On the Frequency Response of Some Different Models Describing the Transient Exchange of Matter between the Atmosphere and the Sea

By PIERRE WELANDER, Institute of Theoretical Physics, University of Stockholm, and International Meteorological Institute in Stockholm

(Manuscript received March 18, 1959)

Abstract

A theoretical study is made of the response characteristics of two "box-models" and two continous models of the sea which could describe an air-sea exchange of matter. It is demonstrated quantitatively that the box-model can depict a transient exchange process correctly only when the time of internal mixing of the sea is small compared to a certain "transfer time". Comparing the two continous models studied, one of which assumes a purely diffusive sea and the other a purely advective sea, one finds some interesting differences in the responce characteristics.

In recent years there has been a great deal of interest in transient air-sea exchange processes involving different radioactive and nonradioactive elements. As one example we may mention the uptake in the sea of radioactive materials produced by atomic bomb explosions. Another example is the extra addition to the sea of carbon dioxide that has been released in the atmosphere during the last half century by combustion of fossil fuels.

In special, this last problem has been subjected to several theoretical investigations (CRAIG 1957, REVELLE-SUESS 1957, BOLIN-ERIKSSON 1959). In these investigations use has been made of so-called "box-models", in which the atmosphere and the sea are considered as separate and internally well-mixed reservoirs. In some cases the sea has been split up in two reservoirs, one comprising the uppermost layer down to about 100 m depth, the "mixed layer", and the other comprising the deep sea.

The general structure of such a box-model is as follows. Each box i is characterized by the amount of material under consideration N_i and by a number of transfer coefficients k_{ij}. The flux of material from the reservoir i to the reservoir j is assumed to be $k_{ij}N_i$ (Fig. 1). One can then set up the continuity equation for the box i as follows:

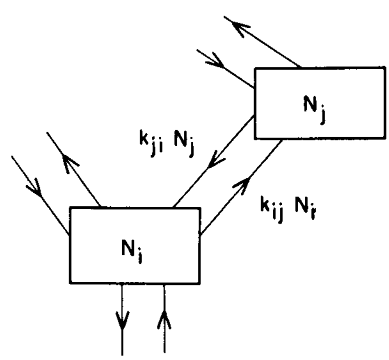

Fig. 1. The general box-model.

$$\frac{dN_i}{dt} = \sum_{\substack{j=1 \\ (j \neq i)}}^{n} k_{ji} N_j - \sum_{\substack{j=1 \\ (i \neq j)}}^{n} k_{ij} N_i + P_i - \gamma N_i \qquad (1)$$

$\frac{dN_i}{dt}$ is the rate of increase of material, P_i is the eventual production in the box and $-\gamma N_i$ represents the decay, if the material is radioactive. Writing down such continuity equations for all the boxes $i = 1, 2, \ldots\ldots n$ gives us a closed system of equations for the determination of the N_i:s as a function of time.

The transient behaviour of the box-model is correct under the assumptions

(i) that the exchange between the reservoirs is effectuated by a steadily working transport mechanism,

(ii) that each reservoir is internally well-mixed.

The first assumption asserts that the model is linear. Increasing all concentrations to the double values should simply double all the fluxes between the reservoirs. The second assumption asserts that these fluxes are proportional to the total amounts of matter in the boxes. While the first assumption on the whole seems to be justified the second assumption is more critical. It requires that the matter is uniformly mixed in the reservoir within a time τ_i^M short compared to the "transfer times" $\tau_{ij} = \frac{1}{k_{ij}}$ characterizing the reservoir. If there is a production and radioactive decay of matter in the reservoir, τ_i^M must also be small compared to a characteristic production time τ_i^P and the decay-time $\tau^D\left(=\frac{1}{\gamma}\right)$. It should be stressed that this is the requirement that the box-model depicts correctly the *transient* change of the reservoir (the deviations from the momentary equilibrium state). When studying the over-all transient change of a system it is not absolutely necessary to consider the transient changes of every reservoir. Those reservoirs which have mixing- and transfer-times very short compared to the period of interest can often be treated as equilibrium reservoirs, and even if they are not well-mixed in the above sense a boxmodel representation may again be justified.

