψ = actual streamfunction in the hole A
Ψ_a = estimated streamfunction in the hole A
$\varepsilon = \Psi_a - \Psi$, the error in Ψ_a
$\eta = \zeta + f$ = the absolute vorticity
$S = d\eta/dt$ = the Lagrangian rate of change of η, due to cyclogenesis

$E = 1/A \int_A \nabla\varepsilon \cdot \nabla\varepsilon dA$, the mean square vector wind error then it was shown that if $\nabla\varepsilon$ refers to its value on the interior side of C:

$$\frac{\partial E}{\partial t} = -\frac{1}{A}\oint_C |V_n| \nabla\varepsilon \cdot \nabla\varepsilon ds + \frac{2}{A}\int_A \varepsilon S dA, \quad (1)$$

where C is the boundary of A, V_n is the normal velocity across C and s is a distance along C. A term has been omitted whose average value is zero and which, on the results of Richardson's "Test" experiments, appears always to be small. The equation states that the mean square vector wind error in A will decrease from its initial value to a new value at which the cyclogenesis term, tending to increase the error, is just balanced by the advection of good data into, and bad data out of, A. The curious thing about this result is that, in so far as we are justified in accepting the approximate equation (1), a decrease in E depends on $\nabla \varepsilon$ being non-zero on C, and yet the analysis nowhere requires this to be so. Therefore if the estimated streamfunction were to be chosen so that it was not only equal to the real streamfunction on C but had equal gradient there, then $\nabla \varepsilon$ would be zero and apparently no improvement in E could be expected (ε and S are positively correlated in general). Since this can be done whenever the external field outside A is known there would appear to be no advantage in carrying out a Lagrangian advection scheme at all if the only improvement that is sought is in the wind field. One of the most straightforward ways of finding a field which satisfies the two boundary conditions (and the way that is used below) is to solve the equation

$$\nabla^4 \Psi = 0$$

within A, subject to given values of Ψ and $\partial\Psi/\partial n$ on the boundary C. The nature of the solutions to this equation are given in later sections. Now, the fields of Ψ obtained this way are judged by only one statistical parameter, namely E. Alternative criteria can be envisaged however which may show that even if E cannot be reduced by a particular forecast scheme, improvements in other respects can still be obtained. In evidence of this it is shown in the last section that the mean square vorticity error in A will decrease if a modified Lagrangian advection is applied, implying that improved representations of the field *are* produced. However, no numerical forecasts have been made to verify this theory, and the present analysis is restricted to comparing the objective solution of the above biharmonic equation with analytic solutions, solutions of the equation $\nabla^2 \Psi = \bar{\zeta}$, the average vorticity, and the forecast solutions given by Richardson. The results show that in general the biharmonic solutions are better than the solutions of $\nabla^2 \Psi = \bar{\zeta}$ when compared with analytic and real meteorological fields, and for those situations considered by Richardson the biharmonic solution gives comparable values of E to those obtained by forecasting.

2. Solution of biharmonic equation for a circle

For comparing the solution of the biharmonic equation $\nabla^4 \Psi = 0$ with known analytic fields, it is convenient to know the method of solution for a circular region when Ψ and $\partial\Psi/\partial R$ (the derivative in the radial direction) are specified on the boundary C. We let the origin lie at the centre of the circle and use polar coordinates (r, φ). The radius of the circle is taken as R.

Then $$\nabla^2 \Psi = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}\Psi + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\Psi. \tag{2}$$

If we assume that Ψ can be represented by a Fourier series: $\Psi = \Sigma A_{mn} r^m e^{in\varphi}$, then insertion in equation (2) shows that m must be related to n by $m-2=n$ or $m=n$. The two series formed by these relations are just adequate to satisfy the known values of

$$\Psi \quad \text{and} \quad \frac{\partial \Psi}{\partial R}\left(=\left(\frac{\partial \Psi}{\partial r}\right)_{r=R}\right)$$

on the circle. Summation yields

$$\Psi(r,\theta) = \frac{(R^2-r^2)^2}{4\pi R}\int_0^{2\pi}\left\{2(R - r\cos(\theta-\varphi))\Psi - (R^2+r^2-2rR\cos(\theta-\varphi))\frac{\partial \Psi}{\partial R}\right\} \times \frac{d\varphi}{(R^2+r^2-2rR\cos(\theta-\varphi))^2}. \tag{3}$$

Writing $S^2 = R^2 + r^2 - 2rR\cos(\theta-\varphi)$

$$\Psi(r,\theta) = -\frac{(R^2-r^2)^2}{4\pi R}\int_0^{2\pi}\frac{\partial}{\partial R}\left(\frac{\Psi}{S^2}\right)d\varphi \tag{4}$$

expressing Ψ at the point (r, θ) within the circle in terms of the boundary values Ψ and $\partial\Psi/\partial R$.

