An Advective Model of the Ocean Thermocline

By PIERRE WELANDER, Institute of Theoretical Physics, University of Stockholm, and International Meteorological Institute in Stockholm[1]

(Manuscript received November 15, 1958)

Abstract

A theoretical study is made of the density field and the associated velocity field produced in an unlimited ocean by a prescribed density distribution at the surface. It is assumed that all motions take place under geostrophic and hydrostatic balance, and that the density is simply advected by the motions occurring. The computation is carried out for a spherical earth. The theory gives a depth of penetration of the surface disturbances of the order of 1000 m, if one assumes a relative density variation of the order 10^{-3} and a characteristic velocity below the boundary layer of the order 1 cm. sec^{-1}. The depth of penetration is proportional to the sine of the latitude. Assuming a stable ocean with a surface density increasing from the equator to the pole the theory gives a meridional distribution of density of the form observed in the real oceans. The associated zonal velocities are westerlies at high latitudes, easterlies near the equator.

To permit a more precise check of the theory by laboratory experiments the corresponding solution is derived for a rotating "dishpan". The solution is found to be of the same type as the one studied in the spherical case, but it is pointed out that fundamental differences between the spherical and parabolic cases are likely to occur in more general solutions than those studied here.

1. Introduction

In the last years there have appeared some interesting papers discussing theoretically the problem of the ocean thermocline (LINEYKIN, 1955; STOMMEL and VERONIS, 1957). In these papers it was assumed that the (turbulent) diffusion of density plays an important role, and the coefficient of diffusion enters as a key parameter in the solution. The density advection terms were not considered in their complete non-linear form but entered the problem only as linearized perturbation terms. The reason for this simplification is, clearly, the mathematical difficulties in handling the non-linear effects.

It should be noticed that the importance of diffusion processes in large-scale ocean dynamics has not yet been proved. It cannot be doubted, however, that density advection plays a fundamental role so that it seems more natural to start out from a purely advective model, in which all diffusion effects are neglected. In the present paper a study of such an advective model is carried out. It appears that the model can explain the main features of the ocean density field below a boundary layer of thickness 100—200 meters. The present computation is carried out for a spherical earth. It is assumed that the motions take place under geostrophic and hydrostatic balance and that all motions disappear at sufficiently great depths. No vertical boundaries are introduced in the model. Accordingly, the model cannot predict "Gulf Streams" and similar boundary phenomena.

[1] The main part of the present study was carried out during the author's stay at the Johns Hopkins University, Civil Engineering Department. The work was spondered by the Office of Naval Research and the U.S. Weather Bureau under contracts N-onr-248(31) and CWB-9504, respectively.

Although the physical model studied here is quite simple the mathematical treatment is still hampered by the non-linearity introduced by the density advection terms. It is possible to derive a single differential equation for a certain density function, but so far it has not been possible to give the general solution of this equation. A particular solution is derived which seems to yield the essential physical results of the model.

As a next step in a systematic approach to the thermocline problem one would include density diffusion, at least in the boundary layer. If one could solve the appropriate boundary layer equations and match the solution to that for the deep water region given by the advective model, it would be possible to determine the density field in the whole ocean in terms of the primary forcing functions acting at the sea surface, such as heating and cooling, evaporation and precipitation. One would also like to investigate such factors as vertical boundaries, finite depth, wind-produced convergence in the Ekman layer etc.

To test the reality of the theoretical solution presented here one would like to set up a corresponding laboratory experiment, using a rotating "dish-pan". One advantage of the laboratory experiment would be the possibility of controlling the effects of the density diffusion. In the laboratory one could work in the regime of laminar motion where only molecular diffusion is important, and one would then avoid the kind of guess-work that enters into all estimates of turbulent diffusion effects.

To permit quantitative comparisons between the theory and such laboratory experiments the present analysis has been extended to the case where the equipotential surfaces are not spherical but paraboloidal. In this case, there occurs in the equation a new term expressing the effect of the latitudinal variation of effective gravity. In the particular solution studied in the present paper this term changes nothing essentially, but it seems likely that it can lead to new types of solutions in more general cases.

2. Formulation of the problem

Consider an ocean of incompressible fluid covering the spherical earth. The earth is assumed to have a radius R and an angular velocity Ω. The sum g of the acceleration of gravity and the centrifugal force is assumed to be constant in magnitude and directed radially. The ocean has a depth h and follows the earth in its rotation except for small motions produced by surface disturbances. To arrive at our model, which is characterized by a geostrophic, hydrostatic and advective balance, the following specific assumptions have to be made:

(a) The horizontal scale of the disturbances is much larger than the depth h.

(b) The non-linear acceleration terms are much smaller than the Coriolis acceleration.

(c) The boundary layer within which friction and diffusion play an important role has a depth much smaller than the depth h.

(d) The Coriolis acceleration is small compared to the product of gravity and the relative density variation.