The purpose of the present note is to compare the behaviour of two different box-models and two continous models of the sea in a simple air-sea exchange problem. In all the cases the atmosphere is treated as an equilibrium reservoir, which seems to be justified in most cases of practical interest. For the sea we assume:

(i) a one-layer box-model
(ii) a two-layer box-model
(iii) a diffusive sea
(iv) an advective sea

The investigation is restricted to the case of a non-radioactive material, it may become actual to study later also the case of radioactive matters.

Contrasting the results obtained for the box-models and the continous models should yield some precise information of the defects of the box-model in such cases where the above-mentioned assumptions are not satisfied. The comparison of the results valid for the two continous models should also be of interest. The question whether matter in the sea is transported mainly by advection or diffusion seems so far not to be settled, and it is possible that the application of theoretical results such as are derived here could cast some further light on this.

In the comparison between the models use is being made of the so-called frequency response diagram, which has found much use in engineering sciences. It should certainly prove useful in discussions on transient exchange and transport problems in geophysics, which are governed by linear equations. An advantage of the technique that should bepointed out is that it can also be applied to observed data. This should be of special value when de aling with such transport links as, say, the biologic transport, for which it seems very hard to set up a specific mathematical model.

Case (i)

The simple box-model representation of the sea is seen in Fig. 2 a. The governing equation for the amount N_s of our matter in the sea is

$$\frac{dN_s}{dt} = k_a N_a - k_s N_s \qquad (2)$$

Considering N_a as a "forcing function" and N_s as a "response function" we like to solve for the ratio $r = \frac{\Delta N_s}{\Delta N_a}$, where ΔN_a and ΔN_s are the complex amplitudes in a harmonically varied state:

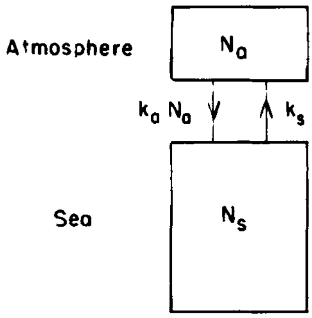

Fig. 2 a. Simple box-model for the air-sea exchange.

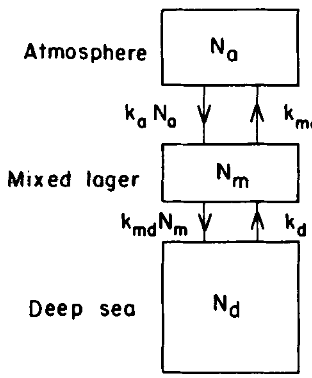

Fig. 2 b. Two-layer box-model for the air-sea exchange.

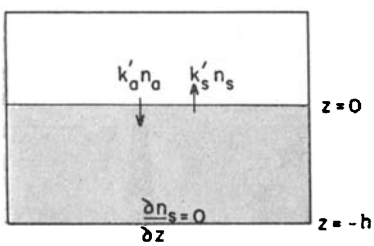

Fig. 2 c. Diffusive sea.

$$\left.\begin{aligned} N_a &= \overline{N}_a + \Delta N_a e^{i\omega t} \\ N_s &= \overline{N}_s + \Delta N_s e^{i\omega t} \end{aligned}\right\} \qquad (3)$$

One finds easily

$$r(\omega) = \frac{\Delta N_s}{\Delta N_a} = \frac{k_a}{k_s + i\omega} = r(0)\,\frac{1}{1 + i\dfrac{\omega}{\omega_s}} \qquad (4)$$

with $r(0) = \frac{k_a}{k_s}$, $\omega_s = k_s$. This complex amplitude ratio can be represented by a curve in the complex plane. In this case the curve is