The solution of the more general equation $\nabla^{2p}\Psi = 0$ (p integer) can be found in exactly the same way, and is

$$\Psi(r,\theta) = (-1)^{p-1}\frac{(R^2-r^2)^p}{2^p\pi(p-1)!}\times\int_0^{2\pi}\left(\frac{1}{R}\frac{\partial}{\partial R}\right)^{p-1}\left(\frac{\Psi}{S^2}\right)d\varphi. \tag{5}$$

TABLE 1. *Comparison of the values of Ψ obtained from the biharmonic equation $\nabla^4\Psi = 0$ and the equation $\nabla^2\Psi = \bar{\zeta}$ with the actual values.*

r	θ	Actual Ψ	Ψ from $\nabla^2\Psi=\bar{\zeta}$	Ψ from $\nabla^4\Psi=0$
0	—	2	2.0124	2.0124
$\frac{R}{4}$	0	2.4615	2.500	2.4804
	$\pi/2$	1.9753	1.9864	1.9857
	π	1.6495	1.6369	1.6796
$\frac{R}{2}$	0	3.0769	3.1238	3.1020
	$\pi/2$	1.9048	1.9110	1.9108
	π	1.3793	1.3523	1.3986
$\frac{3R}{4}$	0	3.9024	4.0347	3.8595
	$\pi/2$	1.7975	1.7961	1.7994
	π	1.1679	1.1409	1.1679

An alternative derivation of (3) is given by R. VON MISES (1925).

In applying these results to known analytic fields in order to test how well such solutions are able to fit a given situation, we may compare the error field with that obtained by fitting a solution of either

$$\nabla^2\Psi = 0 \tag{6}$$

or

$$\nabla^2\Psi = \bar{\zeta},$$

where $\bar{\zeta}$ is the average vorticity obtained from the circulation round C. The latter will clearly give a better solution than the harmonic solution, and can be written in terms of the two boundary conditions Ψ and $\partial\Psi/\partial R$ on C.

$$\Psi = \frac{R^2-r^2}{2\pi}\int_0^{2\pi}\left\{\frac{\Psi}{S^2}-\frac{1}{2R}\frac{\partial\Psi}{\partial R}\right\}d\varphi. \tag{7}$$

An example of a vortex set in an otherwise uniform stream was taken. The streamfunction for this situation was taken as:

$$\Psi = U\left\{y+\frac{10}{1+x^2+y^2}\right\} \tag{8}$$

and the circle had its centre at $(-2,0)$ and had radius $R=1$. In polar coordinates, relative to the centre of the circle, the value of Ψ on C is

$$\Psi = U\left\{\sin\varphi+\frac{S}{3-2\cos\varphi}\right\}. \tag{9}$$

Values of Ψ derived from (3), (7) were compared with the actual values obtained from (8) for various values of x and y within A and it was clear that whilst both solutions gave reasonably good approximations the biharmonic solution was generally the better of the two (see Table 1).

3. Finite difference solutions of the biharmonic equation

In Cartesian coordinates

$$\nabla^4 \equiv \frac{\partial^4}{\partial x^4}+\frac{\partial^4}{\partial y^4}+2\frac{\partial^4}{\partial x^2\partial y^2}.$$

In terms of finite differences when Ψ is given at the vertices of a square grid of spacing Δ and is denoted by Ψ_n^m (where the coordinates of the grid-point are $x=n\Delta, y=m\Delta$)

$$[\nabla^4\Psi]_n^m = \frac{1}{\Delta^4}\Big\{(\Psi_{n+2}^m+\Psi_{n-2}^m+\Psi_n^{m+2}+\Psi_n^{m-2}) + 2(\Psi_{n+1}^{m+1}+\Psi_{n+1}^{m-1}+\Psi_{n-1}^{m+1}+\Psi_{n-1}^{m-1}) - 8(\Psi_{n+1}^m+\Psi_{n-1}^m+\Psi_n^{m+1}+\Psi_n^{m-1}) + 20\,\Psi_n^m\Big\}. \tag{10}$$