If the horizontal scale of the disturbances is of the order L, if the density variations are of the order $\Delta\varrho$, if the characteristic horizontal velocities are of the order U, and if the fluid is characterized by its mean density, $\bar{\varrho}$, and the coefficients of (turbulent) kinematic viscosity and diffusion ν and $\varkappa$, then the above four conditions can be written

$$\frac{h}{L} \ll 1 \tag{1a}$$

$$\frac{U}{L\Omega} \ll 1 \tag{1b}$$

$$\frac{1}{h}\sqrt{\frac{\nu}{\Omega}} \ll 1, \quad \frac{1}{h}\sqrt{\frac{\varkappa L}{U}} \ll 1 \tag{1c}$$

$$\frac{U\Omega}{\dfrac{g\Delta\varrho}{\bar{\varrho}}} \ll 1 \tag{1d}$$

It should be noticed that our model cannot be applied very close to the equator; in this region non-linear accelerations and friction must come into play.

With the above assumptions the equations of motion, the continuity equation, and the density transport equation, valid in the interior of the fluid are

$$-2\Omega \sin\Theta \varrho v = -\frac{1}{r\cos\Theta}\frac{\partial p}{\partial \phi} \tag{2}$$

$$2\Omega \sin \Theta \varrho u = -\frac{1}{r}\frac{\partial p}{\partial \Theta} \tag{3}$$

$$0 = -\frac{\partial p}{\partial r} - g\varrho \tag{4}$$

$$\frac{1}{r\cos\Theta}\frac{\partial}{\partial\phi}(\varrho u) + \frac{1}{r\cos\Theta}\frac{\partial}{\partial\Theta}(\cos\Theta\,\varrho v) + \frac{\partial}{\partial r}(\varrho w) + \frac{2}{r}\varrho w = 0 \tag{5}$$

$$u\frac{1}{r\cos\Theta}\frac{\partial\varrho}{\partial\phi} + v\frac{1}{r}\frac{\partial\varrho}{\partial\Theta} + w\frac{\partial\varrho}{\partial r} = 0 \tag{6}$$

Here r is the radial distance, ϕ is a longitude coordinate and Θ a latitude coordinate, counted positive eastwards and northwards, respectively. u, v, and w are the corresponding velocity components, and p is the pressure.

The above equations will be simplified further by making the realistic assumption that the depth h of the ocean is much smaller than R. In this case the term $\frac{2}{r}\varrho w$ in the continuity equation can be neglected in comparison with the term $\frac{\partial}{\partial r}(\varrho w)$, and one can everywhere replace r by R and dr by dz, z being a vertical coordinate which is zero at the surface (strictly at a horizontal surface just below the boundary layer). After multiplying the density transport equation by ϱ we have then

$$-2\Omega\sin\Theta(\varrho v) = -\frac{1}{R\cos\Theta}\frac{\partial p}{\partial\phi} \tag{2'}$$

$$2\Omega\sin\Theta(\varrho u) = -\frac{1}{R}\frac{\partial p}{\partial\Theta} \tag{3'}$$

$$0 = -\frac{\partial p}{\partial z} - g\varrho \tag{4'}$$

$$\frac{1}{R\cos\Theta}\frac{\partial}{\partial\phi}(\varrho u) + \frac{1}{R\cos\Theta}\frac{\partial}{\partial\Theta}(\cos\Theta\,\varrho v) + \frac{\partial}{\partial z}(\varrho w) = 0 \tag{5'}$$

$$(\varrho u)\frac{1}{R\cos\Theta}\frac{\partial\varrho}{\partial\phi} + (\varrho v)\frac{1}{R}\frac{\partial\varrho}{\partial\Theta} + (\varrho w)\frac{\partial\varrho}{\partial z} = 0 \tag{6'}$$

Our five dependent variables are ϱu, ϱv, ϱw, ϱ and p.

The boundary conditions are

$$\varrho = \varrho_0(\phi, \Theta) \text{ at } z = 0 \tag{7}$$

$$\varrho u = \varrho v = \varrho w = 0 \text{ at } z = -\infty \tag{8}$$

where $\varrho_0(\phi, \Theta)$ is a prescribed function. The condition (8) implies that ϱ approaches a constant value, $\bar{\varrho}$, at $z = -\infty$. Since the physically interesting solutions have a stable stratification, we have $\varrho_0 \leq \bar{\varrho}$.

In the beginning the ocean was assumed to have a finite depth h so that in a strict sense the boundary condition (8) should be applied at the point $z = -h$. In the following we will, however, only deal with the case where the depth of penetration of the surface disturbances is small compared to h. One may then introduce the boundary condition in the stated form for the sake of analytical convenience.

It is to be noted that no condition is imposed on the vertical velocity at $z = 0$. In fact, a condition that $w = 0$ at this surface would mean an overspecification of our problem, when we also require that all velocities should vanish at great depths. In a complete solution there should enter also a boundary layer on top within which friction and diffusion become important. With this boundary layer all the boundary conditions can be satisfied. However, we are here satisfied to study only the deep water solution. Accordingly, $z = 0$ will not represent the real free surface of the ocean but rather a horizontal surface just below the boundary layer.