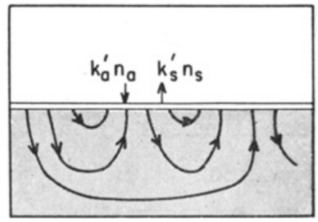

Fig. 2 d. Advective sea.

the half-circle seen in Fig. 3. From this frequency response curve we can read the amplitude and phase of ΔN_s relative to ΔN_a for any frequency. As example, for $\omega = \omega_s$ we see that the amplitude (magnitude) of ΔN_s is $\frac{1}{\sqrt{2}} \sim 0.71$ of the value given by an equilibrium theory ($\omega = 0$), and that the phase difference between ΔN_s and ΔN_a is $-\frac{\pi}{4}$. The frequency response curve obviously contains all information about the model. Every forcing function $N_a(t)$ can be split up in harmonic components, the N_s-responses of which can be read from the response curve, and these harmonic responses can be summed up to give the complete time-response $N_s(t)$ of the model.

Case (ii)

The two-layer box-model of the sea is presented in Fig. 2b. The governing equations are now

$$\left.\begin{aligned} \frac{dN_m}{dt} &= k_a N_a + k_d N_d - (k_{ma} + k_{md}) N_m \\ \frac{dN_d}{dt} &= k_{md} N_m - k_d N_d \\ N_s &= N_m + N_d \end{aligned}\right\} \qquad (5)$$

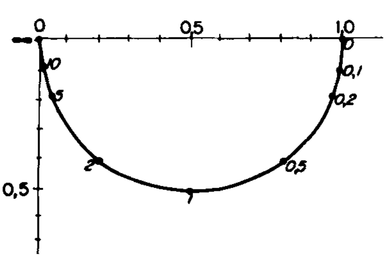

Fig. 3. Frequency response curve for the simple box-model. The figures along the curve give the values of $\frac{\omega}{\omega_s}$.

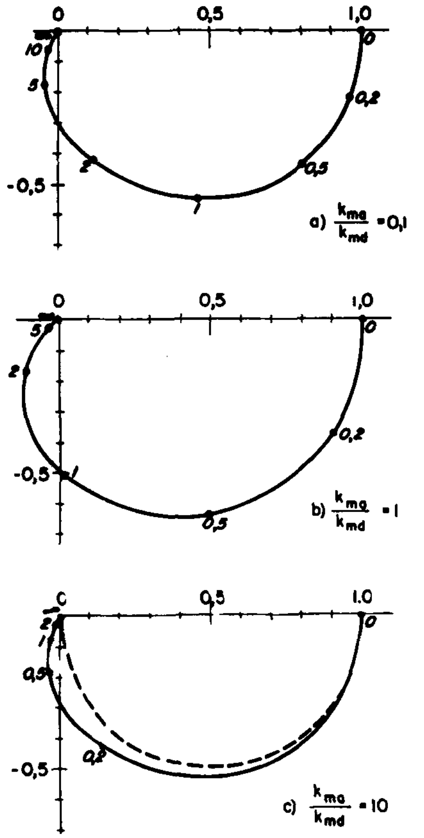

Fig. 4 a—c. Frequency response curves for the two-layer box-model. The figures along the curve give the values of $\frac{\omega}{\omega_s}$. The case $\frac{k_{ma}}{k_{md}} = 0$ corresponds to Fig. 3. The dashed line in Fig. 4 c represents the case $\frac{k_{ma}}{k_{md}} = \infty$. In all cases $\frac{k_d}{k_{md}} = \frac{1}{50}$.