Symbolically then the biharmonic equation $\nabla^4\Psi=0$ can be represented by the block (see KUNZ, 1957).

$$\begin{bmatrix} & & 1 & & \\ & 2 & -8 & 2 & \\ 1 & -8 & 20 & -8 & 1 \\ & 2 & -8 & 2 & \\ & & 1 & & \end{bmatrix}\Psi = 0. \tag{11}$$

For a rectangular "hole" the solution of (11) would clearly require knowing Ψ on two rings of grid-points immediately outside the hole, and this is consistent with the requirements of the continuous solution in which Ψ and $\partial\Psi/\partial n$ are specified on C.

The most obvious way of solving for Ψ in the rectangle is to apply relaxation methods, and apart from possible difficulties connected with convergence of the solution, no drawbacks to this approach are apparent. To investigate the convergence a test field was constructed based on the analytical solution of the biharmonic equation $(x^2+y^2) \log (x^2+y^2)$ over a square 7 grid-points by 7. The bottom left corner of the square was taken at (0,1), and the "hole" had to be restricted to a 3×3 square centred at (3, 4) so that sufficient boundary values would be available. A simple iteration scheme of solution was employed in which at each point the new value of Ψ was calculated entirely from the old values. The convergence was fairly reasonable and steady so that after about 20 iterations the average error was 1% of the average value of Ψ over the hole.

TABLE 2. *Comparison of biharmonic solution and solution of $\nabla^2\varphi = a$ with actual 500 mb contour height fields.*

(a bears the same relation to φ as $\bar{\zeta}$ bears to Ψ). e_4 is the root-mean-square error of the biharmonic solution, e_2 that of $\nabla^2\varphi = a$, and $\Delta\varphi$ is the maximum range of φ across the inner boundary ring of φ.

Date	$e_4/\Delta\varphi$	e_4/e_2
13. 12. 60.	0.1334	0.750
13. 12. 60.	0.0754	0.766
15. 12. 60.	0.0516	0.850
15. 12. 60.	0.0540	0.687
6. 1. 61.	0.0424	0.537
17. 1. 61.	0.0908	0.971
17. 1. 61.	0.0499	0.823
24. 1. 61.	0.1871	0.867
24. 1. 61.	0.0840	0.554
2. 3. 61.	0.0701	0.631
10. 3. 61.	0.1042	1.020
10. 3. 61.	0.1007	0.870
23. 3. 61.	0.1299	0.933
12. 4. 61.	0.0542	0.651
Average	0.0877	0.78

However, for square holes in which a block of 3×3 grid-points are deemed adequate a faster method is available for solving the equation which does not involve iteration. Blocks similar to (11) can be constructed from which the required solutions can be directly evaluated from the boundary values.

For the equation

$$\nabla^2 \Psi = \bar{\zeta}, \qquad (12)$$

$\bar{\zeta}$ is given by $\frac{1}{25}\Sigma\Delta\Psi$ where $\Delta\Psi$ = (outer value–inner value) of the two boundary rings; the extreme corner values of the 7×7 block are not involved and the corner values of the inner (6×6) ring are included twice. The solution of (12) is then given by the following three blocks, which are to be intepreted in the same way as equation (11) is by (10). The unknown values of Ψ, to be determined by these equations, lie within the central square, which is our "hole".