3. Derivation of the *M*-equation

From the system of equations (2')—(6') one can derive a single differential equation for the quantity

$$M(\phi, \Theta, z) = \int_{-\infty}^{z}\int_{-\infty}^{z}\varrho'(dz)^2 \tag{9}$$

where $\varrho' = \varrho - \bar{\varrho}$ is the perturbation density. It is assumed that ϱ' is given at $z = 0$ and that it decays to zero for large negative z rapidly enough to secure the convergence of the integral (9).

Eliminating the pressure in equations (2′) and (3′) by means of the hydrostatic equation (4′), and using the condition (8) one finds

$$\left.\begin{aligned} \varrho u &= \frac{g}{2\Omega R\sin\Theta}\int\limits_{-\infty}^{z}\frac{\partial\varrho}{\partial\Theta}\,dz \\ \varrho v &= -\frac{g}{2\Omega R\sin\Theta\cos\Theta}\int\limits_{-\infty}^{z}\frac{\partial\varrho}{\partial\phi}\,dz \end{aligned}\right\} \quad (10)$$

The continuity equation yields

$$\varrho w = -\frac{g}{2\Omega R^2\sin^2\Theta}\int\limits_{-\infty}^{z}\int\limits_{-\infty}^{z}\frac{\partial\varrho}{\partial\phi}(dz)^2 \quad (11)$$

Replacing $\frac{\partial\varrho}{\partial\Theta}$ by $\frac{\partial\varrho'}{\partial\Theta}$ etc., and introducing the function M one has

$$\left.\begin{aligned} \varrho u &= \frac{g}{2\Omega R\sin\Theta}M_{\Theta z} \\ \varrho v &= -\frac{g}{2\Omega R\sin\Theta\cos\Theta}M_{\phi z} \\ \varrho w &= -\frac{g}{2\Omega R^2\sin^2\Theta}M_{\phi} \end{aligned}\right\} \quad (12)$$

where $M_\Theta = \frac{\partial M}{\partial\Theta}$ etc.; furthermore

$$\left.\begin{aligned} \frac{\partial\varrho}{\partial\phi} &= M_{\phi zz} \\ \frac{\partial\varrho}{\partial\Theta} &= M_{\Theta zz} \\ \frac{\partial\varrho}{\partial z} &= M_{zzz} \end{aligned}\right\} \quad (13)$$

Introducing these expressions into the density transport equation (6′) we get the desired equation for M:

$$\boxed{M_{\phi zz}M_{\Theta z} - M_{\Theta zz}M_{\phi z} = \cot\Theta\, M_{\phi}M_{zzz}} \quad (14)$$

This equation can also be written in the form

$$J(M_{zz}, M_z) = \cot\Theta M_\phi M_{zzz} \quad (14')$$

where J denotes the Jacobian with respect to ϕ and Θ[1].

The boundary conditions for M take on the form

$$\left.\begin{aligned} M_{zz} &= \varrho_0'(\phi,\Theta) \quad \text{at } z=0 \\ M &= 0 \qquad\qquad \text{at } z=-\infty \end{aligned}\right\} \quad (15)$$

where ϱ_0' (ϕ, Θ) is the given surface distribution of the perturbation density.

4. Derivation of a particular solution

To avoid unnecessary mathematical complications we confine our attention only to the half-sphere $0 < \Theta < \frac{\pi}{2}$. Introducing a new latitude coordinate

$$\eta = \log\sin\Theta \quad (16)$$

which runs from O at the pole to $-\infty$ at the equator, equation (14) takes on the simpler form

$$M_{\phi zz}M_{\eta z} - M_{\eta zz}M_{\phi z} = M_\phi M_{zzz} \quad (17)$$

One class of solutions which satisfies equation (17) is obviously

$$M = M(\eta, z) \quad (18)$$

In this case, in fact, both sides of equation (17) vanish separately. The corresponding velocity field is purely zonal, $v = w = 0$. Since there are no vertical velocities the density at the bounday is not advected into the fluid and the solution has little physical interest. For the general case where the surface density distribution depends both on ϕ and η, one may try a solution of the form

$$M = P(\phi,\eta)\ Q(\eta, z) \quad (19)$$

[1] It should be noted that the present M-equation could also have been derived in a so-called β-plane, using rectangular »quasi-coordinates» x, y, z (cf. the derivation in the appendix). In the β-plane the equation becomes

$$M_{xzz}M_{yz} - M_{yzz}M_{xz} = \frac{\beta}{f}M_x M_{zzz}$$

Putting $dx = R\cos\Theta d\phi$, $dy = Rd\Theta$, $f = 2\Omega\sin\Theta$, $\beta = \frac{2\Omega}{R}\cos\Theta$ we get again equation (14).