N_s is again the total amount of our matter in the sea. In this case one finds,

$$r(\omega) = \frac{\Delta N_s}{\Delta N_a} =$$

$$= r(0) \frac{1 + \frac{k_{ma} k_d}{(k_{md}+k_d)^2} i \frac{\omega}{\omega_s}}{1 + \left(1 + \frac{k_{ma}}{k_{md}+k_d}\right) i \frac{\omega}{\omega_s} - \frac{k_{ma} k_{md}}{(k_{md}+k_d)^2} \left(\frac{\omega}{\omega_s}\right)^2} \quad (6)$$

Since the deep sea can store a very large amount of matter compared to the mixed layer ($N_d \gg N_m$) we should have $k_d \ll k_{md}$ and (6) simplifies to

$$r(\omega) = r(0) \frac{1 + \frac{k_{ma}}{k_{md}} \cdot \frac{k_d}{k_{md}} \cdot i \frac{\omega}{\omega_s}}{1 + \left(1 + \frac{k_{ma}}{k_{md}}\right) i \frac{\omega}{\omega_s} - \frac{k_{ma}}{k_{md}} \left(\frac{\omega}{\omega_s}\right)^2} \quad (6\text{a})$$

The form of the response curve depends here on the two parameters $\frac{k_{ma}}{k_{md}}$ and $\frac{k_d}{k_{md}}$. As is seen from (6a), if we let $\frac{k_{ma}}{k_{md}} \to 0$ for a fixed ratio $\frac{k_d}{k_{md}}$ we get again the solution of the simple box-model. This result should be anticipated, since we have here a case where the internal adjustment between the mixed layer and the deep sea becomes infinitely rapid. For finite values of $\frac{k_{ma}}{k_{md}}$ the result may, however, differ considerably from the result of the simple box-model. Assuming that the depth of the mixed layer is about 100 m and the total depth of the oceans about 5,000 m we would guess that $\frac{k_d}{k_{md}} \sim \frac{1}{50}$, and using this value the frequency response curves have been computed for some different values of the parameter $\frac{k_{ma}}{k_{md}}$. The result is seen in Fig. 4.

Case (iii)

In the case of a diffusive sea (Fig. 2c) we assume that the matter introduced at the top of the sea spreads vertically over a depth h by turbulent diffusion. The coefficient of diffusion is assumed to be constant. In the horizontal directions conditions should be uniform everywhere. It is of interest to note that this diffusive model is approached by a multi-layer box-model where the division in horizontal layers is made more and more refined.

The governing equation for the diffusion process is

$$\frac{\partial n_s}{\partial t} = K \frac{\partial^2 n_s}{\partial z^2} \quad (7)$$

n_s is the mass concentration of the matter, K the coefficient of diffusion, and z a vertical coordinate. The boundary conditions are

$$\left.\begin{aligned} &k_a' n_a - k_s' n_s = \varrho_s K \frac{\partial n_s}{\partial z} \\ &\qquad \text{at } z = 0 \text{ (the sea surface)} \\ &\frac{\partial n_s}{\partial z} = 0 \\ &\qquad \text{at } z = -h \text{ (the sea bottom)} \end{aligned}\right\} \quad (8)$$

where k_a', k_s' are transfer coefficients valid at the sea surface and ϱ_s is the density of water. (8) expresses the equality between the flux of mass through unit area of the boundary layer at the sea surface (including both the atmospheric and the oceanic boundary layers) and the turbulent mass flux in the sea just below the boundary layer (note that K is multiplied by ϱ_s to give an absolute mass transport). The transfer coefficients k_a' and k_s' are, of course, related to the coefficients k_a and k_s used earlier. In equilibrium the integrated flux of matter from the atmosphere into the sea is $k_a' n_a A$ and the equally large return flux $k_s' n_s A$, where A is the sea surface area. These fluxes may, however, also be written $k_a N_a$ and $k_s N_s$, respectively. With $N_a = n_a M_a$, $N_s = n_s M_s$, where M_a and M_s are the masses of the atmosphere and the oceans, respectively, we find $k_a = \frac{k_a' A}{M_a}$, $k_s = \frac{k_s' A}{M_s}$. As before we assume a harmonic variation in time:

$$n_a = \bar{n}_a + \Delta n_a e^{i\omega t}$$
$$n_s = \bar{n}_s + \Delta n_s(z) e^{i\omega t}$$