$$\left.\begin{array}{ll}
\left[\begin{array}{c|ccc|c}
 & 1 & 2 & 1 & \\ \hline
1 & 0 & 0 & 0 & 1 \\
2 & 0 & -16 & 0 & 2 \\
1 & 0 & 0 & 0 & 1 \\ \hline
 & 1 & 2 & 1 &
\end{array}\right] & \Psi = 18\bar{\zeta}, \text{ giving the centre value of the hole} \\
\left[\begin{array}{c|ccc|c}
 & 67 & 22 & 7 & \\ \hline
67 & -224 & 0 & 0 & 7 \\
22 & 0 & 0 & 0 & 6 \\
7 & 0 & 0 & 0 & 3 \\ \hline
 & 7 & 6 & 3 &
\end{array}\right] & \Psi = 154\bar{\zeta}, \text{ giving the corner values of the hole} \\
\left[\begin{array}{c|ccc|c}
 & 11 & 37 & 11 & \\ \hline
11 & 0 & -112 & 0 & 11 \\
7 & 0 & 0 & 0 & 7 \\
3 & 0 & 0 & 0 & 3 \\ \hline
 & 3 & 5 & 3 &
\end{array}\right] & \Psi = 98\bar{\zeta}, \text{ giving the remaining four values}
\end{array}\right\} \qquad (13)$$

Thus the first equation reads

$$16\Psi_n^m = -18\bar{\zeta} + \{\Psi_{n-1}^{m-2} + \Psi_{n+1}^{m-2} + \Psi_{n+2}^{m-1} + \Psi_{n+2}^{m+1} + \Psi_{n-1}^{m+2} + \Psi_{n+1}^{m+2}\} + 2\{\Psi_n^{m-2} + \Psi_{n-2}^m + \Psi_{n+2}^m + \Psi_n^{m+2}\}$$

and similarly for the other two. The blocks can be rotated to give the configurations for determining values at grid-points other than those shown.

The corresponding blocks for the biharmonic equation involve two rings of values round the square; the centre value of the hole is found first and is then used to find the other values. This is not essential but is useful in keeping the magnitudes of the numbers down to a minimum. The three blocks are

$$\left.\begin{array}{l}
\left[\begin{array}{cc|ccc|cc}
 & & 13 & 24 & 13 & & \\
 & 26 & -32 & -91 & -32 & 26 & \\
\cline{3-5}
13 & -32 & 0 & 0 & 0 & -32 & 13 \\
24 & -91 & 0 & 316 & 0 & -91 & 24 \\
13 & -32 & 0 & 0 & 0 & -32 & 13 \\
\cline{3-5}
 & 26 & -32 & -91 & -32 & 26 & \\
 & & 13 & 24 & 13 & &
\end{array}\right] \Psi = 0 \\[2ex]
\left[\begin{array}{cc|ccc|cc}
 & & 494 & 224 & 65 & & \\
 & 988 & -3280 & -611 & -56 & 130 & \\
\cline{3-5}
494 & -3280 & 6552 & 0 & 0 & -264 & 65 \\
224 & -611 & 0 & -1332 & 0 & 65 & 16 \\
65 & -56 & 0 & 0 & 0 & 48 & 0 \\
\cline{3-5}
 & 130 & -264 & 65 & 48 & 0 & \\
 & & 65 & 16 & 0 & &
\end{array}\right] \Psi = 0 \\[2ex]
\left[\begin{array}{cc|ccc|cc}
 & & 442 & 1076 & 442 & & \\
 & 884 & -1270 & -6721 & -1270 & 884 & \\
\cline{3-5}
442 & -2232 & 0 & 13{,}923 & 0 & -2232 & 442 \\
114 & 91 & 0 & -6052 & 0 & 91 & 114 \\
0 & 199 & 0 & 0 & 0 & 199 & 0 \\
\cline{3-5}
 & 0 & 56 & 351 & 56 & 0 & \\
 & & 0 & -29 & 0 & &
\end{array}\right] \Psi = 0
\end{array}\right\} \qquad (14)$$

TABLE 3. *Comparison of solutions with the real field* $\Psi = (x^2 + y^2) \log (x^2 + y^2)$

Exact centre-point value	3.4948
Error in solution at 19th iteration in relaxation method	0.0515
Error in solution of $\nabla^2 \Psi = 0$ using finite differences	-1.0190
Error in solution of $\nabla^2 \Psi = \bar{\zeta}$ using finite differences	-0.0011
Error in solution of $\nabla^4 \Psi = 0$ using finite differences	0.0005

Applying these blocks to the analytical form $\Psi = (x^2 + y^2) \log (x^2 + y^2)$ we obtain the results contained in Table 3, which apply to the centre values in the hole. With the exception of the solution of $\nabla^2 \Psi = \bar{\zeta}$ the errors are comparable, but generally larger, than at other points in the hole. For the exception the errors at the other points are about ten times larger than the one quoted here.