It is immediately found that (19) is a solution for arbitrary functions $P(\phi, \eta)$, provided Q satisfies the equation

$$Q_{zz}Q_{\eta z} - Q_z Q_{\eta zz} = QQ_{zzz} \qquad (20)$$

Suitable solutions to (20) are found by putting

$$Q = Q\left(\frac{z}{F(\eta)}\right) \qquad (21)$$

Equation (20) changes then into the form

$$\frac{z}{F(\eta)}\,[Q'Q''' - (Q'')^2] + Q'Q'' - \frac{F(\eta)}{F'(\eta)}\,QQ''' = 0$$

and it is seen that (21) is a possible solution if we choose $\frac{F(\eta)}{F'(\eta)} = \alpha$, or

$$F(\eta) = \text{const}\; e^{\frac{\eta}{\alpha}} \qquad (22)$$

α being an arbitrary constant. Q is now determined by the ordinary differential equation

$$\zeta\,[Q'Q''' - (Q'')^2] + Q'Q'' - \alpha QQ''' = 0 \qquad (23)$$

where $\zeta = \frac{z}{F(\eta)}$ is the argument of Q.

Particular solutions of (23) of the form $Q = \text{const}\;\zeta^a$ are easily found, with possible exponent values $a=0$, $a=1$, and $a = \frac{2\alpha}{\alpha+2}$, but these solutions are not sufficiently well-behaved at $z=0$ and $z=-\infty$ to be of use. For $\alpha=1$, however, the equation has the particular solution

$$Q = \text{const}\; e^{k\zeta} \qquad (24)$$

where k is an arbitrary constant, and this solution represents a possible physical situation If k is positive Q will vanish at $z=-\infty$ and will certainly be well-behaved at $z=0$.

In the following, the discussion will be confined to the solution (24), and we will demonstrate that it is capable of describing correctly the main features of the ocean density field. Nevertheless, it would be of great interest to study more complicated solutions of the equation (23), which possibly could describe finer details in the ocean density field. There seems to be little hope to find the general solution to (23), although it actually can be reduced to a first order equation[1], but one can, of course, resort to numerical methods.

Using the particular solution (24), M takes on the form

$$M = P(\phi, \eta)\,e^{k\zeta} = P(\phi, \eta)\,e^{kze^{-\eta}} \qquad (25)$$

or, in terms of the original coordinates ϕ, Θ, z,

$$\boxed{M = M_0(\phi, \Theta)\,e^{\frac{kz}{\sin\Theta}}} \qquad (26)$$

$M_0(\phi, \Theta)$ is determined by the boundary condition at the surface, while the constant k, which is simply a scale-factor for z, is at our disposal. It can be fixed by prescribing some more physical parameter in the problem, as example the total energy or the total angular momentum of the system.

To estimate the order of magnitude of k in the real oceans we can relate it to the characteristic horizontal velocity. Using the previously derived expressions for the perturbation density and the mass-velocities one finds easily that the order of magnitude of k is given by

$$k \sim \frac{g\frac{\Delta\varrho}{\varrho}}{U\Omega L} \qquad (27)$$

where, again, L is the horizontal scale of the disturbances, U is the characteristic horizontal velocity, and $\Delta\varrho$ is the characteristic density variation. The corresponding depth of penetration of the disturbances is of the order

[1] Introducing, in succession, the following new dependent and independent variables: $Z = \frac{Q'}{Q}$, $Z^* = \zeta Z$, $\zeta^* = \ln\zeta$, equation (23) transforms into the second order equation

$$(Z^* - \alpha)\,(2Z^* - 3Z^{*\prime} + Z^{*\prime\prime}) + (Z^* - 3\alpha)\,(-Z^{*2} + Z^*\cdot Z^{*\prime}) + (1-\alpha)\,Z^{*3} + (2Z^* - Z^{*\prime})\cdot(-Z^* + Z^{*\prime}) = 0$$

Introducing $V = Z^{*\prime}$ as new dependent and $x = Z^*$ as new independent variable this equation is further reduced to the first order equation

$$(x-\alpha)\,V\frac{dV}{dx} - V^2 + (x^2 - 3\alpha x + 3\alpha)\,V - x\,[\alpha x^2 + (1-3\alpha)x + 2\alpha] = 0$$

$$D \sim \frac{1}{k} \sim \frac{U\Omega L}{g\frac{\Delta\varrho}{\bar{\varrho}}} \tag{28}$$

The neglect of the effect of the finite depth requires that $\frac{D}{h} \ll 1$, or

$$\left\{\frac{U}{\sqrt{g\frac{\Delta\varrho}{\bar{\varrho}}h}}\right\}^2 \ll \frac{U}{\Omega L} \tag{29}$$

This inequality states simply that the square of the internal Froude number is much smaller than the Rossby number.