The solution to equation (7) which satisfies the boundary conditions (8) is then

$$\Delta n_s = \frac{k_a' \Delta n_a}{k_s' \cosh \sqrt{\frac{i\omega}{K}}\, h + K \sqrt{\frac{i\omega}{K}} \sinh \sqrt{\frac{i\omega}{K}}\, h} \times \cosh \sqrt{\frac{i\omega}{K}}\,(z+h) \quad (9)$$

Introducing the actual values of ΔN_a and ΔN_s from the relations $\Delta N_a = M_a \Delta n_a$, $\Delta N_s = M_s \overline{\Delta n_s} = M_s \frac{1}{h} \int_{-h}^{0} \Delta n_s \, dz$ and introducing also the transfer coefficients k_a, k_s given earlier we can finally write

$$r(\omega) = \frac{\Delta N_s}{\Delta N_a} = \quad (10)$$

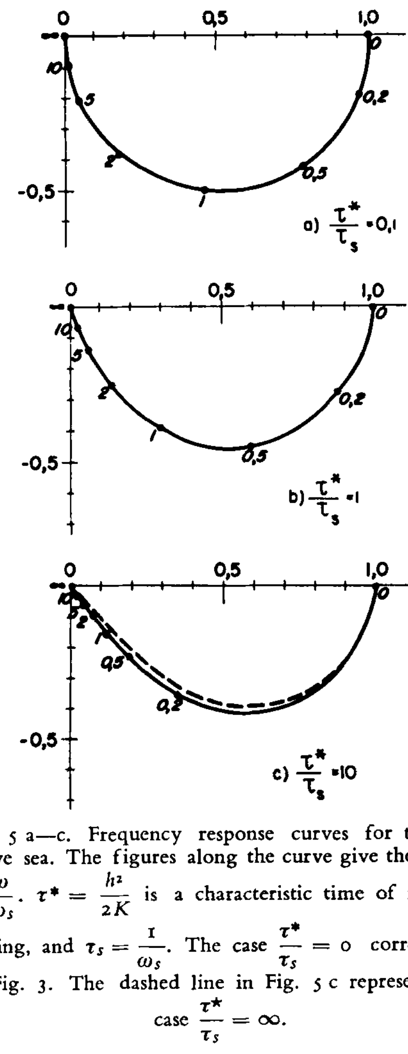

Fig. 5 a—c. Frequency response curves for the diffusive sea. The figures along the curve give the values of $\frac{\omega}{\omega_s}$. $\tau^* = \frac{h^2}{2K}$ is a characteristic time of internal mixing, and $\tau_s = \frac{1}{\omega_s}$. The case $\frac{\tau^*}{\tau_s} = 0$ corresponds to Fig. 3. The dashed line in Fig. 5 c represents the case $\frac{\tau^*}{\tau_s} = \infty$.

$$= r(0) \frac{1}{(1+i)\sqrt{\frac{\tau^*}{\tau_s}}\sqrt{\frac{\omega}{\omega_s}} \coth (1+i) \sqrt{\frac{\tau^*}{\tau_s}}\sqrt{\frac{\omega}{\omega_s}} + i\frac{\omega}{\omega_s}}$$

Here $r(0) = \frac{k_a}{k_s}$, $\omega_s = k_s$, $\tau_s = \frac{1}{k_s}$. τ^* is a new parameter with the value $\tau^* = \frac{h^2}{2K}$. It represents a characteristic mixing time for the sea. If this time is very small compared to τ_s, we find again the response curve of the simple box-model, as expected. For other values of the parameter $\frac{\tau^*}{\tau_s}$ we get deviations from the box-model curve, as seen from Fig. 5.