Although the "real" field is an analytic solution of $\nabla^4 \Psi = 0$, and therefore it is perhaps not surprising that the finite-difference biharmonic solution gives the best fit, it is noteworthy how small the biharmonic error is in this case. However, to discover which of the two solutions (the biharmonic and the solution of $\nabla^2 \Psi = \bar{\zeta}$) gives a better representation of actual meteorological fields a series of comparisons were made on 500 mb contour heights, these being more readily available than streamfunction fields. The holes were chosen 3×3 with grid length 600 km, equal to two normal grid-lengths. The analysis was programmed in autocode and the two error fields were printed out, together with their root-mean-square values and the

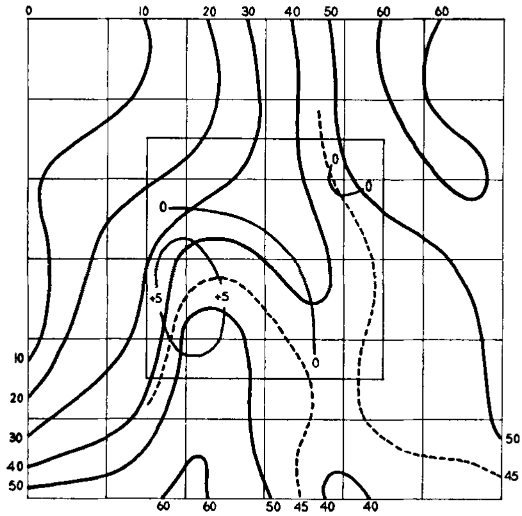

FIG. 1. The 500 mb height field taken relative to the top left-hand corner, showing the hole and the error field of the biharmonic solution.

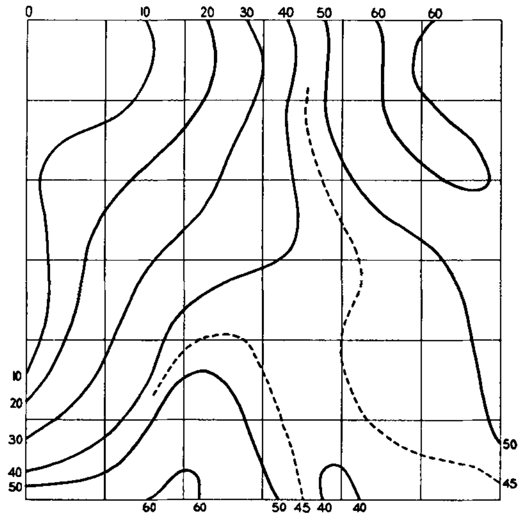

FIG. 3. The resultant biharmonic 500 mb height field taken relative to the top left-hand corner.

ratio of these two. The results for such tests are shown in Table 2. The r.m.s. error of the biharmonic solution is compared with the maximum variation in the field across the inner boundary ring of points. A typical case with error close to that of the average of the set is the second one on the 13th of December. The height and error fields are shown in Figs. 1–3; the first two show the error fields for the solution of $\nabla^4\varphi = 0$ and $\nabla^2\varphi = a$ respectively, superimposed

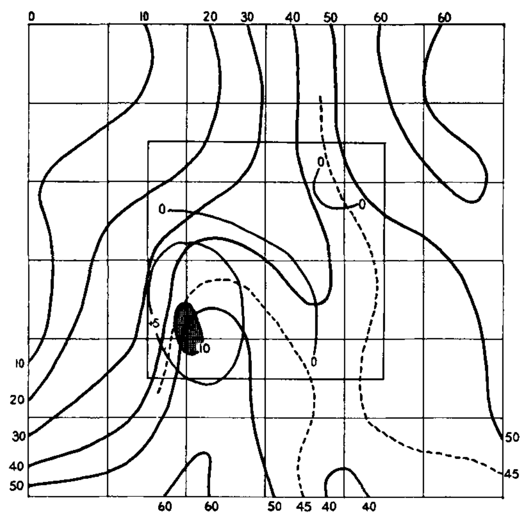

FIG. 2. As Fig. 1 but showing the error field of the solution of $V^2\varphi = a$.

upon the actual height field measured relative to the top left-hand-corner value. On this occasion the error patterns are very similar in shape but have different amplitudes; the greatest error occurring in the region of maximum anticyclonic circulation (i.e. largest negative $\nabla^2\varphi$). The solution of $\nabla^2\varphi = a$, having constant a, tends to underestimate extremes in the field which the biharmonic solution, having a greater degree of freedom, is able to represent provided they are reflected to some extent on the boundary. Fig. 3 shows the solution of the biharmonic equation.