5. Numerical estimate of the characteristic parameters

Assuming $h = 5 \cdot 10^5$ cm, $L = 10^9$ cm, $U = 1$ cm sec^{-1}, $g = 10^3$ cm sec^{-2}, $\frac{\Delta\varrho}{\bar{\varrho}} = 10^{-3}$, $\Omega = 10^{-4}$ sec^{-1}, $\nu = 10^2$ cm^2 sec^{-1}, $\varkappa = 10^2$ cm sec^{-1}, one arrives at the following values for the five characteristic parameters entering in the condition (1):

$$\frac{h}{L} \sim 5 \cdot 10^{-4}, \quad \frac{U}{L\Omega} \sim 10^{-5}, \quad \frac{1}{h}\sqrt{\frac{\nu}{\Omega}} \sim 2 \cdot 10^{-4},$$

$$\frac{1}{h}\sqrt{\frac{\varkappa L}{U}} \sim 6 \cdot 10^{-1}, \quad \frac{U\Omega}{g\frac{\Delta\varrho}{\bar{\varrho}}} \sim 10^{-4}$$

Furthermore, the internal Froude number entering in the condition (29) takes on the value

$$\frac{U}{\sqrt{g\frac{\Delta\varrho}{\bar{\varrho}}h}} \sim 1.4 \cdot 10^{-3}$$

One can, of course, always argue about the precise numerical values to be chosen for such quantities as L, U, $\frac{\Delta\varrho}{\bar{\varrho}}$, ν and $\varkappa$, but unless the values are definitely wrong our assumptions should on the whole be justified. The assumption that the diffusion effects are limited to a thin boundary layer is certainly not justified by the present estimate, since the parameter $\frac{1}{h}\sqrt{\frac{\varkappa L}{U}}$ takes on a value of the order 1, but it should be noted that this value is obtained by using a characteristic velocity for the deep sea. In the boundary layer itself the velocities must be one to two orders of magnitude larger and a more satisfactory value of the boundary layer thickness is then obtained.

With the above numerical values the depth of penetration of the surface disturbances according to (28) becomes

$$D \sim 1000 \text{ m}$$

This value certainly is of the correct order of magnitude (cf. Fig. 1).

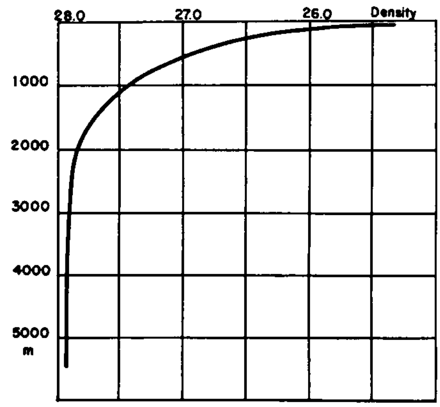

Fig. 1. Example of the vertical density variation in a real ocean. (Average value at 22° S in South Atlantic, computed from DEFANT and WUST, 1936).

6. The density field

With the solution (26) for M the density perturbation field becomes

$$\varrho' = \varrho_0'(\phi, \Theta)\, e^{\frac{kz}{\sin\Theta}} \tag{30}$$

where ϱ_0' (ϕ, Θ) is the surface distribution. According to (30) the perturbation density should decay exponentially downwards from the boundary layer, and such a law will certainly fit the oceanographic observations well (Fig. 1). The depth of penetration should increase from the equator to the pole, varying proportional to the sine of the latitude. This result seems also to be in qualitative agreement with the observations.

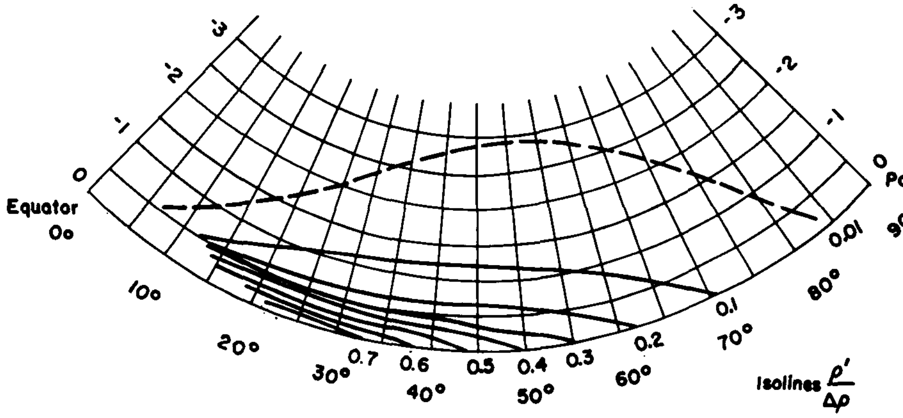

Fig. 2. Theoretical density distribution in a meridional plane. Surface density distribution: $\varrho'_0 = -\Delta\varrho \cos^2 \Theta$. Vertical coordinate: kz.