Case (iv)

In this case all internal transports in the sea are advective. The specific model is sketched in Fig. 2 d. On top we have a thin surface layer, which is assumed to be well mixed horizontally. From this surface layer water is advected into the deep sea, carrying with it our matter, and returning to the surface layer again after a certain circulation time. This time will in general vary from one particle to the next. Consider now a small "flux-tube" having a given circulation time τ. If the mass of water in the tube is dM and the mass flux through it is dF we have

$$dM = \tau dF \tag{11}$$

The "flux-mean" circulation time is

$\bar{\tau} = \frac{1}{F_s}\int_0^{F_s} \tau dF$, where F_s is the total mass flux.

From (11) we find

$$\bar{\tau} = \frac{M_s}{F_s} \tag{12}$$

This time represents a kind of "mixing time" for our system.

Let now $F(\tau)$ be the distribution function for the flux after τ ($F(\tau)$ gives the total mass flux for which the circulation time is smaller then τ). The distribution of M after τ is then also fixed, by equation (11). Consider the balance of matter in the entire surface layer. There is a net transport from the atmosphere $A(k_a'n_a - k_s'n_s)$, as in the diffusive model studied earlier, there is an advective transport $F_s n_s$ into the deep sea, and there is a return transport from the deep sea $\int_0^{F_s} n_s(t-\tau)\,dF(\tau)$. Neglecting the capacity of the thin surface layer we have

$$A(k_a'n_a - k_s'n_s) - F_s n_s + \int_0^{F_s} n_s(t-\tau)\,dF(\tau) = 0 \tag{13}$$

As before, we write $n_a = \bar{n}_a + \Delta n_a e^{i\omega t}$, $n_s = \bar{n}_s + \Delta n_s e^{i\omega t}$. Note that n_s is the momentary concentration in the *surface layer*. From Δn_a and Δn_s we can come over to ΔN_a and ΔN_s by help of the relations $\Delta N_a = M_a \Delta n_a$, $\Delta N_s = \Delta n_s \int_0^{M_s} e^{-i\omega\tau} dM(\tau)$ (from the relation $N_s = \int_0^{M_s} n_s(t-\tau)\,dM(\tau)$). Expressing k_a' and k_s' in terms of k_a and k_s by help of previous formulas and replacing dM by τdF according to (11) we get now

$$r(\omega) = \frac{\Delta N_s}{\Delta N_a} = \frac{k_a \frac{1}{M_s}\int_0^{\tau_m} e^{-i\omega t}\tau \frac{dF}{d\tau} d\tau}{k_s + \frac{1}{\bar{\tau}}\left[1 - \frac{1}{F_s}\int_\tau^{\tau_m} e^{-i\omega t}\frac{dF}{d\tau} d\tau\right]} \tag{14}$$

τ_m represents here the maximum circulation time of the sea. To come further we must make some specific assumption regarding the function $F(\tau)$. The simplest reasonable assumption we can make is that it is linear, running from 0 to F_s when τ runs from 0 to τ_m:

$$\frac{F(\tau)}{F_s} = \frac{\tau}{\tau_m} \tag{15}$$

In this case $\tau_m = 2\bar{\tau}$. We can now write

$$r(\omega) = r(0)\frac{\left(1 + 2i\frac{\bar{\tau}}{\tau_s}\frac{\omega}{\omega_s}\right)e^{-2i\frac{\bar{\tau}}{\tau_s}\frac{\omega}{\omega_s}} - 1}{\frac{i\omega}{\omega_s}\left(1 - e^{-2i\frac{\bar{\tau}}{\tau_s}\frac{\omega}{\omega_s}}\right) + 2\frac{\bar{\tau}}{\tau_s}\left(\frac{\bar{\tau}}{\tau_s} + 1\right)\left(\frac{\omega}{\omega_s}\right)^2} \tag{16}$$

where $r(0) = \frac{k_a}{k_s}$, $\omega_s = k_s$.