4. Comparison with the Lagrangian-Advection forecasts

As stated in the Introduction the main purpose of this analysis was to test whether the Lagrangian advection scheme first suggested by Thompson and put into operation by Richardson gave significantly better fields than those which depended on solving a specified boundary value problem. Richardson made one forecast for a 3×3 grid-length hole in which the comparison-field was constructed by a barotropic Lagrangian advection forecast scheme over the whole region using all the data available to construct the initial Ψ-field (see Richardson's Fig. 7). In this particular case Richardson put

TABLE 4. *Comparison of the biharmonic solutions, the solutions of* $\nabla^2\Psi = \bar{\zeta}$ *and the forecast field with the streamfunction fields obtained by barotropic Lagrangian advection of vorticity without the hole (Richardson's so-called Test fields: see his Fig. 7).*

The units of E are (metres $sec^{-1})^2$.

Time (hr)	Root-mean-square error in 10^{-5} Ψ: Biharmonic	$\nabla^2\Psi = \bar{\zeta}$	Ratio	Forecast field	E, mean square wind velocity error: Biharmonic	$\nabla^2\Psi = \bar{\zeta}$	Forecast field
0	3.507	8.264	0.424	?	3.33	15.92	29.78
6	4.476	8.263	0.542	13.83	2.36	10.74	19.61
12	5.034	8.218	0.613	8.58	2.61	9.00	8.03
19	4.266	7.394	0.577	4.63	1.86	8.00	2.76
24	3.295	6.568	0.502	2.45	1.19	7.26	0.91
30	2.176	6.126	0.355	1.20	0.59	6.94	0.33
36	2.275	5.770	0.394	0.82	0.63	5.94	0.17
42	1.972	5.090	0.387	1.10	0.50	5.11	0.25
48	1.767	4.423	0.400	1.33	0.53	4.02	0.30

the initial average vorticity inside the hole zero, and then smoothed with the exterior field, in accordance with Richardson's section 3, before carrying out the first advection. The field inside the hole was then removed at each 45 minute time step and replaced by an "approximate" field obtained using Thompson's method in its form described by Richardson. The error in Ψ was defined as the difference of these two fields, and the mean square velocity error, E, as given in Table 4 as the average of all the

$$\left[\frac{1}{m}\Delta(\Psi - \Psi_a)\right]^2,$$

where Δ implies the difference of the streamfunction error at one of the internal grid-points with one of its immediate neighbours distance m away. Since the comparison-field was not complicated by the effects of cyclogenesis, the variation of E with time was believed to be attributable to the improvement in the constructed field arising from the advection of data across the boundary of the hole. That this is not the sole reason is apparent from Table 4, which compares the respective errors at the nine time steps at which data was printed out by the machine, and in which both the biharmonic solution and the solution of $\nabla^2\Psi = \bar{\zeta}$ both show decreasing errors as time increases. It would seem reasonable to deduce that the method by which the comparison fields were constructed tends to smooth out small-scale irregularities and so produce fields in the hole more analytically related to the fields around the boundary.

Again the biharmonic solution is significantly better at fitting the comparison field than the solution of $\nabla^2\Psi = \bar{\zeta}$. Initially it is also better than the forecast field but after about 18 hours the forecast error is less than the biharmonic error, although at these times both are quite small and perhaps the difference in the two solutions is not significant.

A second forecast for another occasion, but using the same 3×3 hole, was carried out by Richardson using the *observed* streamfunction field at each time step for comparison purposes (we refer to the case denoted by $3 \times 3_{12}$). This time the effects of cyclogenesis are apparent in the more erratic variations in the error fields, and in the comparatively high level below which the error does not fall. Here again the biharmonic solution is significantly better than the solution of $\nabla^2\Psi = \bar{\zeta}$ and also for the complete duration of the forecast (72 hours) better than the forecast solution (except for two times when the forecast error in Ψ is very slightly better than the biharmonic error). Comparing this case with the other examples given by Richardson it is not clear how typical this particular case is; generally E falls below its initial value and then oscillates about some level principally dependent on the size of the hole. In this respect the present example conforms, and certainly the behaviour of E shows no unusual tendencies. Consequently we can only

TABLE 5. *Comparison of the biharmonic solutions, the solutions of $\nabla^2\Psi = \bar{\zeta}$ and the forecast fields with successive observed streamfunction fields.*

The case considered corresponds to the final one quoted in Richardson's Table 1, denoted there, and in Fig. 10, by $3 \times 3_{12}$.