From (30) the latitudinal variation of ϱ' is found to be

$$\frac{\partial \varrho'}{\partial \Theta} = \left\{\frac{\partial \varrho_0'}{\partial \Theta} - \varrho_0' \cot \Theta \frac{kz}{\sin \Theta}\right\} e^{\frac{kz}{\sin \Theta}}$$

According to this formula $\frac{\partial \varrho'}{\partial \Theta}$ has the sign of $\frac{\partial \varrho_0'}{\partial \Theta}$ at high latitudes, while at low latitudes it has the sign of ϱ_0', at points below the surface. In a realistic case one would have $\varrho_0' < 0$, $\frac{\partial \varrho_0'}{\partial \Theta} > 0$, and $\frac{\partial \varrho'}{\partial \Theta}$ should be positive at high latitudes but negative at low latitudes, at points below the surface. Thus the density isosurfaces should lie at maximum depth at middle latitudes and rise towards the surface both at the low and the high latitudes. This result seems also to be supported by the observations.

As a numerical example the meridional density distribution corresponding to the surface distribution $\varrho_0' = -\Delta\varrho \cos^2 \Theta$ has been constructed (Fig. 2). In this case ϱ_0' runs from a negative value $-\Delta\varrho$ at the equator to a value 0 at the pole, in approximate agreement with conditions in the real oceans. (One should assume here that $\Delta\varrho$ shows some longitudinal variation so that the degenerated axially symmetric case is avoided.) For comparison we show the mean meridional density distribution in the South Atlantic, as computed from the "Meteor"-data (Fig. 3).

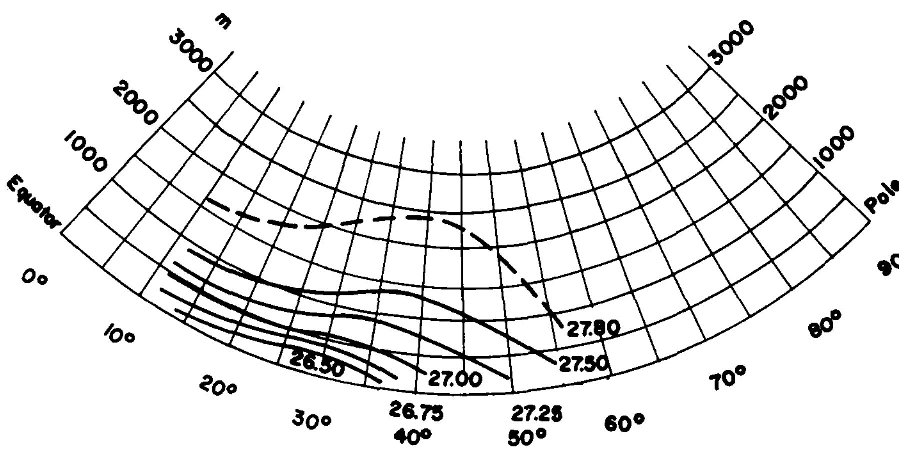

Fig. 3. Example of the meridional density distribution in a real ocean. (Average values for South Atlantic, computed from DEFANT and WUST, 1936).

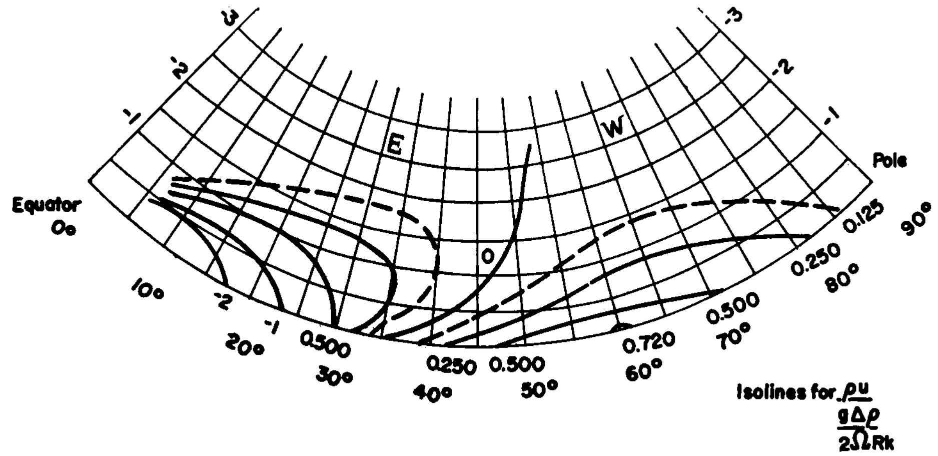

Fig. 4. Theoretical distribution of the zonal velocities in a meridional plane. Surface density distribution: $\varrho'_0 = -\Delta\varrho \cos^2 \Theta$. Vertical coordinate: kz.