Again, making our "mixing time" $\bar{\tau}$ small compared to τ_s we will get the result of the simple box-model, while for larger values of $\frac{\bar{\tau}}{\tau_s}$ the result will be quite different. Curves for some different values of $\frac{\bar{\tau}}{\tau_s}$ are seen in Fig. 6.

From the discussion of the above four models we may verify what was stated in the beginning, namely that the simple box-model is a good approximation only when the "mixing time" is small compared to the "transfer time". When the mixing time becomes comparable to or larger than the transfer time, the result deviates from that of the simple box-model in the way shown in the diagrams. It is of interest to note that the largest deviations generally occur at high values of $\frac{\omega}{\omega_s}$, since such high values may be found in cases of practical interest. As example, in the problem of the carbon dioxide uptake in the sea the period of interest $T\left(\sim\frac{1}{\omega}\right)$ is of the order of a few decades while $\tau_s = \frac{1}{\omega_s}$ certainly is expected to take on values of the order of several hundred years.

It is also of interest to note the considerable differences in the response characteristics of the diffusive and the advective sea, when the "mixing time" takes on large values. In the diffusive model the phase angle will be relatively small and negative, reaching a minimum value of about $-\frac{\pi}{4}$. The amplitude ratio is finite and is more or less independent of the ratio of the mixing time and the transfer time, once this ratio is large enough. All changes will then take place in a thin surface layer and the actual capacity of the reservoir is irrelevant. In the advective model the phase angle decreases indefinitely for increasing frequencies. Making the ratio of the mixing time and the transfer time large the amplitude ratio tends to zero.

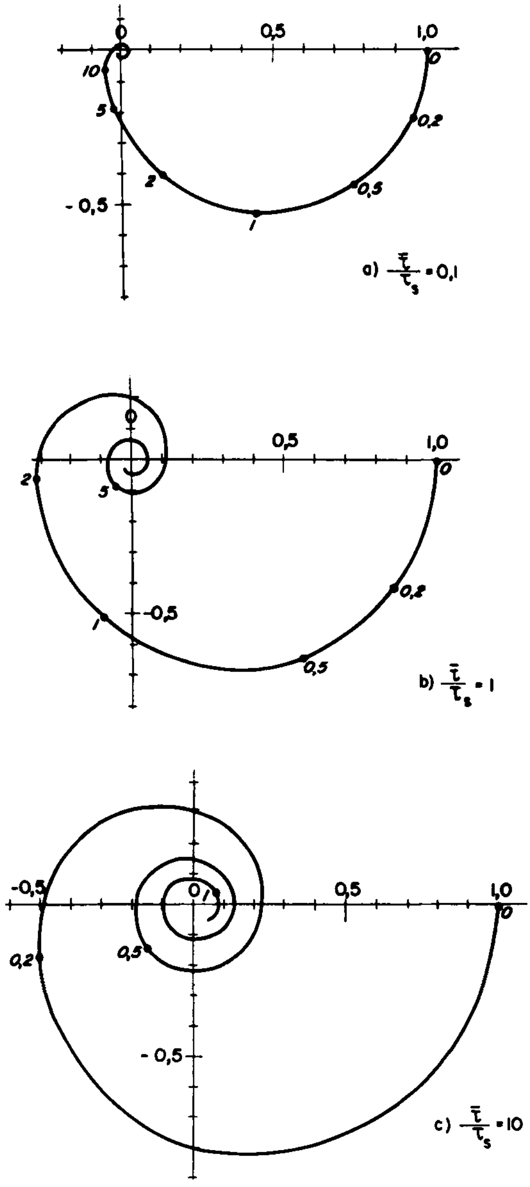

Fig. 6 a—c. Frequency responce curves for the advective sea. The figures along the curve give the values of $\frac{\omega}{\omega_s}$. $\bar{\tau}$ is the mean circulation time, and $\tau_s = \frac{1}{\omega_s}$. The case $\frac{\bar{\tau}}{\tau_s} = 0$ corresponds to Fig. 3.

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