	Root-mean-square error in $10^{-5}\ \Psi$				E, mean square wind velocity error (metres sec^{-1})^2		
Time (hr)	Biharmonic	$\nabla^2\Psi = \bar{\zeta}$	Ratio	Forecast field	Biharmonic	$\nabla^2\Psi = \bar{\zeta}$	Forecast field
0	7.074	9.128	0.755	?	9.22	14.37	20.00
12	5.512	9.021	0.611	11.12	5.58	10.97	16.26
24	6.476	9.435	0.686	6.43	6.59	19.00	9.61
36	8.657	8.948	0.967	8.94	10.33	11.88	9.83
48	6.779	8.874	0.764	11.54	8.72	14.56	16.33
60	4.111	4.842	0.849	6.90	3.39	5.08	7.30
72	9.232	9.122	1.012	9.22	12.69	12.42	16.72

conclude that the biharmonic solution is capable of giving at least as good a representation of the streamfunction field, for this size hole, as Thompson's advection scheme. The results for this operational experiment are given in Table 5.

5. An alternative Lagrangian advection scheme

For specified $g(x,y)$ within the hole A, a solution of $\nabla^4\Psi = g(x,y)$ can be obtained satisfying given boundary conditions for Ψ and $\partial\Psi/\partial n$ on C. Therefore merely to be given these boundary conditions is insufficient to determine Ψ uniquely unless g is well defined. The scheme suggested here is one which attempts to improve the g-field at every time step by two means, firstly by modifying the old field so that the new field becomes consistent with the new boundary conditions and secondly by the advection of new data into A to replace the unknown field advected out. Basically we advance in time by assuming that vorticity is conserved in a Lagrangian sense, but it should be noted that each element is not allowed to keep its original guessed vorticity but is given improved values as more is known of the field.

To be more specific, at $t = 0$ we estimate Ψ_a in A by solving $\nabla^4\Psi_a = 0$ consistent with the boundary conditions. The vorticity $\zeta_a = \nabla^2\Psi_a$ is formed and advected by $\mathbf{V} = \mathbf{k} \wedge \nabla\Psi$ to a new position at $t = \Delta t$. The function $g(x,y,\Delta t)$ is defined as $\nabla^2\zeta_a$, as given at $t = \Delta t$, and the equation $\nabla^4\Psi = g(x,y,\Delta t)$ solved with the new known boundary conditions. The process is then repeated at each time step.

The change in E, the mean square vector wind error, with time can therefore be considered in two parts—a change at each time step when the vorticity field is modified, and a continuous change due to the flux of data across C. However, in the latter, the value of $\nabla\varepsilon$ is either zero or very small on the boundary because of the conditions Ψ satisfied at the beginning of each time step, and therefore by Thompson's theory *no improvement can be* expected due to this process, and changes in E will have to arise from the modifications in the field and from cyclogenesis. The mean square vorticity error F, on the other hand, will change as a result of all these effects. To show the effect of advection on F we will ignore the successive modifications at each time step but assume that both ε and $\nabla\varepsilon$ are zero on C. Moreover, we will assume that the error vorticity $\nabla^2\varepsilon$ can be written

$$\nabla^2\varepsilon = \mu + \lambda\varepsilon + \nabla^2 e, \tag{15}$$

where μ is some constant, ε and $\nabla^2 e$ are uncorrelated, and λ is a second constant

$$F = \frac{1}{A}\int_A (\nabla^2\varepsilon)^2 dA, \tag{16}$$

$$\frac{\partial F}{\partial t} = \frac{2}{A}\int_A \nabla^2\varepsilon \frac{\partial}{\partial t}\nabla^2\varepsilon dA. \tag{17}$$