7. The velocity field

The velocity field corresponding to the solution (26) is given by

$$\left.\begin{aligned} \varrho u &= (\varrho u)_0 \left\{1 - \alpha(\phi, \Theta) \cdot \frac{kz}{\sin\Theta}\right\} e^{\frac{kz}{\sin\Theta}} \\ \varrho v &= (\varrho v)_0 e^{\frac{kz}{\sin\Theta}} \\ \varrho w &= (\varrho w)_0 e^{\frac{kz}{\sin\Theta}} \end{aligned}\right\} \quad (31)$$

where

$$\alpha = \frac{1}{1 + \operatorname{tg}\Theta \dfrac{1}{\varrho'_0} \dfrac{\partial \varrho'_0}{\partial \Theta}}$$

and the subscript o denotes surface values. In terms of the surface density the surface velocities become

$$\left.\begin{aligned} (\varrho u)_0 &= \frac{g}{2\Omega Rk} \left\{\frac{\partial \varrho'_0}{\partial \Theta} + \cot\Theta \varrho'_0\right\} \\ (\varrho v)_0 &= \frac{g}{2\Omega Rk \cos\Theta} \frac{\partial \varrho'_0}{\partial \phi} \\ (\varrho w)_0 &= -\frac{g}{2\Omega R^2 k^2} \frac{\partial \varrho'_0}{\partial \phi} \end{aligned}\right\} \quad (32)$$

From the above relations one can conclude:

(a) ϱv and ϱw decay exponentially downwards, while ϱu may show a maximum below the surface. Such a maximum is found when $\alpha > 1$, i. e. when $-\cot\Theta < \dfrac{1}{\varrho'_0} \dfrac{\partial \varrho'_0}{\partial \Theta} < 0$. In the realistic case where $\varrho'_0 < 0$, $\dfrac{\partial \varrho'_0}{\partial \Theta} > 0$ a maximum will always occur at low latitudes.

(b) $(\varrho v)_0$ and $(\varrho w)_0$ are proportional to $\dfrac{\partial \varrho'_0}{\partial \phi}$, while $(\varrho u)_0$ depends both on $\dfrac{\partial \varrho'_0}{\partial \Theta}$ and ϱ'_0. In the realistic case $\varrho'_0 < 0, \dfrac{\partial \varrho'_0}{\partial \Theta} > 0$, one would find $(\varrho u)_0 > 0$ (westerlies) at high latitudes, and $(\varrho u)_0 < 0$ (easterlies) at low latitudes.

A computation of the zonal velocity field has been carried out for the specific case $\varrho'_0 = -\Delta\varrho \cos^2\Theta$, and the result is seen in Fig. 4.

The prediction by the theory of a prevailing westerly surface flow at the high latitudes and easterly surface flow at the low latitudes is certainly in general agreement with observations. We cannot, of course, hope to depict the detailed current systems in the oceans by the present crude model, and we do not give any reference here to actual velocity data.

APPENDIX

Solution for the dish-pan

In order to make possible a numerical comparison between the theory and experiments under laboratory conditions we derive here the M-equation for a fluid with parabolic equipotentials. In this case the apparent gravity varies with "latitude" and some new effects can be expected.

To be able to make a direct comparison with the spherical case studied earlier we introduce similar coordinates: a longitude coordinate ϕ, a latitude coordinate Θ, and the vertical height z (Fig. 5). In the derivation of the M-

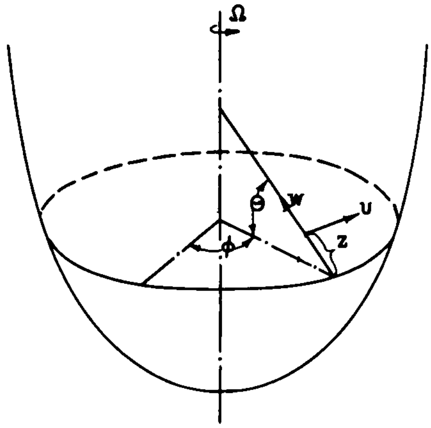

Fig. 5. Coordinates in the paraboloidal system.

equation one may, however, work more practically with "quasi-coordinates" x, y defined by the relations

$$\left.\begin{aligned} dx &= r\,d\phi \\ dy &= R\,d\Theta \end{aligned}\right\} \qquad (33)$$

where r is the normal distance to the central axis and R is the radius of curvature of the paraboloid. r and R are both functions of Θ. Introducing the velocity components u, v, w in the x, y and z-directions, the perturbation pressure p' and the perturbation density ϱ', the Coriolis parameter $f = f(y) = 2\Omega\sin\Theta$, and the apparent acceleration of gravity $g^* = g^*(y) = \frac{g}{\sin\Theta}$, and applying the same physical simplifications as are expressed in the spherical case by the conditions (1a)—(1d) one derives the system of equations

$$-f(y)(\varrho v) = -\frac{\partial p'}{\partial x} \qquad (34)$$

$$f(y)(\varrho u) = -\frac{\partial p'}{\partial y} \qquad (35)$$

$$0 = -\frac{\partial p'}{\partial z} - g^*(y)\varrho \qquad (36)$$

$$\frac{\partial}{\partial x}(\varrho u) + \frac{\partial}{\partial y}(\varrho v) + \frac{\partial}{\partial z}(\varrho w) + 0 \qquad (37)$$

$$(\varrho u)\frac{\partial \varrho}{\partial x} + (\varrho v)\frac{\partial \varrho}{\partial y} + (\varrho w)\frac{\partial \varrho}{\partial z} = 0 \qquad (38)$$