In the advection scheme we envisage we have (as Thompson found)

$$\frac{d}{dt}\nabla^2\varepsilon = -S - \mathbf{k}\nabla\varepsilon \wedge \nabla(f + \nabla^2\Psi_a), \qquad (18)$$

then

$$\frac{1}{A}\int_A \nabla^2\varepsilon \frac{d}{dt}\nabla^2\varepsilon dA = -\frac{1}{A}\int_A S\nabla^2\varepsilon dA$$

$$-\frac{1}{A}\int_A \nabla^2\varepsilon \qquad \{\mathbf{k}\nabla\varepsilon \wedge \nabla(f + \nabla^2\Psi_a)\}dA. \qquad (19)$$

On the basis of (15) the last term in (19) reduces to a similar integral in which the $\nabla^2\varepsilon$ is replaced by $\nabla^2 e$. This must be a very small term because $\nabla^2 e$ is the uncorrelated part of $\nabla^2\varepsilon$, and it will for this reason be omitted.

Therefore

$$\frac{1}{A}\int_A \nabla^2\varepsilon \frac{d}{dt}\nabla^2\varepsilon dA = -\frac{1}{A}\int_A S\nabla^2\varepsilon dA$$

and

$$\frac{\partial F}{\partial t} = \frac{2}{A}\int_A \nabla^2\varepsilon\left(\frac{\partial}{\partial t}\nabla^2\varepsilon - \frac{d}{dt}\nabla^2\varepsilon\right)dA - \frac{2}{A}\int_A S\nabla^2\varepsilon dA. \qquad (20)$$

The first integral may be considered in two parts, a contribution from the boundary C and a contribution from the interior of A. The first of these equals

$$-\lim_{\Delta t\to 0}\frac{2}{A}\oint_C ds \lim_{\delta\to 0}\int_{s-\delta n}^{s+\delta n} \nabla^2\varepsilon(\mathbf{r}) \left[\frac{\nabla^2\varepsilon(\mathbf{r}) - \nabla^2\varepsilon(\mathbf{r}-\mathbf{v}\Delta t)}{\Delta t}\right]d\mathbf{r}$$

where we have followed Thompson in implying the use of forward-extrapolated trajectories. The notation is the same as that used by Smith and Richardson: We integrate over a vanishingly narrow strip, width 2δ, astride the boundary C. Now, since $\nabla^2\varepsilon$ is zero outside the hole, the contribution from the outflow boundary is identically zero and we are left with:

$$-\lim_{\delta\to 0}\frac{2}{A}\int_{C_i} ds \int_{s-\delta n}^{s+\delta n} \nabla^2\varepsilon(\mathbf{r}) v \frac{\partial}{\partial r}\nabla^2\varepsilon(\mathbf{r})d\mathbf{r}$$

(where v is the normal velocity, and c_i is the influx part of C)

$$= -\frac{1}{A}\int_{C_i} v(\nabla^2\varepsilon)^2 ds. \qquad (21)$$

The contribution from the interior of A equals

$$-\frac{2}{A}\int_A \nabla^2\varepsilon\mathbf{V}\cdot\nabla(\nabla^2\varepsilon)dA = -\frac{1}{A}\int_A \nabla\cdot(\nabla^2\varepsilon)^2\mathbf{V}dA$$

since $\mathbf{V}$ is non-divergent

$$= -\frac{1}{A}\oint_C v_n(\nabla^2\varepsilon)^2 ds, \qquad (22)$$

where v_n is the outward normal velocity. Over C_i, v and v_n are opposite in sign and therefore (20) can be written:

$$\frac{\partial F}{\partial t} = -\frac{1}{A}\int_{C_0} v_n(\nabla^2\varepsilon)^2 ds - \frac{2}{A}\int_A S\nabla^2\varepsilon dA. \qquad (23)$$

C_0 is the outflux portion of the boundary C.

Thus the rate of change of mean square error vorticity depends on the out-of-balance of the cyclogenetic term and the term representing the outflux of error vorticity from the hole. In this last term $\nabla^2\varepsilon$ is interpreted as the error vorticity on the immediate interior of the boundary C_0. In general we would therefore expect F to decrease from its initial value to a smaller "equilibrium" value even when ε and $\nabla\varepsilon$ are zero on the boundary and when the mean square wind velocity error would apparently remain unchanged.

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