Expressing ϱu, ϱv, and ϱw in terms of ϱ' one finds

$$\left.\begin{aligned} \varrho u &= \frac{g^*(y)}{f(y)}\int_{-\infty}^{z}\frac{\partial \varrho'}{\partial y}\,dz + \frac{\gamma(y)}{f(y)}\int_{-\infty}^{z}\varrho'\,dz \\ \varrho v &= -\frac{g^*(y)}{f(y)}\int_{-\infty}^{z}\frac{\partial \varrho'}{\partial x}\,dz \\ \varrho w &= -\frac{\beta(y)}{f^2(y)}\cdot g^*(y)\int_{-\infty}^{z}\int_{-\infty}^{z}\frac{\partial \varrho'}{\partial x}(dz)^2 \end{aligned}\right\} \qquad (39)$$

where

$$\beta(y) = \frac{df}{dy}, \quad \gamma(y) = \frac{dg^*}{dy}$$

ϱv and ϱw are seen to have the same form as in the spherical case, where the acceleration of gravity is constant, while ϱu contains an extra term proportional to the vertically integrated perturbation density. Introducing the variable M defined by (9) in Sec. 3 one is led to the equation

$$M_{xzz}M_{yz} - M_{yzz}M_{xz} = \frac{\beta}{f}M_x M_{zzz} - {} - \frac{\gamma}{g^*}M_z M_{xzz} \qquad (40)$$

It is easily shown that for a paraboloid $\frac{\beta}{f} = -\frac{\gamma}{g^*}$. Thus equation (40) is reduced to the form

$$M_{xzz}M_{yz} - M_{yzz}M_{xz} = \frac{\beta}{f}(M_x M_{zzz} + M_z M_{xzz})$$

or, after introducing the coordinates ϕ and Θ,

$$\boxed{\begin{aligned} &M_{\phi zz} M_{\Theta z} - M_{\Theta zz} M_{\phi z} = \\ &= \cot \Theta \,(M_\phi \, M_{zzz} + M_z M_{\phi zz}) \end{aligned}} \qquad (41)$$

This form should be compared with the equation (14) in the spherical case.

Introducing the new variable $\eta = \log \sin \Theta$ (41) simplifies to

$$M_{\phi zz} M_{\eta z} - M_{\eta zz} M_{\phi z} = M_\phi \, M_{zzz} + M_z \, M_{\phi zz} \qquad (42)$$

Attempting a solution of the form $M = P(\phi, \eta) \, Q\,(\eta, z)$ gives

$$Q_{zz} Q_{\eta z} - Q_{\eta zz} Q_z = Q Q_{zzz} + Q_z Q_{zz} \qquad (43)$$

and again solutions of the form $Q = Q\left(\frac{z}{F(\eta)}\right)$ exist provided $F(\eta) = \text{const.}\ e^{\frac{\eta}{\alpha}}$. Q is then to be found from the ordinary differential equation

$$\zeta \,[Q' Q''' - (Q'')^2] + (1 - \alpha)\, Q' Q'' - \alpha Q Q''' = 0 \qquad (44)$$

where $$\zeta = \frac{z}{F(\eta)}$$

In this case an exponential solution of the form $Q = \text{const}\ e^{k\zeta}$ exists if one chooses $\alpha = \frac{1}{2}$. Thus we get

$$M = P(\phi, \eta)\, e^{k\zeta} = P(\phi, \eta)\, e^{kze^{-2\eta}}$$

or, in terms of the original coordinates,

$$\boxed{M = M_0(\phi, \Theta)\, e^{\frac{kz}{\sin^2 \Theta}}} \qquad (45)$$

The solution for the dish-pan is accordingly of essentially the same type as for the sphere, the only difference being that the depth of penetration is now proportional to the square of the sine of the "latitude". This cannot, however be expected in more general cases, due to the occurrence of the new term $M_z M_{\phi zz}$ in the equation (41).

Acknowledgements

The author wants to thank Dr. Benton and Dr. Long of the Johns Hopkins University for their valuable comments and advice during the course of the work.

REFERENCES

LINEYKIN, P. S., 1955: On the determination of the thickness of the baroclinic layer in the sea. *Doklady SSSR Akad. Nauk Tom* **101** p. 461.

STOMMEL, H., and VERONIS, G., 1957: Steady Convective motion in a horizontal layer of fluid heated uniformly from above and cooled non-uniformly from below. *Tellus* **9** pp. 401—407.

DEFANT, A., und WUST, G., 1936: Schichtung und Zirkulation des Atlantischen Ozeans, Wiss. Ergebnisse Deutschen Atlantischen Exp. *Meteor* 1925—27, Bd *VI.*